Minor Summation Formula
and
Classical Group Characters of Nearly Rectangular Shape

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Motivation

We adopt the method (based on the minor-summation formula) developed in

- S. Okada,
  Applications of minor summation formulas to rectangular-shaped representations of classical group,
  J. Algebra **205** (1988), 337–367.

to give alternate proofs and variants of identities in

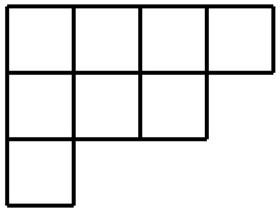
- C. Krattenthaler,
  Identities for classical group characters of nearly rectangular shape,
  J. Algebra **209** (1998), 1–64.

Partitions and half-partitions

A partition of length $\leq n$ is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$$

of nonnegative integers $\lambda_i \in \mathbb{N}$. We represent a partition by its Young diagram. For example, the Young diagram of $(4, 3, 1, 0)$ is



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The size $|\lambda|$ of a partition λ is defined by $|\lambda| = \sum_{i=1}^{n} \lambda_i$. We write $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for $1 \leq i \leq n$.

A half-partition of length n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative half-integers $\lambda_i \in \mathbb{N} + 1/2$.

Rectangular and nearly rectangular shapes

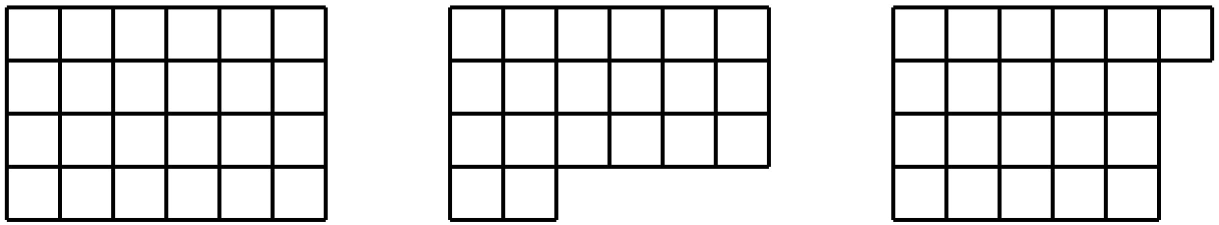
A rectangular shape is a partition or half-partition of the form

$$\lambda = (\underbrace{r, \ldots, r}_{n}) = (r^n).$$

And a nearly rectangular shape is a partition or half-partition of the form

$$\lambda = (\underbrace{r, \ldots, r}_{n-1}, p) = (r^{n-1}, p), \quad \text{or}$$

$$\lambda = (\underbrace{r, \ldots, r}_{p}, \underbrace{r-1, \ldots, r-1}_{n-p}) = (r^p, (r-1)^{n-p}).$$



Schur functions and odd orthogonal characters

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition of length $\leq n$ or a half-partition of length n. We define the Schur function $s_\lambda(\boldsymbol{x})$ and the odd orthogonal character $\mathrm{so}_{2n+1}(\lambda; \boldsymbol{x})$ by

$$s_\lambda(\boldsymbol{x}) = \frac{\det \left(x_i^{\lambda_j+n-j} \right)_{1 \leq i, j \leq n}}{\det \left(x_i^{n-j} \right)_{1 \leq i, j \leq n}},$$

$$\mathrm{so}_{2n+1}(\lambda; \boldsymbol{x}) = \frac{\det \left(x_i^{\lambda_j+n+1/2-j} - x_i^{-(\lambda_j+n+1/2-j)} \right)_{1 \leq i, j \leq n}}{\det \left(x_i^{n+1/2-j} - x_i^{-(n+1/2-j)} \right)_{1 \leq i, j \leq n}},$$

respectively.

Remark $s_\lambda(\boldsymbol{x})$ and $\mathrm{so}_{2n+1}(\lambda; \boldsymbol{x})$ are the irreducible characters of $\mathfrak{gl}_n$ and $\mathfrak{so}_{2n+1}$ respectively.

Identities for rectangular shapes

Theorem (Macdonald) For a nonnegative integer or half-integer r, we have

$$\mathrm{so}_{2n+1}((r^n);\boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \cdot \sum_{\lambda \subset ((2r)^n)} s_\lambda(\boldsymbol{x}),$$

where the sum is taken over all partitions λ such that $\lambda \subset (2r)^n$.

Theorem (Okada) If r and s are nonnegative integers or half-integers such that $r \le s$, then we have

$$\mathrm{so}_{2n+1}((r^n);\boldsymbol{x}) \cdot \mathrm{so}_{2n+1}((s^n);\boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} \mathrm{so}_{2n+1}(\lambda + (s-r)^n;\boldsymbol{x}),$$

where $\lambda + (s-r)^n = (\lambda_1 + s - r, \dots, \lambda_n + s - r)$.

Remark These theorems describe the decompositions of the branching from $\mathfrak{so}_{2n+1}$ to $\mathfrak{gl}_n$ and the tensor product respectively.

These and similar identities can be proved by using the minor-summation formula of Ishikawa and Wakayama.

Identities for nearly rectangular shapes

Theorem (Krattenthaler) If r is a nonnegative integer or half-integer and p is a nonnegative integer such that $p \le r$, then we have

$$\mathrm{so}_{2n+1}((r^{n-1}, r-p); \boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} b'_{\lambda,p} \, s_\lambda(\boldsymbol{x}),$$

where $b'_{\lambda,p}$ is the number of partitions ν satisfying

- λ/ν is a horizontal strip, i.e.

$$\lambda_1 \ge \nu_1 \ge \lambda_2 \ge \nu_2 \ge \lambda_3 \ge \cdots \ge \lambda_n \ge \nu_n;$$

- $|\lambda| - |\nu| = p$;
- the ith cell of λ/ν (counted from left to right) comes before the $(2r - 2p + 2i)$th column for $1 \le i \le p$.

The proof uses the tableau description of $\mathrm{so}_{2n+1}(\lambda; \boldsymbol{x})$ due to Lakshmibai–Musili–Seshadri and a Robinson–Schensted–Knuth-type algorithm.

Identities for nearly rectangular shapes

Theorem (Krattenthaler) If r and s are nonnegative integers or half-integers with $r \le s$ (for simplicity) and p is a nonnegative integer such that $p \le r$, then we have

$$\begin{aligned}\mathrm{so}_{2n+1}((r^{n-1}, r-p); \boldsymbol{x}) \cdot \mathrm{so}_{2n+1}((s^n); \boldsymbol{x}) \\ = \sum_{\lambda \subset ((2r)^n)} b''_{\lambda,p}\, \mathrm{so}_{2n+1}(\lambda + (s-r)^n; \boldsymbol{x}),\end{aligned}$$

where $b''_{\lambda,p}$ is the number of partitions μ of length $\le n$ satisfying

- μ/λ is a horizontal strip;
- $\mu_1 \le 2r$, $\mu_n = \lambda_n$ and

$$\max\{p-\lambda_n, p-\lambda_{n-1}+\lambda_n\} \le |\mu| - |\lambda| \le p,$$
$$2\mu_1 + \cdots + 2\mu_{i-1} + \mu_i \ge 2\lambda_1 + \cdots + 2\lambda_{i-1} + 2p \quad (1 \le i \le n-1).$$

The proof uses Littelmann's extension of the Littlewood–Richardson rule for $\mathfrak{so}_{2n+1}$.

Key observation

Nearly-rectangular-shaped odd orthogonal characters appear in the restriction of rectangular-shaped characters.

Proposition For an integer or a half-integer r, we have

$$\mathrm{so}_{2n+3}((r^{n+1}); x_1, \dots, x_n, u) = \sum_{p=0}^{\lfloor r \rfloor} \left[2(r-p)+1 \right]_u \mathrm{so}_{2n+1}((r^{n-1}, r-p); x_1, \dots, x_n),$$

where

$$[k]_u = \frac{u^{k/2} - u^{-k/2}}{u^{1/2} - u^{-1/2}}.$$

Main results

Theorem A For a partition $\lambda \subset ((2r)^n)$, we define $b_\lambda^{(2r)}(u)$ by putting

$$b_\lambda^{(2r)}(u) = [2r-\lambda_1+1]_u[\lambda_1-\lambda_2+1]_u \cdots [\lambda_{n-1}-\lambda_n+1]_u[\lambda_n+1]_u.$$

(1) For $r \in \mathbb{N} \cup (\mathbb{N}+1/2)$, we have

$$\mathrm{so}_{2n+3}((r^{n+1});\boldsymbol{x},u) = (x_1\cdots x_n)^{-r} \sum_{\lambda\subset((2r)^n)} b_\lambda^{(2r)}(u)\, s_\lambda(\boldsymbol{x}).$$

(2) For r, $s \in \mathbb{N} \cup (\mathbb{N}+1/2)$ with $r \le s$, we have

$$\mathrm{so}_{2n+3}((r^{n+1});\boldsymbol{x},u)\cdot \mathrm{so}_{2n+1}((s^n);\boldsymbol{x}) = \sum_{\lambda\subset((2r)^n)} b_\lambda^{(2r)}(u)\, \mathrm{so}_{2n+1}(\lambda+(s-r)^n;\boldsymbol{x}).$$

Main results

Corollary B If r, $s \in \mathbb{N} \cup (\mathbb{N} + 1/2)$ and $r \leq s$, then we have

$$\mathrm{SO}_{2n+1}((r^{n-1}, r-p); \boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} b^{(2r)}_{\lambda,p} \, s_\lambda(\boldsymbol{x}),$$

$$\mathrm{SO}_{2n+1}((r^{n-1}, r-p); \boldsymbol{x}) \cdot \mathrm{SO}_{2n+1}((s^n); \boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} b^{(2r)}_{\lambda,p} \, \mathrm{SO}_{2n+1}(\lambda + (s-r)^n; \boldsymbol{x}),$$

where $b^{(2r)}_{\lambda,p}$ equals the number of partitions μ of length $\leq n+1$ satisfying

- μ/λ is a horizontal strip of size $|\mu| - |\lambda| = 2r - p$;
- $\mu_1 = 2r$ and
  $$\mu_1 + 2\mu_2 + \cdots + 2\mu_{i-1} + \mu_i \geq 2\lambda_1 + \cdots + 2\lambda_{i-1} \quad (2 \leq i \leq n+1).$$

Remark The coefficient of $s_\lambda(\boldsymbol{x})$ in the branching formula is the same as the coefficient of $\mathrm{SO}_{2n+1}(\lambda + (s-r)^n; \boldsymbol{x})$ in the tensor product formula.

Minor summation Formula

Let $A = (a_{ij})_{0 \le i,\, j \le N}$ be a skew symmetric matrix of degree N, and $T = (t_{ij})_{1 \le i \le n,\, 0 \le j \le N}$ be a $n \times N$ matrix. For an n-element subset $I = \{i_1, \dots, i_n\}$ ($i_1 < \cdots < i_n$) of $[0, N] = \{0, 1, \dots, N\}$, we write

$$A(I) = \left(a_{i_p, i_q} \right)_{1 \le p,\, q \le n}, \quad T([n]; I) = \left(t_{p, i_q} \right)_{1 \le p,\, q \le n}.$$

Theorem (Ishikawa–Wakayama) If n is even, then we have

$$\sum_J \operatorname{Pf} A(J) \cdot \det T\left([n]; J\right) = \operatorname{Pf}\left(T A\, {}^t T \right),$$

where J runs over all n-element subsets of $[0, N]$.

Partitions $\lambda \subset (m^n)$ are in bijection with n-element subsets of $[0, n + m - 1]$ via

$$\lambda \longleftrightarrow I_n(\lambda) = \{\lambda_n, \lambda_{n-1} + 1, \dots, \lambda_2 + n - 2, \lambda_1 + n - 1\}.$$

Minor summation Formula

By applying the minor-summation formula to

$$T = \left(x_i^{j-1} \right)_{\substack{1 \le i \le n \\ 0 \le j \le n+m-1}}, \quad \text{or} \quad \left(x_i^{a+j+1/2} - x_i^{-(a+j+1/2)} \right)_{\substack{1 \le i \le n \\ 0 \le j \le n+m-1}}$$

we obtain

Proposition If $A = (a_{i,j})_{0 \le i, j \le n+m-1}$ is a skew-symmetric matrix, then we have

$$\sum_{\lambda \subset (m^n)} \operatorname{Pf} A(I_n(\lambda))\, s_\lambda(\boldsymbol{x}) = \frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \left(\sum_{k<l} a_{k,l} \det \begin{pmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{pmatrix} \right)_{1 \le i, j \le n},$$

and

$$\sum_{\lambda \subset (m^n)} \operatorname{Pf} A(I_n(\lambda))\, \mathrm{so}_{2n+1}(\lambda + (a^n); \boldsymbol{x})$$

$$= \frac{1}{\Delta^B(\boldsymbol{x})} \operatorname{Pf} \left(\sum_{k<l} a_{k,l} \det \begin{pmatrix} x_i^{k+a+1/2} - x_i^{-(k+a+1/2)} & x_i^{l+a+1/2} - x_i^{-(l+a+1/2)} \\ x_j^{k+a+1/2} - x_j^{-(k+a+1/2)} & x_j^{l+a+1/2} - x_j^{-(l+a+1/2)} \end{pmatrix} \right)_{1 \le i, j \le n}.$$

Strategy of Proof of Theorem A

Step 0: Reduce the proof to the case where n is even.

Step 1: Find a skew-symmetric matrix $A = (a_{i,j})_{0 \le i,\, j \le m+n-1}$ such that

$$\operatorname{Pf} A(I_n(\lambda)) = C \cdot b_\lambda^{(m)}(u) \quad (\lambda \subset (m^n)),$$

for some constant C.

Step 2: Compute the entries of $TA{}^tT$:

$$\sum_{k<l} a_{k,l} \det \begin{pmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{pmatrix}, \quad \sum_{k<l} a_{k,l} \det \begin{pmatrix} x_i^{k+a+1/2} - x_i^{-(k+a+1/2)} & x_i^{l+a+1/2} - x_i^{-(l+a+1/2)} \\ x_j^{k+a+1/2} - x_j^{-(k+a+1/2)} & x_j^{l+a+1/2} - x_j^{-(l+a+1/2)} \end{pmatrix}.$$

Step 3: Transform the Pfaffian $\operatorname{Pf}(TA{}^tT)$ into determinants.

Proof of Theorem A (Step 1)

Proposition Let $A = (a_{i,j})_{0 \le i,\, j \le m+n-1}$ be the skew-symmetric matrix with (i,j) entry give by

$$a_{i,j} = [m+n-j]_u [j-i]_u [i+1]_u \quad (0 \le i < j \le m+n-1).$$

For a partition $\lambda \subset (m^n)$, we have

$$\mathrm{Pf}\, A(I_n(\lambda)) = [m+n+1]_u^{n/2-1} b_\lambda^{(m)}(u).$$

This proposition is obtained from the following lemma by specialization.

Lemma

$$\mathrm{Pf}\left((x_0 - x_i)(x_i - x_j)(x_j - x_{n+1}) \right)_{1 \le i < j \le n} = (x_0 - x_{n+1})^{n/2-1} \prod_{i=0}^{n} (x_i - x_{i+1}).$$

Proof Induction on n by using the Desnanot–Jacobi identity for Pfaffians:

Proof of Theorem A (Step 2)

Let $W^n(\boldsymbol{x};\boldsymbol{a})$ be the $n \times n$ matrix with ith row given by

$$\left(1 + a_i x_i^{n-1} \;\; x_i + a_i x_i^{n-2} \;\; \cdots \;\; x_i^{n-1} + a_i\right).$$

Proposition

$$\sum_{0 \le i < j \le M} [M+1-j]_u [j-i]_u [i+1]_u \det \begin{pmatrix} x^i & x^j \\ y^i & y^j \end{pmatrix}$$

$$= \frac{\det W^3(x,y,u;-x^{M+2},-y^{M+2},-u^{M+2})}{u^{(M-1)/2}(1-x)(1-y)(1-u)(u-x)(u-y)(1-ux)(1-uy)(1-xy)},$$

$$\sum_{0 \le i < j \le M} [M+1-j]_u [j-i]_u [i+1]_u \det \begin{pmatrix} x^{i+1/2+a} - x^{-(i+1/2+a)} & x^{j+1/2+a} - x^{-(j+1/2+a)} \\ y^{i+1/2+a} - y^{-(i+1/2+a)} & y^{j+1/2+a} - y^{-(j+1/2+a)} \end{pmatrix}$$

$$= -\frac{1}{u^{(M-1)/2} x^{M+1/2+a} y^{M+1/2+a} (1-x)(1-y)(1-u)(u-x)(u-y)(1-ux)(1-uy)}$$

$$\times \frac{\det W^2(x,y;-x^{M+1+2a},-y^{M+1+2a}) \det W^3(x,y,u;-x^{M+2},-y^{M+2},-u^{M+2})}{(y-x)(1-xy)}.$$

Proof of Theorem A (Step 3)

The proof of Theorem A can be completed by using the following theorem with $p = 0$, $q = 1$ and

$$a_i = 0, \quad b_i = -x_i^{2r+n+1}, \quad w_1 = u, \quad d_1 = u^{2r+n+1},$$

or

$$a_i = -x_i^{2s-2r}, \quad b_i = -x_i^{2r+n+1}, \quad w_1 = u, \quad d_1 = u^{2r+n+1}.$$

Theorem (Ishikawa–Okada–Tagawa–Zeng) We have

$$\begin{aligned} &\mathrm{Pf}\left(\frac{\det W^{p+2}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det W^{q+2}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{(x_j - x_i)(1 - x_i x_j)} \right)_{1 \le i, j \le n} \\ &= \frac{1}{\prod_{1 \le i < j \le n} (x_j - x_i)(1 - x_i x_j)} \det W^p(\boldsymbol{z}; \boldsymbol{c})^{n/2-1} \det W^q(\boldsymbol{w}; \boldsymbol{d})^{n/2-1} \\ &\quad \times \det W^{n+p}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \det W^{n+q}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}). \end{aligned}$$

Proof of Corollary B (1/2)

It is enough to show that, if

$$b_\lambda^{(2r)}(u) = [2r - \lambda_1 + 1]_u [\lambda_1 - \lambda_2 + 1]_u \cdots [\lambda_{n-1} - \lambda_n + 1]_u [\lambda_n + 1]_u$$
$$= \sum_{p=0}^{\lfloor r \rfloor} b_{\lambda,p}^{(2r)} \left[2(r-p)+1\right]_u,$$

where $[k]_u = \left(u^{k/2} - u^{-k/2}\right) / \left(u^{1/2} - u^{-1/2}\right)$, then the coefficient $b_{\lambda,p}^{(2r)}$ is equal to the number of partitions μ of length $\leq n+1$ satisfying

- $2r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \mu_{n+1} \geq 0$;
- $|\mu| - |\lambda| = 2r - p$;
- $\mu_1 = 2r$ and $\mu_1 + 2\mu_2 + \cdots + 2\mu_{i-1} + \mu_i \geq 2\lambda_1 + \cdots + 2\lambda_{i-1}$ $(2 \leq i \leq n+1)$.

Proof of Corollary B (2/2)

For $k \in \mathbb{N}$, let $V(k)$ be the irreducible $U_q(\mathfrak{sl}_2)$-module of dimension $k+1$ and $B(k)$ the crystal base of $V(k)$. Given a partition λ of length $\leq n$, we can equip the set

$$\mathcal{B} = \left\{ \begin{array}{l} (\mu_1, \dots, \mu_{n+1}) \in \mathbb{N}^{n+1} : \\ \quad 2r \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \mu_{n+1} \geq 0 \end{array} \right\}$$

with an $\mathfrak{sl}_2$-crystal structure such that

$$\mathcal{B} \cong B(2r - \lambda_1) \otimes B(\lambda_1 - \lambda_2) \otimes \cdots \otimes B(\lambda_{n-1} - \lambda_n) \otimes B(\lambda_n),$$

and $b_\lambda^{(2r)}(u^2) = \sum_{\mu \in \mathcal{B}} u^{\mathrm{wt}(\mu)}$. Then we have

$$b_{\lambda,p}^{(2r)} = \#\{\mu \in \mathcal{B} : \widetilde{e}\mu = 0,\ \mathrm{wt}(\mu) = 2(r-p)\},$$

where $\widetilde{e}$ is the Kashiwara operator. We can see that $\widetilde{e}\mu = 0$ if and only if

$$\mu_1 = 2r, \quad \mu_1 + 2\sum_{k=2}^{i-1} \mu_k + \mu_i \geq 2\sum_{k=1}^{i-1} \lambda_k \quad (2 \leq i \leq n+1).$$

Symplectic case

Theorem For r, $s \in \mathbb{N}$ with $r \le s$, we have

$$\mathrm{sp}_{2n}((r^{n-1}, r-p); \boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} c^{(2r)}_{\lambda,p} \, s_\lambda(\boldsymbol{x}),$$

$$\mathrm{sp}_{2n}((r^{n-1}, r-p), \boldsymbol{x}) \cdot \mathrm{sp}_{2n}(s^n; \boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} c^{(2r)}_{\lambda,p} \, \mathrm{sp}_{2n}(\lambda + (s-r)^n; \boldsymbol{x}),$$

where $c^{(2r)}_{\lambda,p}$ is equal to the number of partitions μ of length $\le n+1$ satisfying

- μ/λ is a horizontal strip of size $|\mu| - |\lambda| = 2r - p$;
- $\mu_1, \dots, \mu_{n+1}$ are even, $\mu_1 = 2r$ and

  $$\mu_1 + 2\mu_2 + \cdots + 2\mu_{i-1} + \mu_i \ge 2\lambda_1 + \cdots + 2\lambda_{i-1} \quad (2 \le i \le n+1).$$

Mixed case

Theorem For r, $s \in \mathbb{N}$ with $r > 0$ and $r \le s$, we have

$$\mathrm{o}_{2n+2}((r^{n+1}); \boldsymbol{x}, u) \cdot \mathrm{sp}_{2n}((s^n); \boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} \left(u^{r-c(\lambda)} + u^{-r+c(\lambda)} \right) \mathrm{sp}_{2n}(\lambda + (s-r)^n; \boldsymbol{x}),$$

$$\mathrm{sp}_{2n+2}(((r-1)^{n+1}); \boldsymbol{x}, u) \cdot \mathrm{o}_{2n}(((s+1)^n); \boldsymbol{x}) \cdot (u - u^{-1}) = (-1)^n \sum_{\lambda \subset ((2r)^n)} \left(u^{r-c(\lambda)} - u^{-r+c(\lambda)} \right) \mathrm{sp}_{2n}(\lambda + (s-r)^n; \boldsymbol{x}),$$

where $\mathrm{o}_{2n}(\mu; \boldsymbol{x})$ is the even orthogonal character and $c(\lambda)$ is the number of columns of odd length in the Young diagram of λ.

This theorem can be used to enumerate shifted plane partitions of trapezoidal shape $(2n, 2n-2, \dots, 4, 2)$.

Branching and tensor product multiplicities

Theorem (1) Let $r \in \mathbb{N} \cup (\mathbb{N} + 1/2)$ and μ be a partition or half-partition such that $\mu_1 \le r$. If we expand

$$\mathrm{so}_{2n+1}(\mu; \boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} B_{\mu,\lambda}\, s_\lambda(\boldsymbol{x}),$$

then we have

$$\mathrm{so}_{2n+1}(\mu; \boldsymbol{x}) \cdot \mathrm{so}_{2n+1}((r^n); \boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} B_{\mu,\lambda}\, \mathrm{so}_{2n+1}(\lambda; \boldsymbol{x}).$$

(2) Let $r \in \mathbb{N}$ and μ be a partition such that $\mu_1 \le r$. If we expand

$$\mathrm{sp}_{2n}(\mu; \boldsymbol{x}) = (x_1 \cdots x_n)^{-r} \sum_{\lambda \subset ((2r)^n)} C_{\mu,\lambda}\, s_\lambda(\boldsymbol{x}),$$

then we have

$$\mathrm{sp}_{2n}(\mu; \boldsymbol{x}) \cdot \mathrm{sp}_{2n}((r^n); \boldsymbol{x}) = \sum_{\lambda \subset ((2r)^n)} C_{\mu,\lambda}\, \mathrm{sp}_{2n}(\lambda; \boldsymbol{x}).$$