# Atoms and charge beyond type A

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Séminaire Lotharigien de Combinatoire 90
3-6th September 2023
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$$K_{\lambda,\mu}(q) := \sum_{w \in W} sgn(w)\mathbb{P}_q(w(\lambda+\rho)-\mu-\rho)$$

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$$\sum_{p \geq 0} \dim(J_e^p(V_\mu^\lambda)/J_e^{p-1}(V_\mu^\lambda))q^p$$

- Kazhdan–Lusztig polynomials for the extended affine Hecke algebra.

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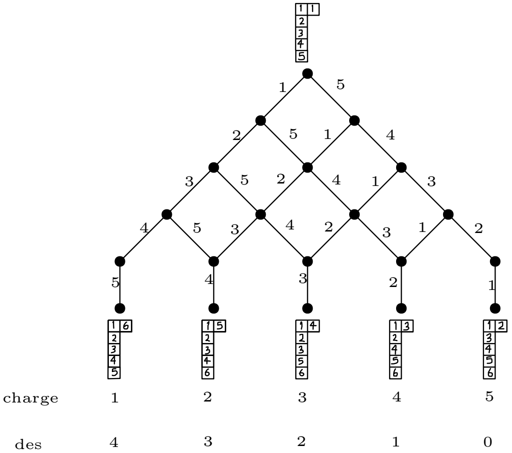
We call such a map a **charge** statistic after Lascoux–Schützenberger, who solved this problem in 1978 for type $A_n$, that is for $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$.

# Fun fact: the classical exponents of type $A_n$



# What is known?

- Lascoux–Leclerc–Thibon's re-interpretation of type $A_n$ charge in terms of crystal operators (1995)
- Conjecture in type $C_n$ due to Lecouvey (2005)
- $\rightsquigarrow$ proven for $\lambda = k\omega_1$ (jt. with M. Dołęga and T. Gerber) (2020)
- Lenart–Lecouvey's formula for the generalized exponents $K_{\lambda,0}$ (2019)
- Patimo's new proof of the charge formula in type $A_n$ using the geometry of the affine grassmannian, Hecke algebras, and combinatorics of twisted Bruhat graphs... (2023)
- New charge statistic in type $C_2$, using the above mentioned methodology (jt. with L. Patimo) (preprint 2023)

Thank you :)