On the enumeration of Shi regions in Weyl cones

Eleni Tzanaki

Department of Mathematics & Applied Mathematics
University of Crete, Greece

joint work with

Aram Dermenjian

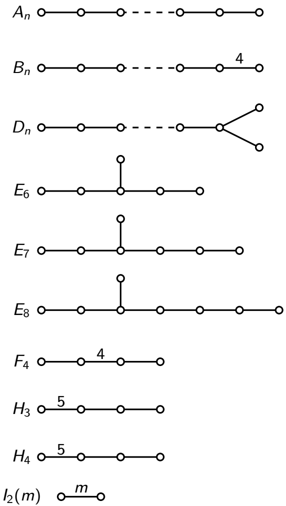
90th Séminaire Lotharingien de Combinatoire
4-6 September 2023, Bad Boll, Germany

Supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.).
Project Number: HFRI-FM20-04537

- Coxeter group: $\langle r_1, \ldots, r_n : (r_i r_j)^{m_{ij}} = \mathbf{1} \rangle$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$
- finite Coxeter groups classified in terms of Coxeter–Dynkin diagrams (Coxeter, 1935)
- A reflection group is a group generated by a set of reflections s_a in a Euclidean space V

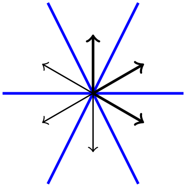
$$s_a(x) = x - 2\frac{\langle x, a \rangle}{\langle a, a \rangle} a \qquad \text{for } a \in V$$

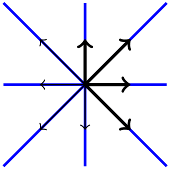
- finite Coxeter groups coincide with finite reflection groups



Group	symmetries of	$\|W\|$	$\|\Phi^+\|$	coxeter number h	exponents
$A_n, n \geq 1$	n-simplex	$(n+1)!$	$\frac{n(n+1)}{2}$	$n+1$	$1, 2, \ldots, n$
$B_n/C_n, n \geq 2$	n-cube or n-cross polytope	$2^n n!$	n^2	$2n$	$1, 3, \ldots, 2n-1$
$D_n, n \geq 4$	-	$2^{n-1} n!$	$n^2 - n$	$2(n-1)$	$1, 3, \ldots, 2n-3, n-1$
E_6	-	$2^7 3^4 5$	36	12	$1, 4, 5, 7, 8, 11$
E_7	-	$2^{10} 3^4 5 7$	63	18	$1, 5, 7, 9, 11, 13, 17$
E_8	-	$2^{14} 3^5 5^2 7$	120	30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	24-cell (icositetrahoron)	1152	24	12	$1, 5, 7, 11$
H_3	icosahedron or dodecahedron	120	15	10	$1, 5, 9$
H_4	a regular 120-sided solid or a regular 600-sided solid	14400	60	30	$1, 11, 19, 29$
$I_2(m), m \geq 3$	regular m-gon	$2m$	m	m	$1, m-1$

- root system Φ associated to W is a set of vectors $\alpha \in \mathbb{R}^n$ s.t.
  - $W = \{s_\alpha : \alpha \in \Phi\}$
  - $\alpha, -\alpha \in \Phi$
  - $s_\alpha \Phi = \Phi$

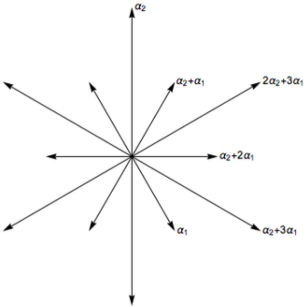




- positive roots Φ^+ of Φ: a subset of Φ satisfying
  - for each $\alpha \in \Phi$ exactly one of $\alpha, -\alpha \in \Phi^+$
  - for any two distinct $\alpha, \beta \in \Phi^+$ if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$
- simple roots Δ of Φ^+: each $\alpha \in \Phi^+$ is an $\mathbb{R}_{\geq 0}$-linear combination of Δ
- Φ is crystallographic if : $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$

  equiv. each $\alpha \in \Phi^+$ is an $\mathbb{N}$-linear combination of Δ
- crystallographic reflection groups are known as Weyl groups
- if Φ is crystallographic then we define the partial order $\preceq$ on Φ^+

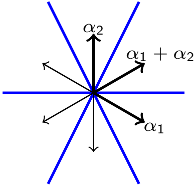
  $a \preceq b$ if and only if $b - a \in \mathbb{N}\Delta$

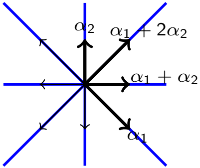


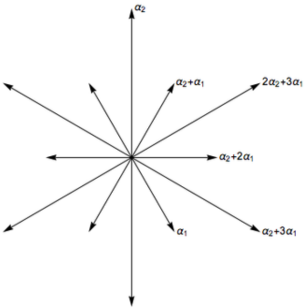
- root system Φ associated to W is a set of vectors $\alpha \in \mathbb{R}^n$ s.t.
  - $W = \{s_\alpha : \alpha \in \Phi\}$
  - $\alpha, -\alpha \in \Phi$
  - $s_\alpha \Phi = \Phi$
- positive roots Φ^+ of Φ: a subset of Φ satisfying
  - for each $\alpha \in \Phi$ exactly one of $\alpha, -\alpha \in \Phi^+$
  - for any two distinct $\alpha, \beta \in \Phi^+$ if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$
- simple roots Δ of Φ^+: each $\alpha \in \Phi^+$ is an $\mathbb{R}_{\geq 0}$-linear combination of Δ
- Φ is crystallographic if : $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$

  equiv. each $\alpha \in \Phi^+$ is an $\mathbb{N}$-linear combination of Δ
- crystallographic reflection groups are known as Weyl groups
- if Φ is crystallographic then we define the partial order $\preceq$ on Φ^+

$$a \preceq b \text{ if and only if } b - a \in \mathbb{N}\Delta$$







Root Systems and root posets: type A_{n-1}

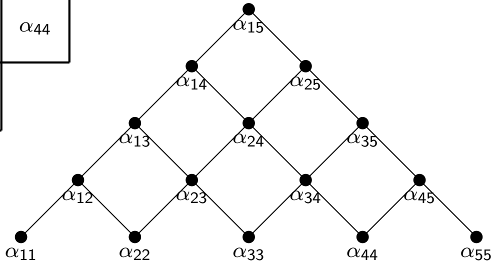
- $e_1, \ldots, e_n$ standard basis of $\mathbb{R}^n$
- positive roots Φ^+:

  $e_i - e_j$ for $1 \leq i < j \leq n$
- simple roots Δ:

  $e_i - e_{i+1}$ for $1 \leq i < n$
- Shorthand:

  $\alpha_{ij} = e_i - e_{j+1}$ for $1 \leq i \leq j < n$

α_{15}	α_{25}	α_{35}	α_{45}	α_{55}
α_{14}	α_{24}	α_{34}	α_{44}	
α_{13}	α_{23}	α_{33}		
α_{12}	α_{22}			
α_{11}				



- positive roots Φ^+

  e_i for $1 \leq i \leq n$

  $e_i \pm e_j$ for $1 \leq i < j \leq n$

- simple roots Δ:

  $e_i - e_{i+1}$ for $1 \leq i < n$

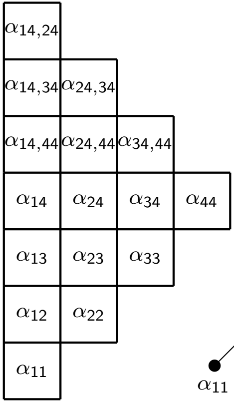
  e_n

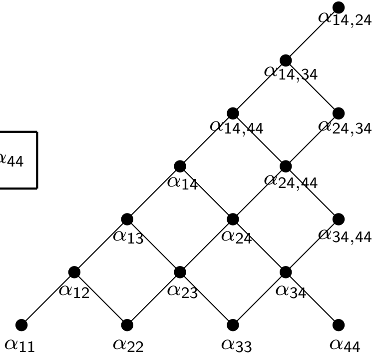
- Shorthand

  $\alpha_{ij} = e_i - e_{j+1}$ for $1 \leq i \leq j < n$

  $\alpha_{in} = e_i$ for $1 \leq i \leq n$

  $\alpha_{in,jn} = \alpha_{in} + \alpha_{jn} = e_i + e_j$ for $1 \leq i < j < n$





- positive roots Φ^+

  e_i for $1 \leq i \leq n$

  $e_i \pm e_j$ for $1 \leq i < j \leq n$

- simple roots Δ:

  $e_i - e_{i+1}$ for $1 \leq i < n$

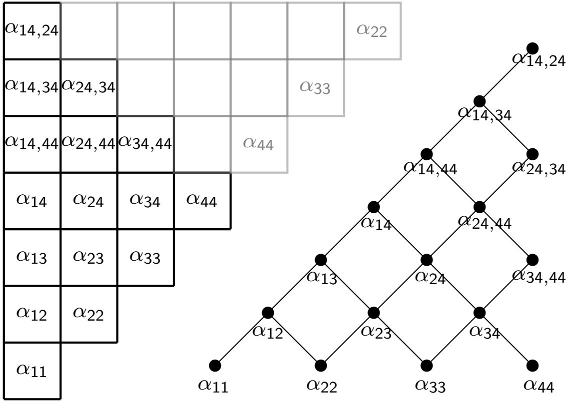
  e_n

- Shorthand

  $\alpha_{ij} = e_i - e_{j+1}$ for $1 \leq i \leq j < n$

  $\alpha_{in} = e_i$ for $1 \leq i \leq n$

  $\alpha_{in,jn} = \alpha_{in} + \alpha_{jn} = e_i + e_j$ for $1 \leq i < j < n$



- positive roots Φ^+

  $e_i \pm e_j$ for $1 \leq i < j \leq n$

- simple roots Δ:

  $e_i - e_{i+1}$ for $1 \leq i < n+1$

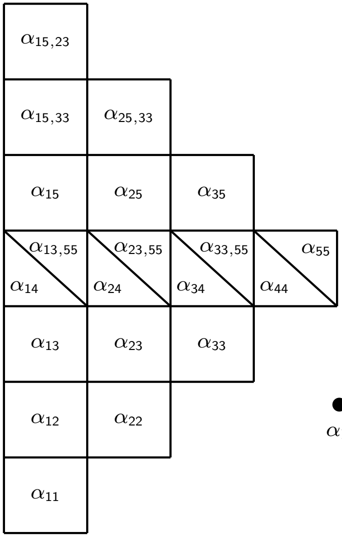
  $e_{n-1} + e_n$

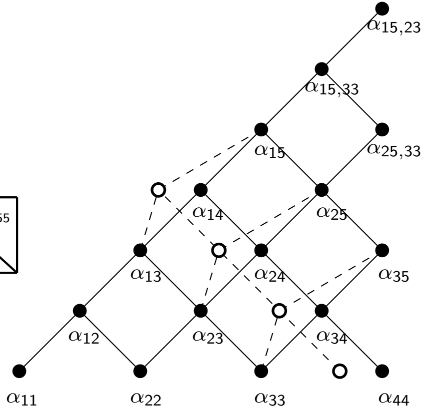
- Shorthand

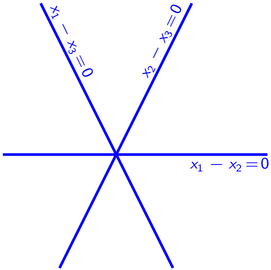
  $\alpha_{ij} = e_i - e_{j+1}$ for $1 \leq i \leq j < n$

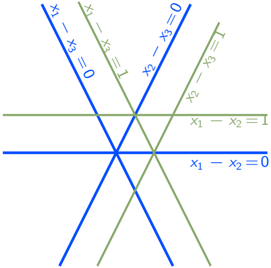
  $\alpha_{nn} = e_{n-1} + e_n$

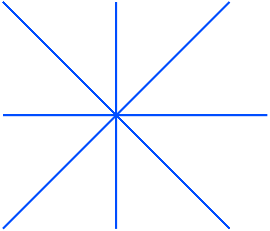
  $\alpha_{ij,k\ell} = \alpha_{ij} + \alpha_{k\ell}$



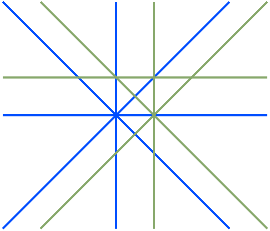






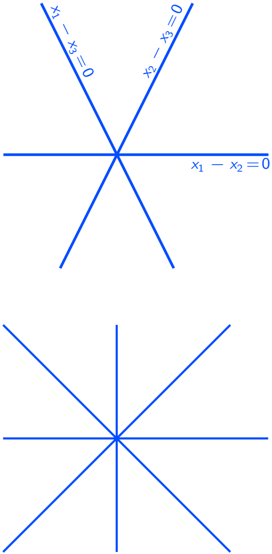


Coxeter arrangement

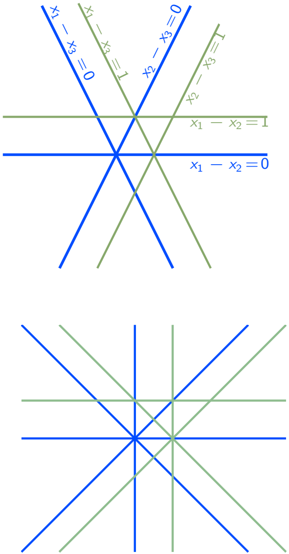


Shi arrangement

- a hyperplane arrangement is a collection of hyperplanes
- the connected components defined by the complements of the hyperplanes are called regions
- Coxeter arrangement: $\langle \alpha, x \rangle = 0$ for all $\alpha \in \Phi^+$
- the regions of the Coxeter arrangement are cones known as Weyl cones
- number of regions of the Coxeter arrangement is $|W|$
- Shi arrangement: $\langle \alpha, x \rangle = 0, 1$ for all $\alpha \in \Phi^+$



Coxeter arrangement



Shi arrangement

- number of regions of the Coxeter arrangement

$$|W| = \prod_{i=1}^{n}(e_i + 1)$$

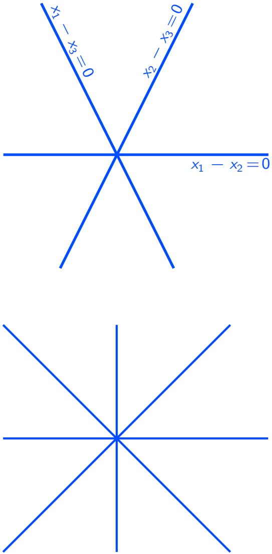
- number of regions of the Shi arrangement

$$\prod_{i=1}^{n}(e_i + h + 1)$$

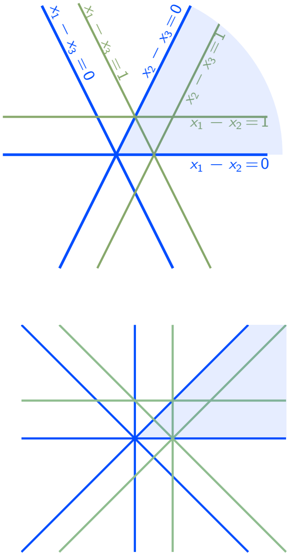
- dominant cone: the intersection $\langle \alpha, x \rangle \geq 0$ for all $\alpha \in \Phi^+$

- number of dominant regions of the Shi arrangement

$$\prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}$$



Coxeter arrangement



Shi arrangement

- number of regions of the Coxeter arrangement

$$|W| = \prod_{i=1}^{n}(e_i + 1)$$

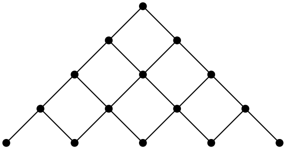
- number of regions of the Shi arrangement

$$\prod_{i=1}^{n}(e_i + h + 1)$$

- dominant cone: the intersection $\langle \alpha, x \rangle \geq 0$ for all $\alpha \in \Phi^+$
- number of dominant regions of the Shi arrangement

$$\prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1}$$

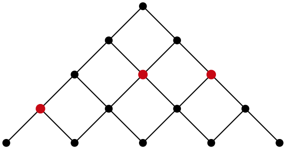
Theorem [Shi'97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.



- an antichain of a poset $(P, \preceq)$ is a subset of mutually incomparable elements
- an ideal of a poset $(P, \preceq)$ are all elements weakly above an antichain

$$\mathcal{A} \text{ antichain} \quad \Leftrightarrow \quad \text{ideal } I_{\mathcal{A}} = \{b \in P : b \succeq a \text{ for some } a \in \mathcal{A}\}$$

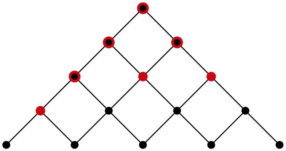
Theorem [Shi'97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.



- an antichain of a poset $(P, \preceq)$ is a subset of mutually incomparable elements
- an ideal of a poset $(P, \preceq)$ are all elements weakly above an antichain

$$\mathcal{A} \text{ antichain} \quad \Leftrightarrow \quad \text{ideal } I_{\mathcal{A}} = \{b \in P : b \succeq a \text{ for some } a \in \mathcal{A}\}$$

Theorem [Shi'97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.

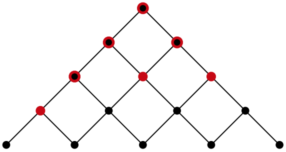


- an antichain of a poset $(P, \preceq)$ is a subset of mutually incomparable elements
- an ideal of a poset $(P, \preceq)$ are all elements weakly above an antichain

$$\mathcal{A} \text{ antichain} \quad \Leftrightarrow \quad \text{ideal } I_{\mathcal{A}} = \{b \in P : b \succeq a \text{ for some } a \in \mathcal{A}\}$$

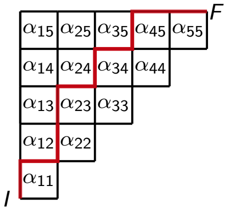
Dominant regions and antichains

Theorem [Shi'97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.



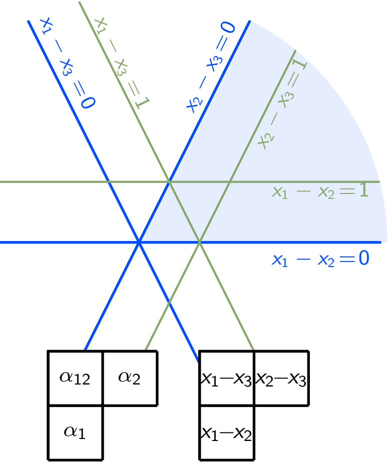
- an antichain of a poset $(P, \preceq)$ is a subset of mutually incomparable elements
- an ideal of a poset $(P, \preceq)$ are all elements weakly above an antichain

$$\mathcal{A} \text{ antichain} \quad \Leftrightarrow \quad \text{ideal } I_{\mathcal{A}} = \{b \in P : b \succeq a \text{ for some } a \in \mathcal{A}\}$$



- In type A_n the ideals of the root poset are in bijection with lattice paths from I to F in a staircase diagram of size n

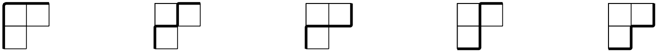
Theorem [Shi '97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.



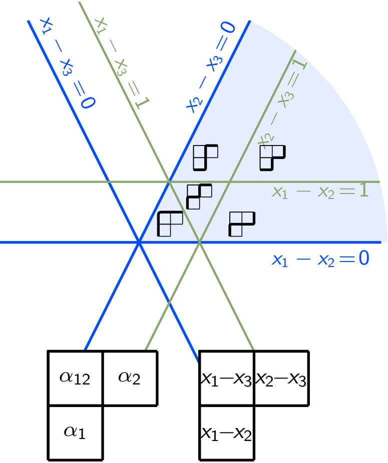
- The bijection:

  if I is an ideal in Φ^+ the corresponding region R_I is defined as

  $$R_I = \{x : \langle \alpha, x \rangle > 1 \text{ if } \alpha \in I \text{ and } 0 < \langle \alpha, x \rangle < 1 \text{ otherwise}\}$$



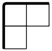
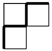
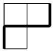
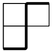
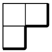
Theorem [Shi '97] There is a bijection between Shi regions in the dominant cone and antichains (ideals) in the root poset.

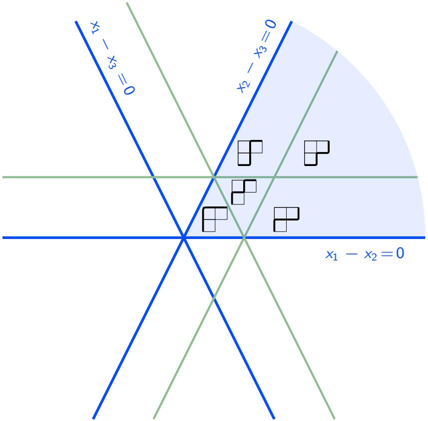


- The bijection:

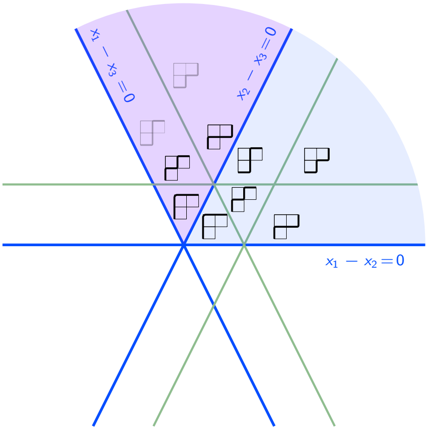
  if I is an ideal in Φ^+ the corresponding region R_I is defined as

$$R_I = \{x : \langle \alpha, x \rangle > 1 \text{ if } \alpha \in I \text{ and } 0 < \langle \alpha, x \rangle < 1 \text{ otherwise}\}$$

				
$0 < x_1 - x_2 < 1$	$0 < x_1 - x_2 < 1$	$0 < x_1 - x_2 < 1$	$x_1 - x_2 > 1$	$x_1 - x_2 > 1$
$0 < x_2 - x_3 < 1$	$0 < x_2 - x_3 < 1$	$x_2 - x_3 > 1$	$0 < x_2 - x_3 < 1$	$x_2 - x_3 > 1$
$0 < x_1 - x_3 < 1$	$x_1 - x_3 > 1$	$x_1 - x_3 > 1$	$x_1 - x_3 > 1$	$x_1 - x_3 > 1$



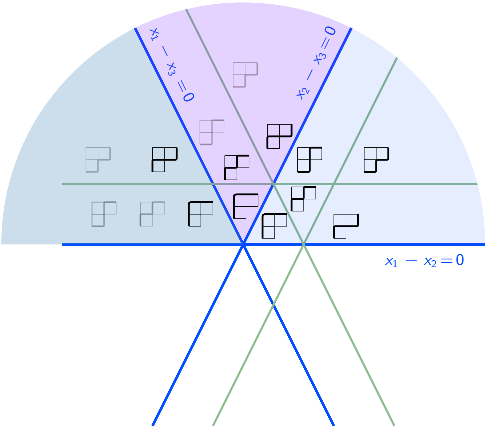
- the dominant cone $\mathcal{C} = \{x : 0 < \langle a, x \rangle$ for all $a \in \Phi^+\}$
- the other Weyl cones are $w\mathcal{C}$ with $w \in W$
- the regions in $w\mathcal{C}$ correspond to *some* ideals of the root poset
- ➠ can we describe them in terms of the root poset?
- ➠ can we count them?



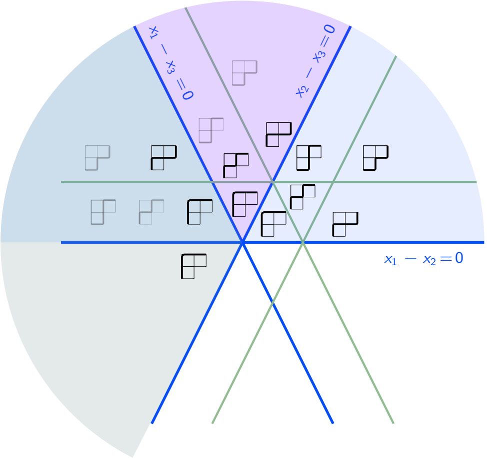
- the dominant cone $\mathcal{C} = \{x : 0 < \langle a, x \rangle \text{ for all } a \in \Phi^+\}$
- the other Weyl cones are $w\mathcal{C}$ with $w \in W$
- the regions in $w\mathcal{C}$ correspond to *some* ideals of the root poset

➠ can we describe them in terms of the root poset?

➠ can we count them?



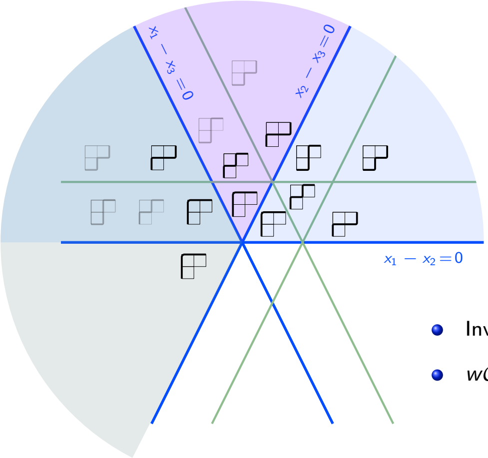
- the dominant cone $\mathcal{C} = \{x : 0 < \langle a, x\rangle \text{ for all } a \in \Phi^+\}$
- the other Weyl cones are $w\mathcal{C}$ with $w \in W$
- the regions in $w\mathcal{C}$ correspond to *some* ideals of the root poset
- ➠ can we describe them in terms of the root poset?
- ➠ can we count them?



- the dominant cone $\mathcal{C} = \{x : 0 < \langle a, x \rangle \text{ for all } a \in \Phi^+\}$
- the other Weyl cones are $w\mathcal{C}$ with $w \in W$
- the regions in $w\mathcal{C}$ correspond to *some* ideals of the root poset

➠ can we describe them in terms of the root poset?

➠ can we count them?



- the dominant cone $\mathcal{C} = \{x : 0 < \langle a, x \rangle \text{ for all } a \in \Phi^+\}$
- the other Weyl cones are $w\mathcal{C}$ with $w \in W$
- the regions in $w\mathcal{C}$ correspond to *some* ideals of the root poset
- ➠ can we describe them in terms of the root poset?
- ➠ can we count them?

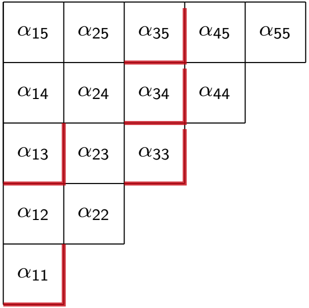
- $\mathrm{Inv}(w) = \{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}$
- $w\mathcal{C} = \{x : 0 < \langle \alpha, x \rangle$ if $\alpha \in \Phi^+ \setminus \mathrm{Inv}(w^{-1})$

  and $\langle \alpha, x \rangle < 0$ if $\alpha \in \mathrm{Inv}(w^{-1})\}$

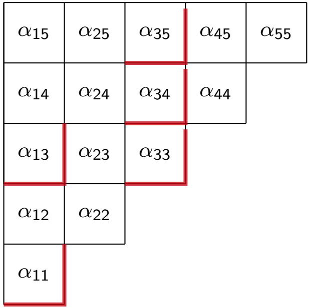
Theorem [Armstrong, Reiner & Rhoades, '14] For every $w \in W$, the Shi regions in the Weyl cone $w\mathcal{C}$ are in bijection with ideals in the subposet of the root poset Φ^+ restricted to $\Phi^+ \setminus \mathrm{Inv}(w^{-1})$.

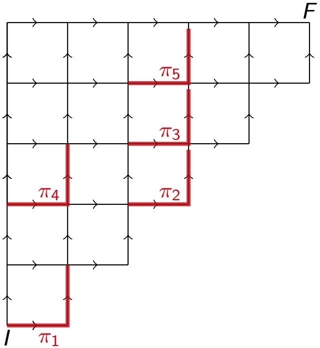
example in $W = A_5$

- If $w = s_5 s_2 s_4 s_3 s_1$ then $w\mathcal{C} = \{x \in \mathbb{R}^6 : x_2 > x_4 > x_1 > x_5 > x_6 > x_3\}$
- inversion set: $\mathrm{Inv}(w^{-1}) = \{\alpha_{11}, \alpha_{13}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$
- The number of Shi regions in $w\mathcal{C}$ is equal to the number of ideals in $\Phi^+ \setminus \{\alpha_{11}, \alpha_{13}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$

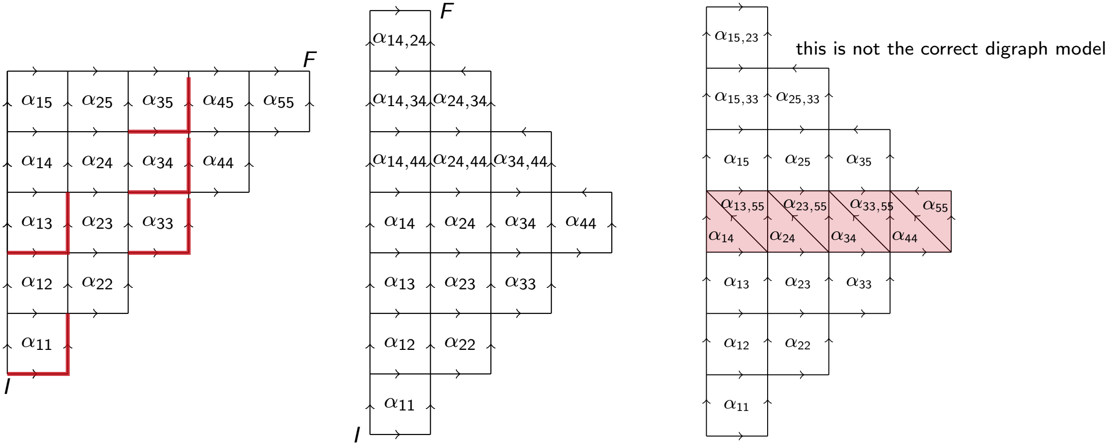


- If $w = s_5 s_2 s_4 s_3 s_1$ then $w\mathcal{C} = \{x \in \mathbb{R}^6 : x_2 > x_4 > x_1 > x_5 > x_6 > x_3\}$
- inversion set: $\mathrm{Inv}(w^{-1}) = \{\alpha_{11}, \alpha_{13}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$
- The number of Shi regions in $w\mathcal{C}$ is equal to the number of ideals in $\Phi^+ \setminus \{\alpha_{11}, \alpha_{13}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$
- The number of Shi regions in $w\mathcal{C}$ is equal to the number of paths from I to F in the directed acyclic graph Γ which do not contain any of the subpaths $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$





- The general situation: We have an acyclic directed graph G from I to F, $\Pi = \{\pi_1, \dots, \pi_n\}$ is a subset of non-overlapping subpaths of G
- We want to count all paths from I to F which do not contain any subpath from the set Π



Theorem [Dermenjian T. '23] Let I and F be the source and the sink of an acyclic directed graph Γ. Let $\Pi = \{\pi_1, \ldots, \pi_n\}$ be a collection of non-overlapping paths in Γ. Then

$$|I \to F : \pi \text{ contains no subpath from the set } \Pi|$$

$$= \begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |F_3 \to I_1| & \cdots & |F_n \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |F_3 \to I_2| & \cdots & |F_n \to I_2| & |I \to I_2| \\ |F_1 \to I_3| & |F_2 \to I_3| & |I_3 \to I_3| & \cdots & |F_n \to I_3| & |I \to I_3| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ |F_1 \to I_n| & |F_2 \to I_n| & |F_3 \to I_n| & \cdots & |I_n \to I_n| & |I \to I_n| \\ |F_1 \to F| & |F_2 \to F| & |F_3 \to F| & \cdots & |F_n \to F| & |I \to F| \end{vmatrix}$$

where I_i and F_i are the initial and terminal point of π_i, and

$|A \to B|$ is the number of paths from A to B in Γ

Theorem [Dermenjian T. '23] Let I and F be the source and the sink of an acyclic directed graph Γ. Let $\Pi = \{\pi_1, \dots, \pi_n\}$ be a collection of non-overlapping paths in Γ. Then

$$|I \to F : \pi \text{ contains no subpath from the set } \Pi|$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & |I \to I_1| \\ |F_1 \to I_2| & 1 & 0 & \cdots & 0 & |I \to I_2| \\ |F_1 \to I_3| & |F_2 \to I_3| & 1 & \cdots & 0 & |I \to I_3| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ |F_1 \to I_n| & |F_2 \to I_n| & |F_3 \to I_n| & \cdots & 1 & |I \to I_n| \\ |F_1 \to F| & |F_2 \to F| & |F_3 \to F| & \cdots & |F_n \to F| & |I \to F| \end{vmatrix}$$

where I_i and F_i are the initial and terminal point of π_i, and

$|A \to B|$ is the number of paths from A to B in Γ

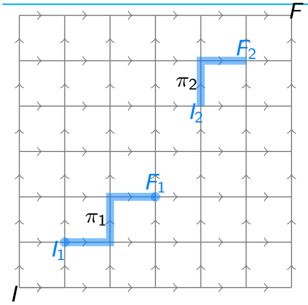
Idea of the proof

- $\det(A) = \sum_{\sigma \in S_{n+1}} \left(sgn(\sigma) \prod_{i \in [n+1]} a_{i,\sigma(i)} \right) = \sum_{\substack{\sigma = c_1 c_2 \cdots c_k \\ \sigma \in S_{n+1}}} \left(sgn(\sigma) \prod_{i \in c_1} a_{i,c_1(i)} \prod_{i \in c_2} a_{i,c_2(i)} \cdots \right)$
- If a non trivial cycle $c \in S_{n+1}$ does not contain $n+1$ then $\prod_{i \in c} a_{i,c(i)} = 0$
- If $c = (k)$ is a trivial cycle with $1 \leq k \leq n$ then $a_{kk} = 1$
- If a cycle $c = (n+1\, i_1 i_2 \cdots i_\ell)$ is a cycle $\in S_{n+1}$ that contains $n+1$ then

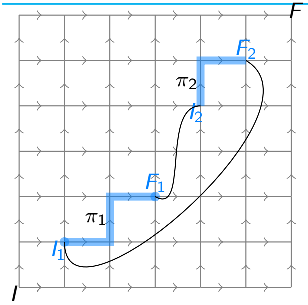
$$\prod_{i \in c} a_{i\, c(i)} = |I \to F : \pi \text{ contains the subpaths } \pi_{i_1}, \pi_{i_\ell}, \ldots, \pi_{i_2}|$$

- inclusion exclusion

Counting with determinants



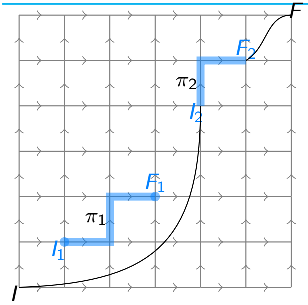
$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$



$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$

$$|I \to F| \Big(1 - |F_1 \to I_2||F_2 \to I_1| \Big)$$

(1) $|I \to F|$ (1') $|F_1 \to I_2||F_2 \to I_1| = |I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1| = 0$



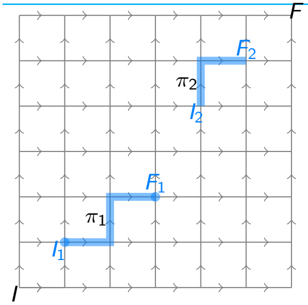
$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$

$$|I \to F|\Big(1 - |F_1 \to I_2||F_2 \to I_1|\Big)$$

$$- |I \to I_2|\Big(|F_2 \to F| - |F_1 \to F||F_2 \to I_1|\Big)$$

(1) $|I \to F|$ (1') $|F_1 \to I_2||F_2 \to I_1| = |I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1| = 0$

(2) $|I \to I_2||F_2 \to F| = |I \to I_2 \xrightarrow{\pi_2} F_2 \to F| = |I \to F :$ contains $\pi_2|$



$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$

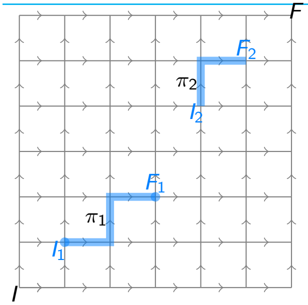
$$|I \to F|\Big(1 - |F_1 \to I_2||F_2 \to I_1|\Big)$$

$$- |I \to I_2|\Big(|F_2 \to F| - |F_1 \to F||F_2 \to I_1|\Big)$$

(1) $|I \to F|$ (1') $|F_1 \to I_2||F_2 \to I_1| = |I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1| = 0$

(2) $|I \to I_2||F_2 \to F| = |I \to I_2 \xrightarrow{\pi_2} F_2 \to F| = |I \to F :\ \text{contains } \pi_2|$

(2') $|I \to I_2||F_2 \to I_1||F_1 \to F| = |I \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1 \xrightarrow{\pi_1} F_1 \longrightarrow F| = |I \to F :\ \text{contains } \pi_2 \text{ and } \pi_1|$



$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$

$$|I \to F|\Big(1 - |F_1 \to I_2||F_2 \to I_1|\Big)$$

$$- |I \to I_2|\Big(|F_2 \to F| - |F_1 \to F||F_2 \to I_1|\Big)$$

$$+ |I \to I_1|\Big(|F_1 \to I_2||F_2 \to F| - |F_1 \to F|\Big)$$

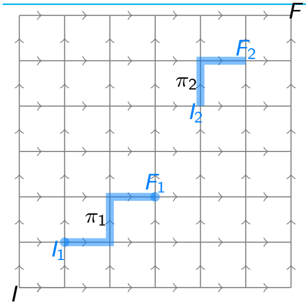
(1) $|I \to F|$ (1') $|F_1 \to I_2||F_2 \to I_1| = |I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1| = 0$

(2) $|I \to I_2||F_2 \to F| = |I \to I_2 \xrightarrow{\pi_2} F_2 \to F| = |I \to F : \text{ contains } \pi_2|$

(2') $|I \to I_2||F_2 \to I_1||F_1 \to F| = |I \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1 \xrightarrow{\pi_1} F_1 \longrightarrow F| = |I \to F : \text{ contains } \pi_2 \text{ and } \pi_1|$

(3) $|I \to I_1||F_1 \to I_2||F_2 \to F| = |I \to I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow F| = |I \to F : \text{ contains } \pi_1 \text{ and } \pi_2|$

(3') $|I \to I_1||F_1 \to F| = |I \longrightarrow I_1 \xrightarrow{\pi_1} F_1 \longrightarrow F\Big) = |I \to F : \text{ contains } \pi_1|$



$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |I \to I_2| \\ |F_1 \to F| & |F_2 \to F| & |I \to F| \end{vmatrix} =$$

# of paths from I to F with no subpaths in $\{\pi_1, \pi_2\}$

$$|I \to F|\Big(1 - |F_1 \to I_2||F_2 \to I_1|\Big)$$

$$- |I \to I_2|\Big(|F_2 \to F| - |F_1 \to F||F_2 \to I_1|\Big)$$

$$+ |I \to I_1|\Big(|F_1 \to I_2||F_2 \to F| - |F_1 \to F|\Big)$$

(1) $|I \to F|$ (1') $|F_1 \to I_2||F_2 \to I_1| = |I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1| = 0$

(2) $|I \to I_2||F_2 \to F| = |I \to I_2 \xrightarrow{\pi_2} F_2 \to F| = |I \to F : \text{ contains } \pi_2|$

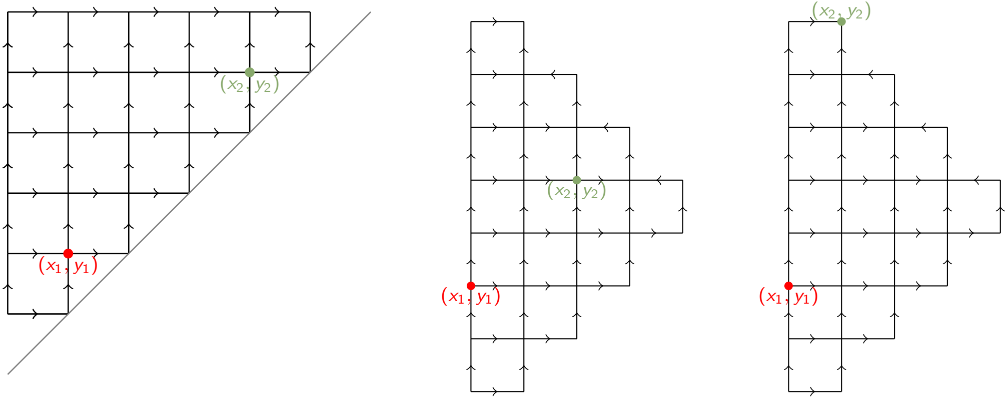
(2') $|I \to I_2||F_2 \to I_1||F_1 \to F| = |I \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow I_1 \xrightarrow{\pi_1} F_1 \longrightarrow F| = |I \to F : \text{ contains } \pi_2 \text{ and } \pi_1|$

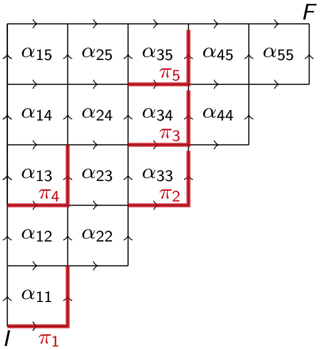
(3) $|I \to I_1||F_1 \to I_2||F_2 \to F| = |I \to I_1 \xrightarrow{\pi_1} F_1 \longrightarrow I_2 \xrightarrow{\pi_2} F_2 \longrightarrow F| = |I \to F : \text{ contains } \pi_1 \text{ and } \pi_2|$

(3') $|I \to I_1||F_1 \to F| = |I \longrightarrow I_1 \xrightarrow{\pi_1} F_1 \longrightarrow F\Big) = |I \to F : \text{ contains } \pi_1|$

Lemma [André 1887] Let Γ be the infinite digraph of $\mathbb{Z}^2$ with vertical edges pointing north and horizontal edges pointing east. Then

$$|(x_1,y_1)\to(x_2,y_2):\pi \text{ weakly above } x=y| = \begin{cases}\binom{x_2+y_2-x_1-y_1}{y_2-y_1}-\binom{x_2+y_2-x_1-y_1}{y_2-x_1+1} & \text{if } x_1\le x_2 \text{ and } y_1\le y_2\\ 0 & \text{otherwise}\end{cases}$$



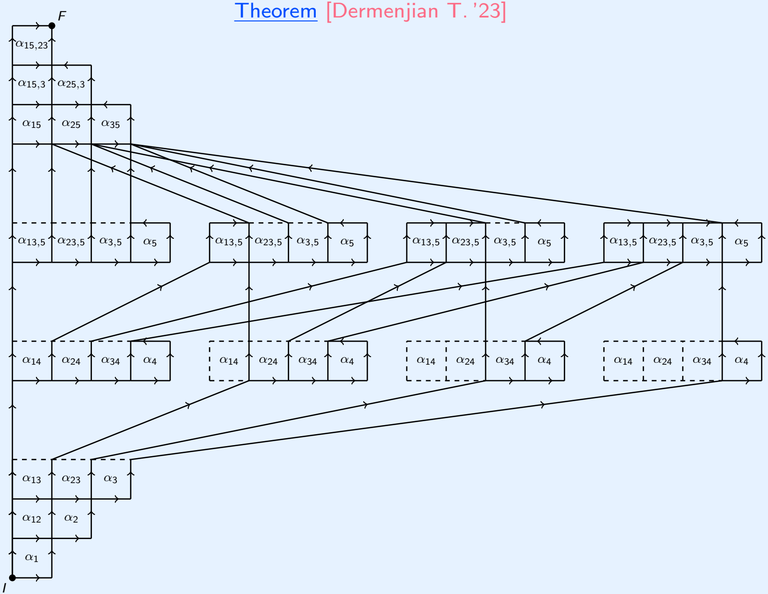


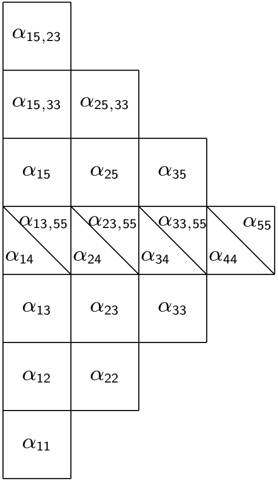
$$\begin{vmatrix} |I_1 \to I_1| & |F_2 \to I_1| & |F_3 \to I_1| & |F_4 \to I_1| & |F_5 \to I_1| & |I \to I_1| \\ |F_1 \to I_2| & |I_2 \to I_2| & |F_3 \to I_2| & |F_4 \to I_2| & |F_5 \to I_2| & |I \to I_2| \\ |F_1 \to I_3| & |F_2 \to I_3| & |I_3 \to I_3| & |F_4 \to I_3| & |F_5 \to I_3| & |I \to I_3| \\ |F_1 \to I_4| & |F_2 \to I_4| & |I_3 \to I_4| & |F_4 \to I_4| & |F_5 \to I_4| & |I \to I_4| \\ |F_1 \to I_5| & |F_2 \to I_5| & |I_3 \to I_5| & |F_4 \to I_5| & |F_5 \to I_5| & |I \to I_5| \\ |F_1 \to F| & |F_2 \to F| & |F_3 \to F| & |F_4 \to F| & |F_5 \to F| & |I \to F| \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \binom{0}{0}-\binom{0}{2} \\ \binom{2}{1}-\binom{2}{3} & 1 & 0 & 0 & 0 & \binom{4}{2}-\binom{4}{4} \\ \binom{3}{2}-\binom{3}{4} & 0 & 1 & \binom{1}{0}-\binom{1}{4} & 0 & \binom{5}{3}-\binom{5}{5} \\ 0 & 0 & 0 & 1 & 0 & \binom{2}{2}-\binom{2}{4} \\ \binom{4}{3}-\binom{4}{5} & 0 & 0 & \binom{2}{1}-\binom{2}{5} & 1 & \binom{6}{4}-\binom{6}{6} \\ \binom{8}{4}-\binom{8}{6} & \binom{4}{2}-\binom{4}{4} & \binom{3}{1}-\binom{3}{4} & \binom{6}{2}-\binom{6}{6} & \binom{2}{0}-\binom{2}{4} & \binom{9}{4}-\binom{9}{7} \end{vmatrix} = 38$$

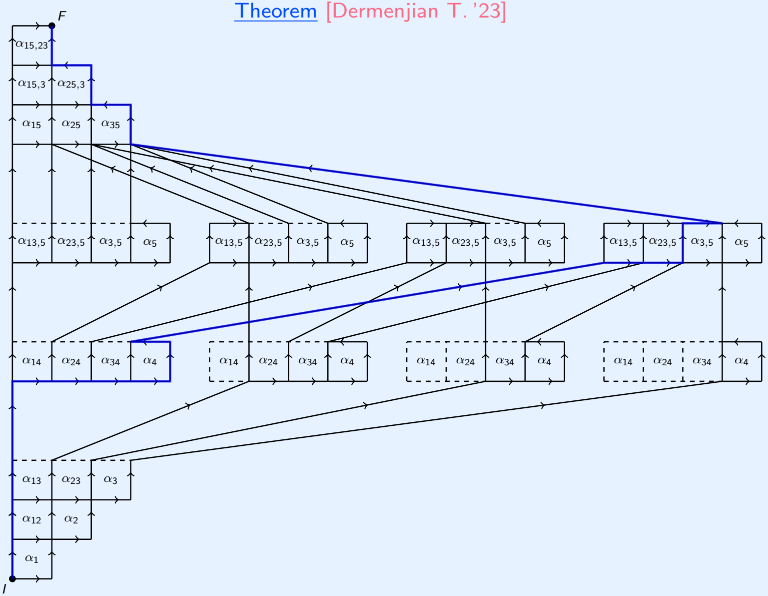
$$w\mathcal{C} = \{x \in \mathbb{R}^6 : x_2 > x_4 > x_1 > x_5 > x_6 > x_3\}$$

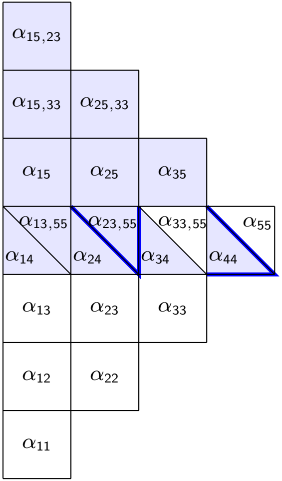
Theorem [Dermenjian T. '23]





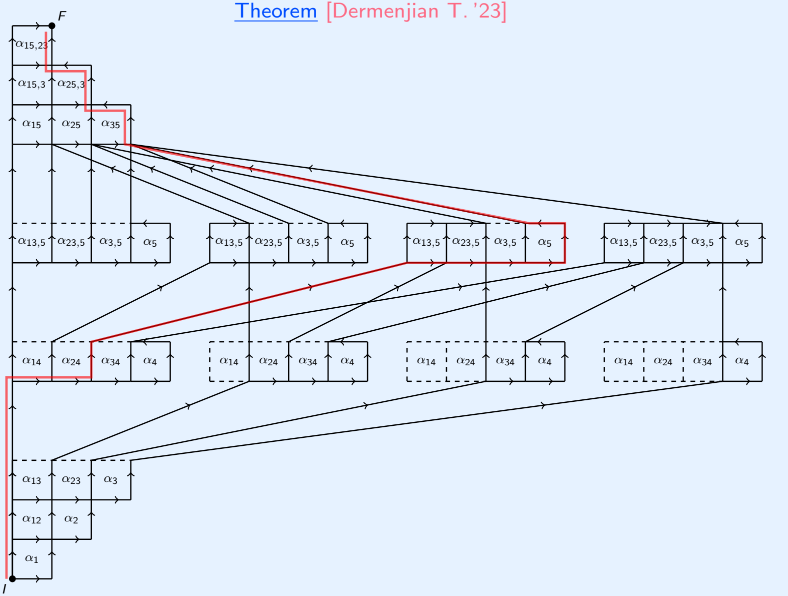
Theorem [Dermenjian T. '23]

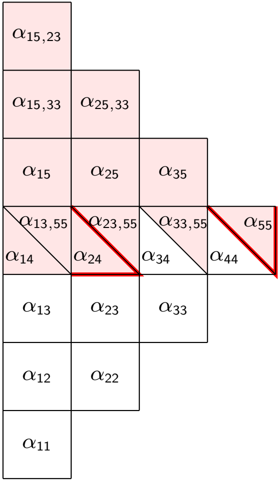




digraph for type D

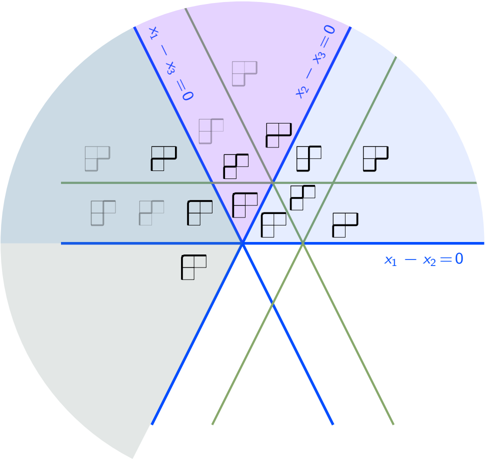
Theorem [Dermenjian T. '23]





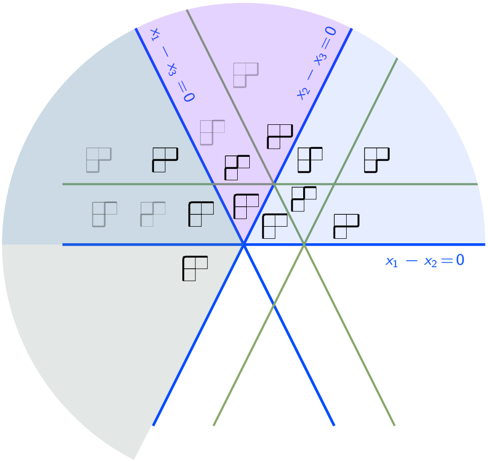
- Count ideals in the subposet $\Phi^+ \setminus \mathrm{Inv}(w^{-1})$ w.r.t. minimal elements
- Count regions in $w\mathcal{C}$ w.r.t. number of separating walls

$$\gamma(I \to J) = \sum_{\substack{\pi \text{ path in } \Gamma \\ \text{from } I \text{ to } J}} t^{|corners(\pi)|}$$



- Count ideals in the subposet $\Phi^+ \setminus \mathrm{Inv}(w^{-1})$ w.r.t. minimal elements
- Count regions in $w\mathcal{C}$ w.r.t. number of separating walls

$$\gamma(I \to J) = \sum_{\substack{\pi \text{ path in } \Gamma \\ \text{from } I \text{ to } J}} t^{|corners(\pi)|}$$

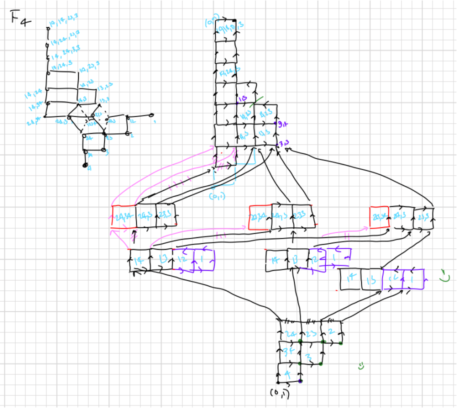


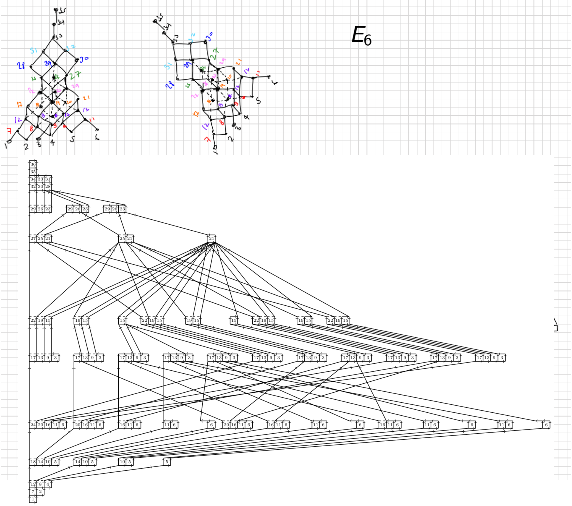
Theorem[Dermenjian T. '23]

$$\gamma(I \to F | \pi \text{ has no subpath in } \Pi) =$$

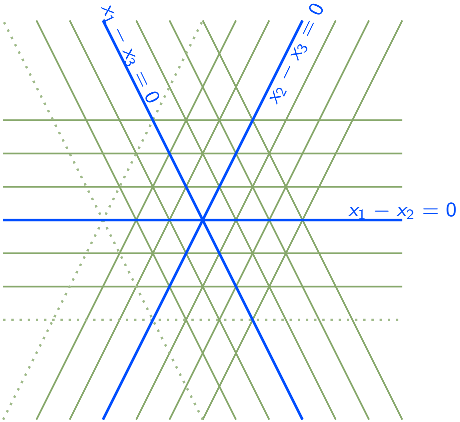
$$\begin{vmatrix} 1 & t\cdot\gamma(F_2 \to I_1) & \cdots & t\cdot\gamma(F_n \to I_1) & \gamma(I \to I_1) \\ t\cdot\gamma(F_1 \to I_2) & 1 & \cdots & t\cdot\gamma(F_n \to I_2) & \gamma(I \to I_2) \\ t\cdot\gamma(F_1 \to I_3) & t\cdot\gamma(F_2 \to I_3) & \cdots & t\cdot\gamma(F_n \to I_3) & \gamma(I \to I_3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\cdot\gamma(F_1 \to I_n) & t\cdot\gamma(F_2 \to I_n) & \cdots & 1 & \gamma(I \to I_n) \\ t\cdot\gamma(F_1 \to F) & t\cdot\gamma(F_2 \to F) & \cdots & t\cdot\gamma(F_n \to F) & \gamma(I \to F) \end{vmatrix}$$

① Find the appropriate acyclic directed graphs for the exceptional reflection groups





② Count regions of the *m*-Shi arrangement in each $w\mathcal{C}$



- *m*-extended Shi arrangement

  $\langle \alpha, x \rangle = k$ for all $\alpha \in \Phi^+$ and $-m < k \leq m$

- # dominant regions: $\displaystyle\prod_{i=1}^{n} \frac{e_i + mh + 1}{e_i + 1}$

- # regions: $\displaystyle\prod_{i=1}^{n} (e_i + mh + 1)$

Thank you for your attention!

ευχαριστώ για την προσοχή σας