A diagrammatic representation of the Temperley-Lieb algebra of type $\widetilde{B}$

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joint work with R. Biagioli[1] and G. Fatabbi[2]

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A brief overview

How it started

- In 1971, N. **Temperley** e E. **Lieb** introduced the TL algebra to solve a statistical mechanics problem.
- **Penrose** (1971) and **Kauffman** (1987) showed that the TL algebra can be realized as a diagram algebra.
- In 1987, **Jones** presented the TL algebra in terms of abstract generators and relations.
- In 1995, **Graham** defined the so-called *generalized Temperley–Lieb algebra* TL(Γ) as a quotient of the Iwahori-Hecke algebra of a Coxeter system of arbitrary type Γ.

Definition

We define a **Coxeter system** to be a pair (W, S) where

- S is a set of generators;
- $W = \langle S \mid (st)^{m_{st}} = 1 \rangle$ where $m_{st} \in \mathbb{N}$, $m_{st} = m_{ts}$ for all $s, t \in S$ and $m_{st} = 1$ iff $s = t$.

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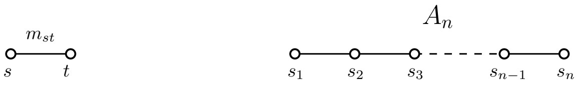
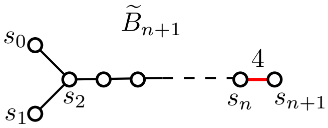
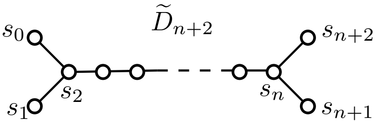
Coxeter graph Γ




$\widetilde{B}_{n+1}$
s_0
4
s_1
s_2
s_n
s_{n+1}


$\widetilde{D}_{n+2}$
s_0
s_{n+2}
s_1
s_2
s_n
s_{n+1}

The relations

More explicitly

- $s^2 = 1$ for all $s \in S$;
- $st = ts$ if $m_{st} = 2$ *(commutation relation)*;
- $\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}}$, if $3 \leq m_{st} < \infty$ *(braid relation)*.

Reduced expressions

For $w \in W$, the **length** of w, denoted by $\ell(w)$, is the minimum length l of an expression $s_1 \cdots s_l$ of w with $s_i \in S$. The expressions of length $\ell(w)$ are called **reduced**.

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Matsumoto's Theorem

Any reduced expression of w can be obtained from any other reduced expression of w using only braid and commutation relations.

Fully commutative elements

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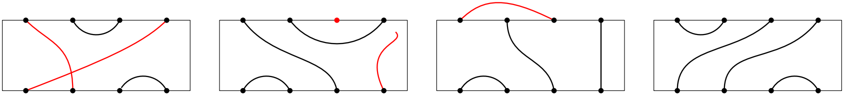
For example, let $\mathbf{w} = s_1 s_4 s_3 s_2 s_1$ be a reduced expression of $w \in W(A_4)$. Then

$$s_4 s_3 s_1 s_2 s_1 \text{ and } s_4 s_3 s_2 s_1 s_2$$

are both reduced expression of w, so w is not fully commutative.

Diagrams

A **k-diagram** consists of a finite number of disjoint plane curves, called *edges*, embedded in a box having k nodes on the top (*north*) face and k nodes on the bottom (*south*) face.



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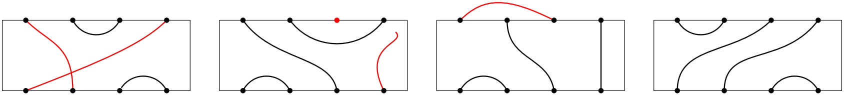
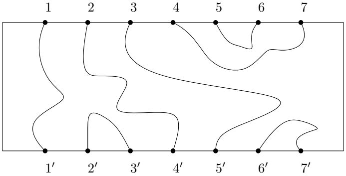
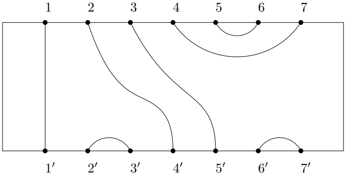
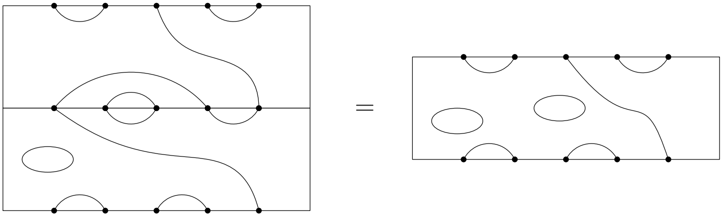






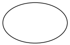
Product

We can define an associative product by concatenation.



TL algebra

The **(diagram) Temperley-Lieb algebra** $\mathbb{D}(A_n)$ is the $\mathbb{Z}[\delta]$-algebra having the diagrams as basis with the multiplication defined above subject to the following relation.

 $= \delta$

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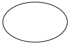 $= \delta$

Example

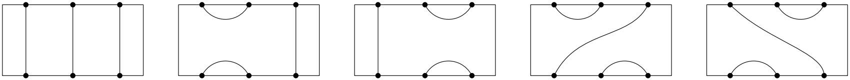

Figure: Basis diagrams of $\mathbb{D}(A_2)$.

Question

What about the generalized TL algebras?

Decorated diagrams

Consider $\Omega = \{\bullet, \circ, \triangle\}$. We can decorate a k-diagram with the elements of Ω^*.

(D0) All decorated edges can be deformed so as to take $\bullet$-decorations to the left wall of the diagram and $\circ$ or $\triangle$-decorations to the right wall simultaneously without crossing any other edges.

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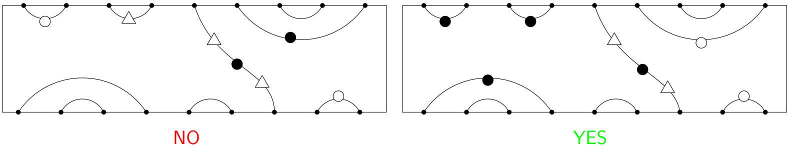
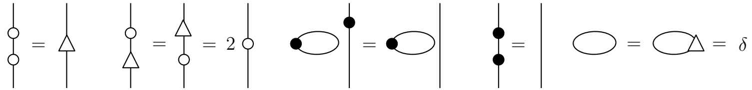


Diagram algebra

Now we define

- a set of relations



(r1) (r2) (r3) (r4) (r5)

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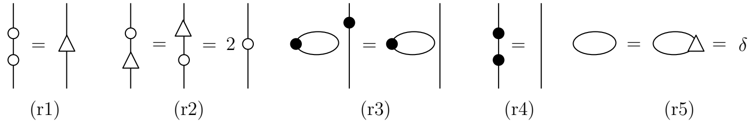
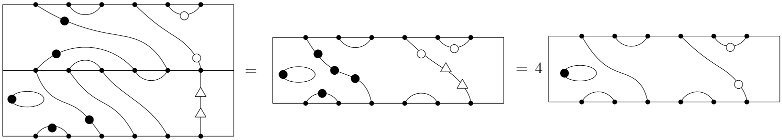


- an associative diagram product by concatenation.



We consider a subfamily of the decorated diagrams, called **admissible diagrams**.

Figure: Admissible loops.

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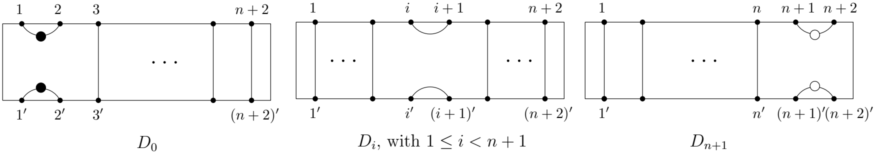
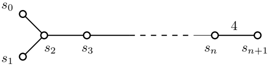
The admissible diagrams form a $\mathbb{Z}[\delta]$-algebra, denoted by $\mathbb{D}(\widetilde{B}_{n+1})$, which is generated by the **simple diagrams** defined as follows.



TL algebra of type $\widetilde{B}$



$\tilde{B}_{n+1}$

The **Temperley–Lieb algebra** of type $\widetilde{B}_{n+1}$, $\mathrm{TL}(\widetilde{B}_{n+1})$, is the $\mathbb{Z}[\delta]$-algebra generated by $\{b_0, b_1, \ldots, b_{n+1}\}$ with defining relations:

(1) $b_i^2 = \delta b_i$ for all $i \in \{0, 1, \ldots, n+1\}$;

(2) $b_i b_j = b_j b_i$ if $|i - j| > 1$ and $\{i, j\} \neq \{0, 2\}$, or $\{i, j\} = \{0, 1\}$;

(3) $b_i b_j b_i = b_i$ if $i, j \in \{1, \ldots, n\}$ and $|i - j| = 1$, or $\{i, j\} = \{0, 2\}$;

(4) $b_i b_j b_i b_j = 2 b_i b_j$ if $\{i, j\} = \{n, n+1\}$.

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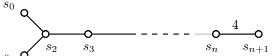

The **Temperley–Lieb algebra** of type $\widetilde{B}_{n+1}$, TL($\widetilde{B}_{n+1}$), is the $\mathbb{Z}[\delta]$-algebra generated by $\{b_0, b_1, \ldots, b_{n+1}\}$ with defining relations:

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The set $\{b_w \mid w \in \mathrm{FC}(\widetilde{B}_{n+1})\}$ is a basis for TL($\widetilde{B}_{n+1}$), called **monomial basis**, where $b_w := b_{i_1} \cdots b_{i_k}$ for $s_{i_1} \cdots s_{i_k}$ any reduced expression of w.

The main result

Consider the $\mathbb{Z}[\delta]$-algebra homomorphism

$$\tilde{\theta}_B : \mathrm{TL}(\widetilde{B}_{n+1}) \longrightarrow \mathbb{D}(\widetilde{B}_{n+1})$$
$$b_i \mapsto D_i$$

for $i \in \{0, 1, \dots, n+1\}$.

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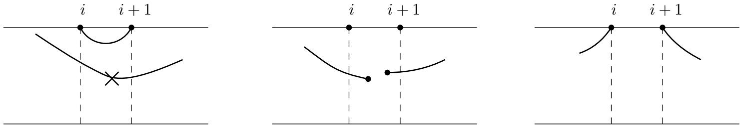
Theorem

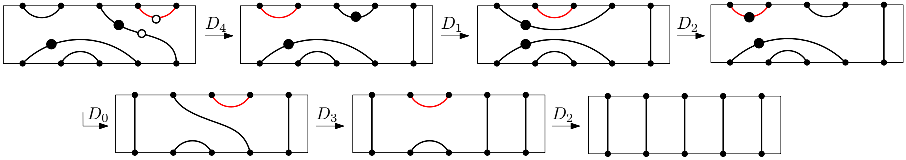
The map $\tilde{\theta}_B$ is an algebra **isomorphism**. Moreover, each admissible diagram corresponds to a unique monomial basis element.

An ingredient for the proof

The **cut and paste operation** for factorizing admissible diagrams into a product of simple diagrams.



An example.



$$D = D_4 D_1 D_2 D_0 D_3 D_2$$

What about $\widetilde{D}$?

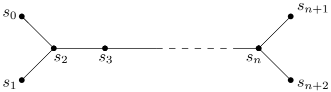
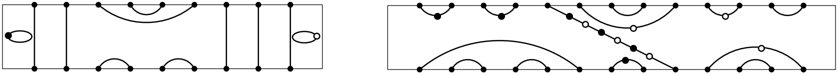


Figure: Coxeter graph of type $\widetilde{D}_{n+2}$.



$$TL(\widetilde{D}_{n+2}) \overset{\tilde{\theta}_D}{\cong} \mathbb{D}(\widetilde{D}_{n+2})$$

Thank you!