Enumeration of colored permutations by the parity of descent positions

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Joint work with Qiongqiong Pan and Jiang Zeng

Outline of the talk

- Eulerian polynomial and q-analogue
- Coxeter group and Wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$
- Enumeration results
- Application: Signed alternating Eulerian polynomial

Eulerian polynomials - Euler's definition $(t \cdot \frac{d}{dt})$

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Leonhard Euler (1707-1783)

Euler's triangle

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1

Euler's definition

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Euler's exponential generating function formula

$$\sum_{n\geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{(t-1)z} - t}$$

$\mathfrak{S}_n$: Set of permutations of $[n] := \{1, \ldots, n\}$.

Definition (descent,excedance)

For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$,

$$\mathrm{des}(\sigma) := |\{i \in [n-1] : \sigma_i > \sigma_{i+1}\}|,$$
$$\mathrm{exc}(\sigma) := |\{i \in [n-1] : \sigma_i > i\}|.$$

Example

$$\mathrm{des}(3715624) = |\{2,5\}| = 2$$
$$\mathrm{exc}(3715624) = |\{1,2,4,5\}| = 4$$

Major Percy Alexander MacMahon (1854-1929)

P. A. MacMahon, Combinatory analysis,
vols 1 and 2, Cambridge University Press, 1916.

A combinatorial interpretation of Eulerian numbers

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
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$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{des}(\sigma)} = 1 + 4t + t^2$$

$$\sum_{\sigma\in\mathfrak{S}_3} t^{\mathrm{exc}(\sigma)} = 1 + 4t + t^2$$

Euler's triangle

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Euler's triangle

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Eulerian polynomial

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

MacMahon (1905) showed equidistribution of des and exc.
Carlitz and Riordan (1955) showed equals $A_n(t)$.

Let $\sigma \in \mathfrak{S}_n$,

Inversion number:

$$\mathrm{inv}(\sigma) := |\{(i,j) : \sigma(i) > \sigma(j),\ 1 \leq i < j \leq n\}|.$$

$$\mathrm{inv}(3142) = |\{(3,1),(3,2),(4,2)\}| = 3$$

Define q-analogue of n and n factorial:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$$

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q$$

$\mathfrak{S}_3$	inv
123	0
132	1
213	1
231	2
312	2
321	3

$$\sum_{\sigma\in\mathfrak{S}_3} q^{\mathrm{inv}(\sigma)}$$
$$= 1 + 2q + 2q^2 + q^3$$
$$= (1 + q + q^2)(1 + q)$$

Theorem (Rodrigues 1839)

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$$

Theorem (Stanley 1976)

We have

$$\sum_{n\geq 0}\frac{t^n}{[n]!_q}\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)}q^{\mathrm{inv}(\sigma)}=\frac{(1-t)\exp(t(1-x);q)}{1-t\exp(t(1-x);q)},$$

where

$$\exp(t;q)=\sum_{n\geq 0}\frac{t^n}{[n]_q!}.$$

When $q=1$, it reduces to Euler's formula. Note that Carlitz studied his q-analogue — $(\mathrm{des},\mathrm{maj})$.

R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, J. Combin. Thy. A, **20** (1976), 336–356.

Let $\sigma \in \mathfrak{S}_n$,

Odd descent set:

$$\mathrm{ODES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1} \text{ and } i \text{ is odd}\}$$

$$\sigma = 32541, \ \mathrm{ODES}(\sigma) = \{1, 3\}$$

Even descent set:

$$\mathrm{EDES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1} \text{ and } i \text{ is even}\}$$

$$\sigma = 32541, \ \mathrm{EDES}(\sigma) = \{4\}$$

Alternating descent set:

$$\mathrm{ALDES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1} \text{ and } i \text{ is odd or } \sigma_i < \sigma_{i+1} \text{ and } i \text{ is even}\}.$$

$$\sigma = 32541, \ \mathrm{ALDES}(\sigma) = \{1, 2, 3\}$$

Refinement of Eulerian polynomials

In 1973, Carlitz and Scoville considered the four-variable polynomials

$$P_n(x_0, x_1, y_0, y_1) = \sum_{\sigma \in \mathfrak{S}_n} x_0^{\mathrm{easc}(\sigma)} x_1^{\mathrm{oasc}(\sigma)} y_0^{\mathrm{edes}(\sigma)} y_1^{\mathrm{odes}(\sigma)}.$$

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Theorem (Carlitz and Scoville 1973)

Let $\alpha=\sqrt{(y_0-x_0)(y_1-x_1)}$, *we have*

$$\sum_{n\geq 1} P_n(x_0,x_1,y_0,y_1)\frac{t^n}{n!}=\frac{(x_1+y_1)\sum_{n\geq 1}\frac{\alpha^{n-1}t^{2n}}{(2n)!}+\sum_{n\geq 1}\frac{\alpha^{n-1}t^{2n-1}}{(2n-1)!}}{1-(x_0y_1+x_1y_0)\sum_{n\geq 1}\frac{\alpha^{n-1}t^{2n}}{(2n)!}}.$$

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$$\sum_{n \geq 1} P_n(x_0, x_1, y_0, y_1) \frac{t^n}{n!} = \frac{(x_1 + y_1) \sum_{n\geq 1} \frac{\alpha^{n-1} t^{2n}}{(2n)!} + \sum_{n \geq 1} \frac{\alpha^{n-1} t^{2n-1}}{(2n-1)!}}{1 - (x_0 y_1 + x_1 y_0) \sum_{n \geq 1} \frac{\alpha^{n-1} t^{2n}}{(2n)!}}.$$

In 2008, Chebikin considered the alternating Eulerian polynomials

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{aldes}(\sigma)}.$$

In 2018, Sun considered the bivariate Eulerian polynomials

$$\sum_{\sigma \in \mathfrak{S}_n} p^{\mathrm{odes}(\sigma)} q^{\mathrm{edes}(\sigma)}.$$

A q-analogue of Carlitz and Scoville's formula

Pan and Zeng considered

$$P_n(x_0, x_1, y_0, y_1, q) := \sum_{\sigma \in \mathfrak{S}_n} x_0^{\mathrm{easc}(\sigma)} x_1^{\mathrm{oasc}(\sigma)} y_0^{\mathrm{edes}(\sigma)} y_1^{\mathrm{odes}(\sigma)} q^{\mathrm{inv}(\sigma)}.$$

Theorem (Pan and Zeng 2022)

Let $\alpha = \sqrt{(y_0 - x_0)(y_1 - x_1)}$, *we have*

$$\sum_{n \geq 1} P_n(x_0, x_1, y_0, y_1, q) \frac{t^n}{[n]!_q}$$

$$= \frac{(x_1 + y_1)\cosh(\alpha t; q) + \alpha \sinh(\alpha t; q) - y_1(\cosh^2(\alpha t; q) - \sinh^2(\alpha t; q)) - x_1}{x_0 x_1 - (x_0 y_1 + x_1 y_0)\cosh(\alpha t; q) + y_0 y_1(\cosh^2(\alpha t; q) - \sinh^2(\alpha t; q))}.$$

When $q = 1$, it reduces to Carlitz and Scoville's formula.

Q. Q. Pan and J. Zeng,
Enumeration of permutations by the parity of descent position, Discrete Math. **346** (2023), no. 10, Paper No. 113575.

Coxeter group

Define a Coxeter system to be a pair (W, S), where W is a group, and S is a minimal generating set of W.

For every s, t in S, satisfying the following relations:

- $st = e$ if and only if $s = t$, and if $s \neq t$;
- $(st)^{m(s,t)} = (ts)^{m(s,t)} = e$ for some integer $m(s, t) > 1$.

We say such a group W is a Coxeter group of rank $r = |S|$.

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For $n \geq 2$, the symmetric group $\mathfrak{S}_n$ has a Coxeter presentation with generators $\{\tau_1, \ldots, \tau_{n-1}\}$ and defining relations

- $\tau_i^2 = e$ for $1 \leq i \leq n-1$;
- $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ for $1 \leq i \leq n-2$;
- $\tau_i \tau_j = \tau_j \tau_i$ for $\mid i - j \mid \geq 2$.

where τ_i is meaning that swapping the entries in positions i and $i+1$.

Coxeter group of type A, what's more, type B, D, E...

Analogy with symmetric group (length, descent)

For $w \in W$, define

Length: $\ell(w) = \min\{k \in \mathbb{N} : w = s_1 \cdots s_k, s_i \in S\}$.

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Proposition

When $W = \mathfrak{S}_n$ (type A),

$$\mathrm{inv}(w) = \ell(w),$$
$$\mathrm{DES}(w) = \{s_i \in S : w(i) > w(i+1)\}.$$

When $W = \mathcal{B}_n$ (type B),

$$\ell_B(w) = \mathrm{inv}(w) - \sum_{w(i)<0} w(i),$$

$$\mathrm{DES}_B(w) = \{s_i^B \in S_B : w(i-1) > w(i)\},$$

where $w(0) := 0$.

A. Björner and F. Brenti, Combinatorics of Coxeter groups, G.T.M 231, Springer-Verlag, New York, 2005.

Type B analogue of Carlitz and Scoville's formula

Remark

$$\mathrm{edes}_B(\gamma) + \mathrm{easc}_B(\gamma) = \lfloor (n+1)/2 \rfloor,$$
$$\mathrm{odes}_B(\gamma) + \mathrm{oasc}_B(\gamma) = \lfloor n/2 \rfloor.$$

Pan and Zeng considered

$$B_n(x, y) := \sum_{\sigma \in \mathcal{B}_n} x^{\mathrm{odes}_B(\sigma)} y^{\mathrm{edes}_B(\sigma)}.$$

In a follow up, Dey et al. gave a q-analogue version

$$B_n(x, y, q) := \sum_{\sigma \in \mathcal{B}_n} x^{\mathrm{odes}_B(\sigma)} y^{\mathrm{edes}_B(\sigma)} q^{\ell_B(\sigma)}.$$

H. K. Dey, U. Shankar and S. Sivasubramanian,
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Note that Coxeter group of type A and B are special cases of Wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$, isomorphic.

Our results will unify the formulas of type A and B.

Wreath product and r-colored permutations

Wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$

$$G(r,n) := \{(c_1,\dots,c_n;\sigma) \mid c_i \in [0,r-1], \sigma \in \mathfrak{S}_n\}.$$

Multiplication:

$$(c;\sigma)\cdot(c';\tau) := ((c_1 + c'_{\tau^{-1}(1)},\dots,c_n + c'_{\tau^{-1}(n)});\sigma\circ\tau),$$

where addition $+$ is in $\mathbb{Z}_r$ and composition $\circ$ is in $\mathfrak{S}_n$.

Window notation:

$$\gamma = [\sigma(1)^{c_1},\dots,\sigma(n)^{c_n}] \xlongequal{\textit{simplification}} [\gamma(1),\dots,\gamma(n)]$$

Viewed as a r-colored permutation, introduced by Steingrímsson.

The entry c_i is called the color of the $\sigma(i)$.

Example

$\gamma = [3^1, 2, 1^3, 4^2, 6^2, 5^1] \in G(5,6)$.

E. Steingrímsson. Permutation statistics of indexed permutations, Eur. J. Combin. **15**, (1994), 187–205.

Generator set, length and descent set

Generator set: $S_G := \{s_0, s_1, \ldots, s_{n-1}\}$, where for $i \in [n-1]$

$$s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ and } s_0 := [1^1, 2, \ldots, n].$$

For $\gamma \in G(r, n)$,

Length:

$$\ell_G(\gamma) := \min\{r \in \mathbb{N} : \gamma = s_{i_1} \cdots s_{i_r}, \text{ for some } s_{i_j} \in S_G\}.$$

Descent set:

$$\mathrm{DES}_G(\gamma) := \{s \in S_G : \ell_G(\gamma s) < \ell_G(\gamma)\}$$

and its size will be denoted by $\mathrm{des}_G(\gamma)$.

Combinatorial definition of length and descent

Linear order over colored set $\{0, 1, \ldots, n, 1^1, \ldots, n^1, \ldots, 1^{r-1}, \ldots, n^{r-1}\}$,

$$n^{r-1} < \ldots < n^1 < \ldots < 1^{r-1} < \ldots < 1^1 < 0 < 1 < \ldots < n.$$

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$$n^{r-1} < \ldots < n^1 < \ldots < 1^{r-1} < \ldots < 1^1 < 0 < 1 < \ldots < n.$$

For $\gamma \in G(r, n)$,

Length:

$$\ell_G(\gamma) := \mathrm{inv}(\gamma) + \sum_{c_i \neq 0} (\sigma(i) + c_i - 1),$$

Descent set:

$$\mathrm{DES}_G(\gamma) := \{i \in [0, n-1] : \gamma(i) > \gamma(i+1)\},$$

where $\gamma(0) := 0$. Note that $0 \in \mathrm{DES}_G(\gamma)$ iff $c_1 > 0$.

A q-analogue of exponential series over wreath product

For $n \in \mathbb{N}$, define the standard q-factorial notation

$$(a;q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1. \end{cases}$$

Theorem

$$G_{(r,n)}(a,q) := \sum_{\gamma \in G(r,n)} a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)} = [n]_q!(-aq[r-1]_q;q)_n,$$

where $\mathrm{col}(\gamma) := |\{i \in [n] : c_i > 0\}|$.

q-analogue of exponential series over wreath product, that is

$$\exp_{G(r)}(t;a,q) := \sum_{n \geq 0} \frac{t^n}{G_{(r,n)}(a,q)}$$

and write

$$\cosh_{G(r)}(t;a,q) = \frac{\exp_{G(r)}(t;a,q) + \exp_{G(r)}(-t;a,q)}{2},$$

$$\sinh_{G(r)}(t;a,q) = \frac{\exp_{G(r)}(t;a,q) - \exp_{G(r)}(-t;a,q)}{2}.$$

Main result

Define $G_{(r,n)}(x,y,a,q) := \sum_{\gamma \in G(r,n)} x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)}$

Theorem

For any $n \in \mathbb{N}$, we have

$$\sum_{n\geq 0} G_{(r,2n)}(x,y,a,q) \frac{t^{2n}}{G_{(r,2n)}(a,q)} = (1-y) \cdot \frac{\Big((1 - x\cosh(Mt;q)) \cosh_{G(r)}(Mt;a,q) + x \sinh(Mt;q) \sinh_{G(r)}(Mt;a,q) \Big)}{1 - (x+y)\cosh(Mt;q) + xy\exp(Mt;q)\exp(-Mt;q)},$$

$$\sum_{n\geq 0} G_{(r,2n+1)}(x,y,a,q) \frac{t^{2n+1}}{G_{(r,2n+1)}(a,q)} = M \cdot \frac{\Big((1 - y\cosh(Mt;q)) \sinh_{G(r)}(Mt;a,q) + y \sinh(Mt;q) \cosh_{G(r)}(Mt;a,q) \Big)}{1 - (x+y)\cosh(Mt;q) + xy\exp(Mt;q)\exp(-Mt;q)}.$$

where $M = \sqrt{(1-x)(1-y)}$.

Special cases

- When $r = 1$, $a = 0$, we can obtain the formula of Pan and Zeng about type A.
- When $r = 2$, $a = 1$, we can obtain the formula of Dey et al. about type B.
- When $r = 2$, $x = y$, $q = 1$, then substituting $t \leftarrow 2t$, we obtain Brenti's formula,

$$\sum_{n \geq 0} G_{(2,n)}(x, x, a, 1) \frac{t^n}{n!} = \frac{(1-x)\exp(t(1-x))}{1 - x\exp(t(1-x)(1+a))}.$$

Some notations and mapping f

$\binom{[n]}{m}^r$: the set of all r-colored subsets of size m of $[n]$.

$G_i(r,n)$: set of colored permutations with the last $n-i$ elements ordered increasingly

$$G_i(r,n) := \{\gamma \in G(r,n) : \gamma(i+1) < \cdots < \gamma(n-1) < \gamma(n)\}.$$

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$$G_i(r,n) := \{\gamma \in G(r,n) : \gamma(i+1) < \cdots < \gamma(n-1) < \gamma(n)\}.$$

If $\gamma = [\sigma(1)^{c_1}, \sigma(2)^{c_2}, \cdots, \sigma(n-m)^{c_{n-m}}] \in G(r, n-m)$ and $A^r = \{a_1^{c_1}, a_2^{c_2}, \cdots, a_m^{c_m}\}_< \in \binom{[n]}{m}^r$ with $A \subseteq [n]$ and $[n] \setminus A = \{s_1, s_2, \ldots, s_{n-m}\}_<$, define a mapping $f : G(r, n-m) \times \binom{[n]}{m}^r \to G_{n-m}(r,n)$ by

$$f(\gamma, A^r) = \gamma|_{[n]\setminus A} * [A^r].$$

where

$$[A^r] = [a_1^{c_1}, a_2^{c_2}, \cdots, a_m^{c_m}],$$

and

$$\gamma|_{[n]\setminus A} := [s_{\sigma(1)}^{c_1}, s_{\sigma(2)}^{c_2}, \ldots, s_{\sigma(n-m)}^{c_{n-m}}].$$

$*$ means concatenation.

Mapping f's properties

The mapping $f : G(r, n-m) \times \binom{[n]}{m}^r \to G_{n-m}(r, n)$ satisfies

$$\mathrm{col}(f(\gamma, A^r)) = \mathrm{col}(\gamma) + \mathrm{col}(f(e, A^r)),$$
$$\ell_G(f(\gamma, A^r)) = \ell_G(\gamma) + \ell_G(f(e, A^r)),$$

where e is the identity permutation in $G(r, n-m)$.
The second property can be proved by using the alternative definition of length.

Definition

For $\gamma = [\gamma(1), \ldots, \gamma(n)] = [\sigma(1)^{c_1}, \ldots, \sigma(n)^{c_n}] \in G(r, n)$, we have

$$\ell_G(\gamma) = \mathrm{inv}(\gamma) + \mathrm{inv}_c(\gamma) + \sum_{c_i \neq 0} c_i.$$

Note that

$$\mathrm{inv}_c(\gamma) = \sum_{1 \leq i < j \leq n} |\{(i,j) : \gamma_c(i) > \gamma(j)\}|,$$

where $\gamma_c(i)$ is defined by

$$\gamma_c(i) = \begin{cases} |\gamma(i)| & \text{if } c_i \neq 0; \\ |\gamma(i)|^1 & \text{if } c_i = 0. \end{cases}$$

Main Lemma

Lemma

Let $0 \leq m \leq n$. *For any* $\gamma \in G(r, n-m)$, *we have*

$$\sum_{A^r \in \binom{[n]}{m}^r} a^{\mathrm{col}(f(\gamma, A^r))} q^{\ell_G(f(\gamma, A^r))} = a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)} \binom{n}{m}_q (-aq^{n-m+1}[r-1]_q; q)_m.$$

Proof.

It suffices to prove the identity for $\gamma := e$. Let

$$F_m(a,q) := \sum_{A^r \in \binom{[n]}{m}^r} a^{\mathrm{col}(f(e,A^r))} q^{\ell_G(f(e,A^r))}.$$

Then we have to prove the following identity

$$[n-m]_q!(-aq[r-1]_q; q)_{n-m} \, F_m(a,q) \, [m]_q! = [n]_q!(-aq[r-1]_q; q)_n.$$

□

Sketch of the Proof

Proof.

We define a mapping $\Phi : G(r, n-m) \times \binom{[n]}{m}^r \times \mathfrak{S}_m \to G(r, n)$ by

$$\Phi(\gamma, A^r, \sigma) = \gamma|_{[n]\setminus A} * \sigma(A^r),$$

where $\gamma|_{[n]\setminus A} := [s_{\sigma(1)}^{c_1}, s_{\sigma(2)}^{c_2}, \ldots, s_{\sigma(n-m)}^{c_{n-m}}]$, and $\sigma(A^r) := [a_{\sigma_1}^{c'_{\sigma_1}}, a_{\sigma_2}^{c'_{\sigma_2}}, \ldots, a_{\sigma_m}^{c'_{\sigma_m}}]$.
Φ satisfies

$$\mathrm{col}\big(f(e, A^r)\big) + \mathrm{col}(\gamma) = \mathrm{col}\big(\Phi(\gamma, A^r, \sigma)\big);$$
$$\ell_G\big(f(e, A^r)\big) + \ell_G(\gamma) + \mathrm{inv}(\sigma) = \ell_G\big(\Phi(\gamma, A^r, \sigma)\big).$$

□

Example

For $\gamma = [4^1, 5, 1^2, 3^1, 2^3] \in G(4, 5)$, $A^4 = \{4^3, 6^1, 2^1, 1\}_< \in \binom{[9]}{4}^4$ and $\sigma = 3412 \in \mathfrak{S}_4$, we have

$$\Phi\Big([4^1, 5, 1^2, 3^1, 2^3], \{4^3, 6^1, 2^1, 1\}, 3412\Big) \overset{f}{\Longrightarrow} ([8^1, 9, 3^2, 7^1, 5^3 | 4^3, 6^1, 2^1, 1], 3412)$$
$$\overset{\sigma}{\Longrightarrow} [8^1, 9, 3^2, 7^1, 5^3, 2^1, 1, 6^1, 4^3].$$

$$\sum_{\gamma\in G(r,i)} \sum_{A^r\in\binom{[n]}{n-i}^r} x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}(f(\gamma,A^r))} q^{\ell_G(f(\gamma,A^r))}$$

$$= G_{(r,i)}(x,y,a,q) \binom{n}{n-i}_q (-aq^{i+1}[r-1]_q;q)_m.$$

Lemma

1. *If i is odd,*

$$\sum_{\gamma\in G_i(r,n)} w(\gamma) = x\frac{G_{(r,i)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!} - (x-1)\sum_{\gamma\in G_{i-1}(r,n)} w(\gamma).$$

2. *If i is even,*

$$\sum_{\gamma\in G_i(r,n)} w(\gamma) = y\frac{G_{(r,i)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!} - (y-1)\sum_{\gamma\in G_{i-1}(r,n)} w(\gamma).$$

where $w(\gamma) = x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)}$.

Type D

Denote by $\mathcal{D}_n$ the subgroup of $\mathcal{B}_n$ (i.e. $G(2,n)$) consisting of all the colored permutations having an even number of colored entries, that is,

$$\mathcal{D}_n := \{\gamma \in \mathcal{B}_n : \mathrm{col}(\sigma) \equiv 0 \pmod 2\}.$$

Define the enumerative polynomials

$$D_n(x,y,a,q) := \sum_{\gamma \in \mathcal{D}_n} x^{\mathrm{odes}_D(\gamma)} y^{\mathrm{edes}_D(\gamma)} a^{\mathrm{col}(\gamma)} q^{\ell_D(\gamma)},$$

Theorem

For any $n \in \mathbb{P}$, We have

$$D_{2n}(x,y,a,q)$$
$$= \frac{1}{2}\left(B_n(-a,q)\langle t^n\rangle \widehat{B}_0(t;x,y,-\frac{a}{q},q) + B_n(a,q)\langle t^n\rangle \widehat{B}_0(t;x,y,\frac{a}{q},q) \right),$$
$$D_{2n+1}(x,y,a,q)$$
$$= \frac{1}{2}\left(B_n(-a,q)\langle t^n\rangle \widehat{B}_1(t;x,y,-\frac{a}{q},q) + B_n(a,q)\langle t^n\rangle \widehat{B}_1(t;x,y,\frac{a}{q},q) \right),$$

where the notation $\langle t^n\rangle f(t)$ stands for the coefficient of t^n in $f(t)$.

Signed Eulerian polynomial

In 1989 Loday conjectured, Désarménien and Foata first investigated the relationship between a signed Eulerian polynomial and the classic Eulerian polynomial in 1992,

$$\sum_{\pi \in \mathfrak{S}_{\mathbf{2}n}} (-1)^{\ell(\pi)} t^{\mathrm{des}(\pi)} = (1-t)^n \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$$

and

$$\sum_{\pi \in \mathfrak{S}_{\mathbf{2}n+\mathbf{1}}} (-1)^{\ell(\pi)} t^{\mathrm{des}(\pi)} = (1-t)^n \sum_{\pi \in \mathfrak{S}_{n+\mathbf{1}}} t^{\mathrm{des}(\pi)}.$$

In 1992, Wachs proposed an elegant involution proof.

J.L. Loday,
Opérations sur l'homologie cyclique des algèbres commutatives.

J. Désarménien, D. Foata,
The signed Eulerian numbers.

M. Wachs,
An involution for signed Eulerian numbers.

Dey and Sivasubramanian computed the signed alternating descent enumeration in classical Weyl groups of types A and B.

H. K. Dey, and S. Sivasubramanian, Signed alternating descent enumeration in classical Weyl groups,
Discrete Math. **346** (2023), no. 10, Paper No. 113540.

We now evaluate the signed alternating descent enumeration for colored permutations.

Define q-alternating descent polynomials over wreath product by

$$\mathrm{Alt}_n^{G(r)}(x,q) := \sum_{\gamma \in G(r,n)} x^{\widehat{d}_G(\gamma)} q^{\ell_G(\gamma)},$$

and let $\mathrm{Alt}_n^{G(r)}(x) := \mathrm{Alt}_n^{G(r)}(x,-1)$.

Counting signed alternating descent

Theorem

If $r \geq 2$ is an even integer, then

$$\mathrm{Alt}_n^{G(r)}(x) = (-1)^{\lfloor (n+1)/2 \rfloor}(1-x)^n \quad \text{for } n \geq 0.$$

Theorem

If $r \geq 1$ is an odd integer, then $\mathrm{Alt}_1^{G(r)}(x) = x$ and

$$\mathrm{Alt}_n^{G(r)}(x) = \begin{cases} x(1-x)^m A_m(x) & \text{if } n = 2m \ (m \in \mathbb{N}^*); \\ \frac{2x^2}{1+x}(1-x)^{2m} A_{2m}(x) & \text{if } n = 4m+1 \ (m \in \mathbb{N}^*); \\ 0 & \text{if } n = 4m+3 \ (m \in \mathbb{N}). \end{cases}$$

When $r = 1, 2$, the above formulas reduce to Dey and Sivasubramanian's results.

Sketch of the proof

Note that $\mathrm{Alt}_n^{G(r)}(x,q) := x^{\lfloor (n+1)/2 \rfloor} G_{(r,n)}(x, 1/x, 1, q)$.

Lemma

Let $\mathrm{Alt}_0^{G(r)}(x;q) = 1$ *and* $M = 1 - x$*. For positive integer* r*, we have*

$$\sum_{n\geq 0} \frac{\mathrm{Alt}_{2n}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n}} \cdot \frac{t^{2n}}{[2n]_q!} = (x-1)\times \frac{(1 - x\cos(Mt;q))\cos_{G(r)}(Mt;q) - x\sin(Mt;q)\sin_{G(r)}(Mt;q)}{x - (x^2+1)\cos(Mt;q) + x\exp(iMt;q)\exp(-iMt;q)},$$

$$\sum_{n\geq 0} \frac{\mathrm{Alt}_{2n+1}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n+1}} \cdot \frac{t^{2n+1}}{[2n+1]_q!} = (x-1)\times \frac{(x - \cos(Mt;q))\sin_{G(r)}(Mt;q) + \sin(Mt;q)\cos_{G(r)}(Mt;q)}{x - (x^2+1)\cos(Mt;q) + x\exp(iMt;q)\exp(-iMt;q)}.$$

Multiplying the two sides by

$$x - (x^2 + 1)\cos((1-x)t; q) + x\exp(i(1-x)t; q) \cdot \exp(-i(1-x)t; q),$$

then taking the limit $q \to -1$. By the limits of q-binomial coefficients:

$$\lim_{q \to -1} \binom{2n}{2m+1}_q = 0,$$

$$\lim_{q \to -1} \binom{2n}{2m}_q = \lim_{q \to -1} \binom{2n+1}{2m}_q = \lim_{q \to -1} \binom{2n+1}{2m+1}_q = \binom{n}{m},$$

$$\lim_{q \to -1} [r-1]_q = \lim_{q \to -1} \frac{1-q^{r-1}}{1-q} = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ 1, & \text{if } r \text{ is even.} \end{cases}$$

For $n = 4m+3$, we construct a weight preserving and sign reversing involution to prove that $\mathrm{Alt}_{4m+3}^{G(r)}(x, -1) = 0$.

The final slide

Thanks