Some real-rooted polynomials in algebraic combinatorics

Katerina Kalampogia-Evangelinou

Department of Mathematics, National and Kapodistrian University of Athens, Greece

joint work with Christos Athanasiadis

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Let L be a finite Partially Ordered Set (poset) and $c_k(L)$ be the number of k-element chains of L.
We consider the chain polynomial of L

$$p_L(x) = f(\Delta(L), x) = \sum_{k \geq 0} c_k(L) x^k$$

which is the f-polynomial of the order complex $\Delta(L)$ of L.

- $\Delta(L)$ is the set of all chains of L (it is a simplicial complex)

For some purposes we may focus on the corresponding h-polynomial of the poset.

$$h_L(x) = h(\Delta(L), x) = (1-x)^n p_L\left(\frac{x}{1-x}\right) =$$

$$= \sum_{k\geq 0} c_k(L) x^k (1-x)^{n-k}$$

where n is the largest size of a chain in L.

$$h_L(x) = h_0 + h_1 x + \cdots + h_n x^n$$

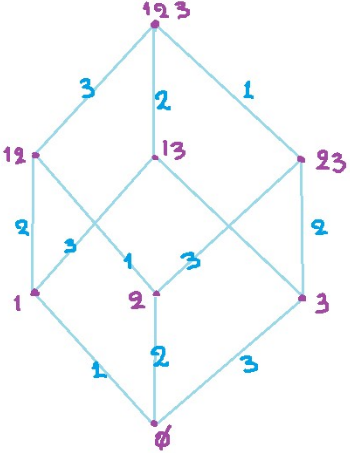
where the coefficients add to the number of n-chains of L.

- If L is Cohen-Macaulay the h-polynomial has nonnegative coefficients
- If L has an R-labeling the coefficients of $h_L(x)$ are given a combinatorial interpretation

- Let $\lambda : C(L) \to \mathbb{N}$ be an edge labeling of the Hasse diagram of L.
- We say that λ is an R-labeling of L if in each closed interval $[x, y]$ of L there exists a unique increasing maximal chain.

If λ is an R-labeling of L we get that $h_L(x) = \sum_c x^{\mathrm{des}(w)}$
where c runs in the maximal chains of L

Example: Let L_n be the lattice of subsets of [n]
(Boolean algebra of rank n)



$$h_{L_n}(x) = A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)}$$

which is the classical n-th Eulerian Polynomial
that counts descents in the Symmetric Group

$$A_n(x) = \begin{cases} 1 & n = 1 \\ 1 + x & n = 2 \\ 1 + 4x + x^2 & n = 3 \\ 1 + 11x + 11x^2 + x^3 & n = 4 \\ 1 + 26x + 66x^2 + 26x^3 + x^4 & n = 5 \end{cases}$$

The Eulerian polynomial $A_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ is

- unimodal

$$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_{n-1}$$

- palindromic

$$a_k = a_{n-1-k}$$

- log-concave

$$a_k^2 \geq a_{k-1} a_{k+1}$$

- gamma-positive

$$A_n(x) = \sum_{k \geq 0} \gamma_k x^k (1+x)^{n-1-2k}$$

$\gamma_k \geq 0$ for every k

- real-rooted

every root of $A_n(x)$ is real

These are properties we often encounter in algebraic and geometric combinatorics.
We focus on the property of real-rootedness
which has strong implications for a polynomial.
Let $f(x) = f_0 + f_1 x + \cdots + f_n x^n$ be a polynomial with nonnegative integer coefficients.
If it is real-rooted, then:

- it is unimodal
- it is log-concave
- if it is also palindromic, it is gamma-positive

Question: For which finite posets does the chain polynomial (equivalently the h-polynomial) have only real roots?

It is not true for all finite posets!

Conjecture (Neggers, 1979)

The chain polynomial of every finite distributive lattice is real-rooted

- equivalent to the poset conjecture
- it was finally disproved by Brändén and Stembridge (2007)

- $(3+1)-$avoiding posets (Stanley, 1998)
- the face lattices of simplicial (and simple) polytopes (Brenti-Welker, 2008)
- the face lattices of cubical polytopes (Athanasiadis, 2020)

Theorem (Athanasiadis, K. E. 2022-2023)

- *the face lattice of Pyr(P) and Prism(P), when the face lattice of P has a real-rooted h-polynomial*
- *the lattices of flats of near-pencils and uniform matroids*
- *the subspace lattice* $L_n(q)$
- *the partition lattices* Π_n *and* Π_n^B
- *the noncrossing partition lattice* $NC(W)$ *for every finite irreducible Coxeter group* W
- *rank-selected subposets of Cohen-Macaulay simplicial posets*

- the face lattice of Pyr(P) and Prism(P), when the face lattice of P has a real-rooted h-polynomial

This gives new families of convex polytopes whose face lattice has real-rooted chain polynomial

Question: Is it true for all convex polytopes? (Brenti-Welker 2008)

- the lattices of flats of near-pencils and uniform matroids
- the subspace lattice $L_n(q)$
- the partition lattices Π_n and Π_n^B

These are examples of geometric (atomic and semimodular) lattices.

Geometric Lattices are exactly the lattices of flats of matroids!

Conjecture

Every finite geometric lattice has a real-rooted chain polynomial.

It would give an affirmative answer to the challenging open problem of the unimodality of the h-polynomial of a geometric lattice.

Theorem (Nyman-Swartz, 2004)

Let L be a geometric lattice of rank n and let

$$h_L(x) = h_0 + h_1 x + \cdots + a_{n-1} x^{n-1}$$

Then:

- $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{n-1}{2} \rfloor}$
- $h_i \leq h_{n-1-i}$ *for* $0 \leq i \leq h_{\lfloor \frac{n-1}{2} \rfloor}$

- the noncrossing partition lattice $NC(W)$ for every finite irreducible Coxeter group W
- rank-selected subposets of Cohen-Macaulay simplicial posets

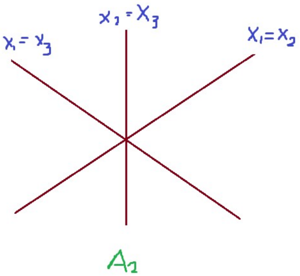
Our main question fails for all Cohen-Macaulay posets. But is true for many families of doubly Cohen-Macaualay posets.

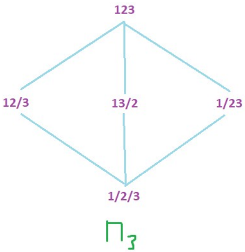
Is it true for all doubly Cohen-Macaulay posets?

In this talk, we will present results for these two classes of posets:

- the Partition Lattices
- the Noncrossing Partition Lattices

The partition lattice Π_n consists of all set partitions of $[n]$, partially ordered by reverse refinement.





It is the intersection lattice of the type A braid arrangement

$$h_{\Pi_n}(x) = \begin{cases} 1 \\ 1 + 2x \\ 1 + 11x + 6x^2 \\ 1 + 47x + 108x^2 + 24x^3 \\ 1 + 197x + 1268x^2 + 1114x^3 + 120x^4 \\ 1 + 870x + 13184x^2 + 29383x^3 + 12542x^4 + 720x^5 \end{cases}$$

The number of maximal chains of Π_n is $\frac{n!(n-1)!}{2^{n-1}}$

Theorem (Athanasiadis, K. E. 2022-2023)

The chain polynomial of the partition lattice is real-rooted.

$$h_{\Pi_n}(x) = \sum_{\sigma \in A_n} x^{\mathrm{des}(\sigma)}$$

$$A_n = \{1\} \times \{1,1,2\} \times \cdots \times \{1,\cdots,1,\cdots,n-2,n-2,n-1\}$$

A_4 *consists of*

- $1|1|1$ *with multiplicity 6*
- $1|12$ *with multiplicity 4*
- $1|13$ *with multiplicity 2*
- $12|1$ *with multiplicity 3*
- $12|2$ *with multiplicity 2*
- 123 *with multiplicity 1*

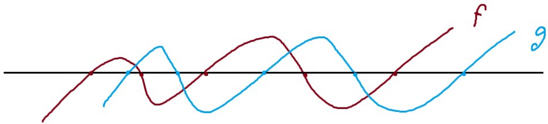
- *combinatorial interpretation: we used Gessel's R-labeling.*

If y covers x in Π_n then y is obtained from x by merging two blocks of x, B_1 and B_2.
We set $\lambda(x, y) = \max\{\min(B_1), \min(B_1)\}$

- *real-rootedness: we used the method of interlacing polynomials*

Let $f(x), g(x)$ be real-rooted polynomials.

- f(x) interlaces g(x) ($f(x) \preceq g(x)$) :
  their roots are interpolating with g(x) having the largest root



- if $f(x) \preceq g(x)$ then f(x)+g(x) is real-rooted

For $k \in [n-1]$ we set $h_{n,k}(x) = \sum_{\sigma \in A_{n,k}} x^{\mathrm{des}(\sigma)}$

where $A_{n,k}$ is the multiset of words $(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}) \in A_n$ with $\sigma_{n-1} = k$

$$h_{\Pi_n}(x) = h_{n+1,n}(x) = \sum_{k=1}^{n-1} h_{n,k}(x)$$

$$h_{n+1,k}(x) = (n+1-k)\left(\sum_{i=1}^{k-1} h_{n,i}(x) + x \sum_{i=k}^{n-1} h_{n,i}(x)\right)$$

by induction on n we prove that:

$(h_{n,n-1}(x), h_{n,n-2}(x), \cdots, h_{n,1}(x))$

is an interlacing sequence of polynomials

so $h_{\Pi_n}(x) = \sum_{k=1}^{n-1} h_{n,k}(x)$ is real-rooted

The partition lattice Π_n^B can be defined as the intersection lattice of the type B braid arrangement.

The number of maximal chains of Π_n^B *is* $(n!)^2$

$$h_{\Pi_n^B}(x) = \begin{cases} 1 + 3x \\ 1 + 20x + 15x^2 \\ 1 + 111x + 359x^2 + 105x^3 \\ 1 + 642x + 5978x^2 + 6834x^3 + 945x^4 \\ 1 + 4081x + 92476x^2 + 268236x^3 + 143211x^4 + 10395x^5 \end{cases}$$

Theorem (Athanasiadis, K. E. 2022-2023)

The h-polynomial of the Partition Lattice of type B is real-rooted.

$$h_{\Pi_n^B}(x) = \sum_{\sigma \in B_n} x^{\mathrm{des}(\sigma)}$$

$$B_n = \{1\} \times \{1, 1, 1, 2\} \times \cdots \times \{1, \cdots, 1, \cdots, n-1, n-1, n-1, n\}$$

B_3 consists of

- $1|1|1$ *with multiplicity 15*
- $1|12$ *with multiplicity 9*
- $1|13$ *with multiplicity 3*
- $12|1$ *with multiplicity 5*
- $12|2$ *with multiplicity 3*
- 123 *with multiplicity 1*

Let W be a finite irreducible Coxeter group.

Definition (absolute order)

We define a partial order $\leq$ on W, called the absolute order, by letting $a \leq b$ if there exists a shortest factorization of a in reflections which is a prefix of a shortest factorization of b into reflections.

We set $NC(W) = [e, \gamma]$ where e is the identity and γ is a Coxeter element of W.
It is called the Noncrossing Partition Lattice.

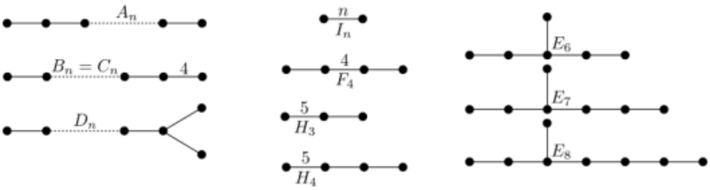
Noncrossing partitions are central objects of study in Catalan-Coxeter combinatorics.

Chains enumeration in noncrossing partition lattices has been studied by:

- Edelman (1980)
- Reiner (1997)
- Stanley (1997)
- Athanasiadis-Reiner (2004)
- Reading (2008)
- Kim (2011)
- Chapuy-Douvropoulos (2022)

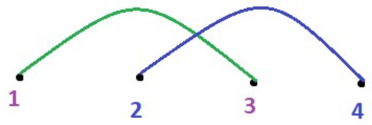
Theorem (Athanasiadis, K. E. 2022-2023)

The noncrossing partition lattice $NC(W)$ has a real-rooted chain polynomial for every irreducible finite Coxeter group W.



Let $W = A_{n-1}$ (symmetric group of degree n)

- The elements of $NC(A_{n-1})$ can be viewed as the set partitions of $[n]$ that "do not cross"



$\{\{1,3\},\{2,4\}\}$ is not in $NC(A_3)$

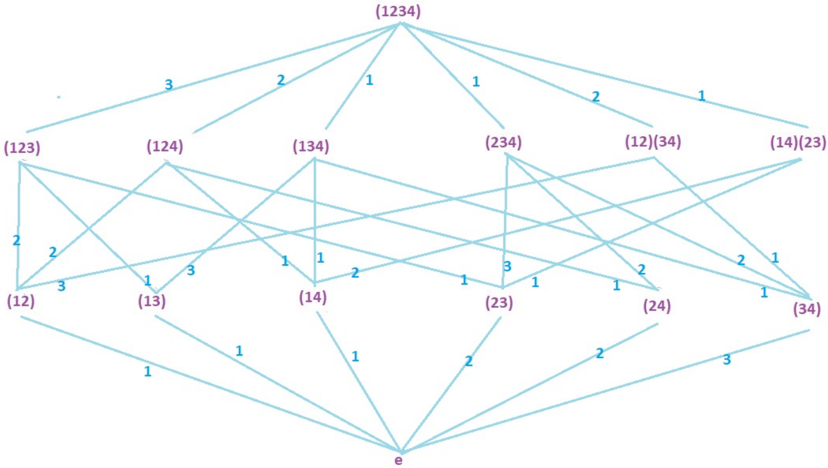
- The cardinality of $NC(A_{n-1})$ is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

- $NC(A_{n-1})$ has n^{n-2} maximal chains
- $h_{NC(A_{n-1})}(x)$ counts chains on the set of parking functions of length n-1 (P_{n-1})

$$h_{NC(A_{n-1})}(x) = \sum_{w \in P_{n-1}} x^{\mathrm{des}(w)} = \frac{1}{n} \sum_{w \in [n]^{n-1}} x^{\mathrm{des}(w)}$$

P_3

$123, 13|2, 2|13, 23|1, 3|12, 3|2|1$ $1|12, 12|1, 2|1|1,$
$1|13, 13|1, 3|1,$ $12, 2|12, 2|2|1,$ $1|1|1$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Let $W = B_n$ (the hyperoctahedral group of degree n)

- $NC(B_n)$ has n^n maximal chains

$$h_{NC(B_n)}(x) = \sum_{w \in [n]^n} x^{\mathrm{des}(w)}$$

$$n = 3$$

$$123, 13|2, 2|13, 23|1, 3|12, 3|2|1$$

$$1|12, 12|1, 2|1|1, 1|13, 13|1, 3|1|1, 2|23, 23|2, 3|2|2$$

$$1|1|1, 2|2|2, 3|3|3$$

$$12|2, 2|12, 2|2|1, 23|3, 3|23, 3|3|2, 13|3, 3|13, 3|3|1$$

- $NC(D_n)$ has $2(n-1)^n$ maximal chains

$$h_{NC(D_n)}(x) = \sum_{w \in D_n} x^{\mathrm{des}(w)}$$

$$D_n = \{(\pm w_1, w_2, \cdots, w_n) : w_1, w_2, \cdots, w_n \in [n-1]\}$$

and k is considered a descent if $|w_k| > |w_{k+1}|$ or $w_k = w_{k+1} > 0$

$$n = 3$$

$1\|12$	$12\|1$	$2\|1\|1$	$12\|2$	$2\|12$	$2\|2\|1$	$1\|1\|1$	$2\|2\|2$
-112	$-12\|1$	$-2\|1\|1$	$-12\|2$	$-2\|12$	$-22\|1$	$-11\|1$	$-22\|2$

$$h_{NC(W)}(x) = \begin{cases} 1 + (n-1)x & \text{if } W = I_n \\ 1 + 28x + 21x^2 & \text{if } W = H_3 \\ 1 + 275x + 842x^2 + 232x^3 & \text{if } W = H_4 \\ 1 + 100x + 265x^2 + 66x^3 & \text{if } W = F_4 \\ 1 + 826x + 10778x^2 + 21308x^3 + 8141x^4 + 418x^5 & \\ & \text{if } W = E_6 \\ 1 + 4152x + 110958x^2 + 446776x^3 + 412764x^4 & \\ +85800x^5 + 2431x^6 & \text{if } W = E_7 \\ 1 + 25071x + 1295238x^2 + 9523785x^3 + 17304775x^4 & \\ +8733249x^5 + 1069289x^6 + 17342x^7 & \text{if } W = E_8 \end{cases}$$

- Can we use the methods we develop it this research to attack other interesting problems?
- The h^*-polynomials come from counting lattice points in lattice polytopes and behave similarly to the h-polynomials.

Question: Are the h^*-polynomials of matroid polytopes unimodal? Are they real-rooted?

Thank you for your attention!