An Introduction to Combinatorial Species

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The main reference for the theory of combinatorial species is the book Combinatorial Species and Tree-Like Structures by François Bergeron, Gilbert Labelle, and Pierre Leroux.

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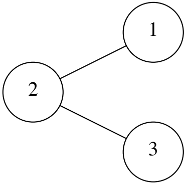
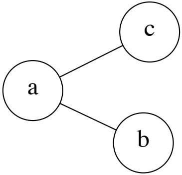
If a structure has label set U and we have a bijection $\sigma\colon U \to V$ then we can replace each label $u \in U$ with its image $\sigma(u)$ in V.



$$1 \mapsto c$$
$$2 \mapsto a$$
$$3 \mapsto b$$



What are species good for?

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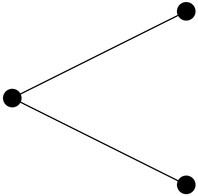
More interestingly, it allows us to count unlabeled versions of labeled structures (unlabeled structures). If we have a bijection $A \to A$ then we also get a bijection from the set of structures with label set A to itself, so we have an action of the symmetric group on A acting on these structures. The orbits of these structures are the unlabeled structures.



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The theory of species also sheds some light on actions of symmetric groups and symmetric functions.

Definition of a species

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This means that if F is a species then for every finite set U, there is a finite set $F[U]$ (the set of F-structures on U), and for any bijection $\sigma\colon U \to V$ there is a bijection $F[\sigma]\colon F[U] \to F[V]$.

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Moreover, we have the functorial properties

- If $\sigma\colon U \to V$ and $\tau\colon V \to W$ then $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$.
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Think of $F[U]$ as some sort of graph with label set U, even though there are no "labels" in the definition.

Examples of species

- The species E of sets: $E[U] = \{U\}$.
- The species E_n of n-sets:

$$E_n[U] = \begin{cases} \{U\} & \text{if } |U| = n \\ \varnothing & \text{if } |U| \neq n \end{cases}$$

- We write X for E_1, the species of singletons.
- The species Par of set partitions
- The species L of linear orders
- The species S of permutations (bijections from a set to itself).
- The species C of cyclic permutations
- the species $\mathcal{G}$ of graphs
- the species $\mathcal{G}^c$ of connected graphs

Isomorphism of species

Let F and G be species. An isomorphism α from F to G is a family of bijections $\alpha_U : F[U] \to G[U]$ for every finite set U such that for every bijection $\sigma : U \to V$, and every $s \in F[U]$ we have $G[\sigma](\alpha_U(s)) = \alpha_V(F[\sigma](\sigma))$.

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For example, the subset $\{1, 3, 4\}$ of $[5]$ corresponds to the ordered partition $(\{1, 3, 4\}, \{2, 5\})$.

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Let's see what happens for $n = 2$. Here we have $|S[2]| = |L[2]| = 2$ and

$$S[2] = \{(1)(2), (1\,2)\}, \qquad L[2] = \{12, 21\}$$

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What happens if apply the bijection $[2] \to [2]$ that switches 1 and 2? Both elements of $S[2]$ are fixed, but the two elements of $L[2]$ switch. So S and L can't be isomorphic.

Operations on species

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So an $(F+G)$-structure is either an F-structure or a G-structure.

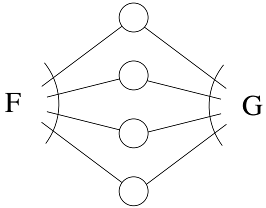
We can also have infinite sums, as long as they "converge"

$$E = \sum_{n=0}^{\infty} E_n$$

Next is Cartesian product:

$$(F \times G)[U] = F[U] \times G[U]$$

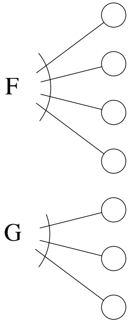
So an $(F \times G)$-structure is an F-structure and a G-structure on the same set of points.



The ordinary product FG is more useful than the Cartesian product, but the definition is more complicated:

$$(FG)[U] = \sum_{U_1, U_2} F[U_1] \times G[U_2],$$

where the sum is over all decompositions of U into U_1 and U_2, so that $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \varnothing$.



Note that $(FG)[U]$ is not the same as $(GF)[U]$, but the species FG and GF are isomorphic. We usually identify species that are isomorphic.

We can define powers inductively, and we find that the species L_n of linear orders of n-sets is isomorphic to X^n, and

$$L = \sum_{n=0}^{\infty} X^n.$$

(Note that $X^0 = E_0$.)

Finally, we have composition or substitution of species, $F \circ G$. An element of $(F \circ G)[U]$ consists of a partition of U into (not necessarily nonempty) blocks, a G-structure on each block, and an F-structure on the set of blocks.

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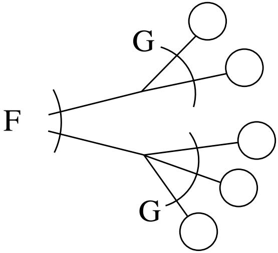

The most important special case is $F = E$, the species of sets, or $F = E_n$, the species of n-sets. Then $E \circ G$ is the species of sets of G-structures and $E_n \circ G$ is the species of n-sets of G-structures.

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Generating functions for species

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The unlabeled generating function is

$$\widetilde{F}(x) = \sum_{n=0}^{\infty} \tilde{f}_n x^n,$$

where $\tilde{f}_n$ is the number of unlabeled F-structures on $[n]$.

These generating functions are compatible with addition and multiplication:

$$(F+G)(x) = F(x) + G(x) \qquad (\widetilde{F+G})(x) = \widetilde{F}(x) + \widetilde{G}(x)$$

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as long as $G(x)$ has no constant term; i.e., $G[\varnothing] = \varnothing$.

However, $(\widetilde{F \circ G})(x)$ cannot be computed from $\widetilde{F}(x)$ and $\widetilde{G}(x)$.

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For the species C of cyclic permutations,

$$C(x) = \sum_{n=0}^{\infty} (n-1)! \frac{x^n}{n!} = \log\left(\frac{1}{1-x}\right) \quad \text{and} \quad \widetilde{C}(x) = \frac{x}{1-x}.$$

For the species $S = E \circ S$ of permutations,

$$S(x) = \exp(C(x)) = \frac{1}{1-x} = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} \text{ and } \widetilde{S}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

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For the species $\mathrm{Par} = E \circ E^+$ of partitions, we have

$$\mathrm{Par}(x) = \exp(E^+(x)) = e^{e^x - 1}$$

$$\widetilde{\mathrm{Par}}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

The cycle index series

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The cycle index Z_F of F is the characteristic of this action of $\mathfrak{S}_n$.

For each π in $\mathfrak{S}_n$, let $\operatorname{fix} F[\pi]$ be the number of elements of $F[n]$ fixed by $F[\pi]$. Let $c_i(\pi)$ be the number of cycles of π of length i. Then we define

$$Z_F = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \operatorname{fix} F[\pi] \, p_1^{c_1(\pi)} p_2^{c_2(\pi)} \ldots,$$

where p_j is the power sum symmetric function $x_1^j + x_2^j + x_3^j + \cdots$.

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Since $\operatorname{fix} F[\pi]$ depends only on the cycle type of π, we can write this formula in another way.

Let $\lambda = (1^{m_1} 2^{m_2} \cdots)$ be a partition of n. The number of permutations in $\mathfrak{S}_n$ of cycle type λ is $n!/z_\lambda$, where

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots .$$

Let $\operatorname{fix} F[\lambda] = \operatorname{fix} F[\pi]$ where π is any permutation in $\mathfrak{S}_n$ of cycle type λ. Then

$$Z_F = \sum_{\lambda \vdash n} \operatorname{fix} F[\lambda] \frac{p_\lambda}{z_\lambda}.$$

where $p_\lambda = p_1^{m_1} p_2^{m_2} \ldots$.

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More generally, let's take $F = E_n$. Then $E_n[n]$ has only one element, $[n]$, and it's fixed by every element of $\mathfrak{S}_n$. So for every partition λ of n, we have $\operatorname{fix} E_n[\lambda] = 1$, so

$$Z_{E_n} = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}.$$

This is equal to the complete symmetric function

$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For the species $L_n = X^n$ of linear orders of size n, only the identity element fixes anything, and it fixes all $n!$ linear orders, so

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For the species C_n of n-cycles, a permutation π doesn't fix anything unless π consists of n/d d-cycles for some d dividing n. It's not too hard to show that

$$Z_{C_n} = \frac{1}{n} \sum_{d|n} \varphi(d) p_d^{n/d}$$

where φ is Euler's function.

For species that are not homogeneous, the cycle index is the sum of the cycle indices of the homogeneous components. So

$$Z_E = \sum_{n=0}^{\infty} Z_{E_n} = \sum_{n=0}^{\infty} h_n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i} = \exp\left(\sum_{j=1}^{\infty} \frac{p_j}{j}\right)$$

and

$$Z_L = \sum_{n=0}^{\infty} Z_{L^n} = \sum_{n=0}^{\infty} p_1^n = \frac{1}{1 - p_1}$$

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$\widetilde{F}(x)$ is obtained from Z_F be replacing each p_i with x^i, or equivalently, replacing x_1 with x and x_i with 0 for $i > 1$.

Species operations and the cycle index

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$$p_\lambda * p_\mu = z_\lambda \delta_{\lambda,\mu}\, p_\lambda.$$

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This is because

$$\operatorname{fix}(F \times G)[\pi] = \operatorname{fix} F[\pi]\ \operatorname{fix} G[\pi].$$

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Plethysm can be defined in several equivalent ways. The most intuitive way to define $f \circ g$ when g has positive integer coefficients, is to write g as a sum of monic terms and substitute them for the variables of f.

For composition of species, we have a corresponding operation on symmetric functions called composition or plethysm:

$$Z_{F \circ G} = Z_F \circ Z_G.$$

Plethysm can be defined in several equivalent ways. The most intuitive way to define $f \circ g$ when g has positive integer coefficients, is to write g as a sum of monic terms and substitute them for the variables of f.

But if f and g expressed in terms of the p_i, a more efficient procedure is to first define $p_j \circ g$ to be the result of replacing each p_i in g with p_{ij}, and then replacing each p_j in f with $p_j \circ g$.

Pólya's theorem and the coefficients of the cycle index

There is a simple and sometimes useful interpretation for the coefficients of the cycle index. We know that the coefficient of x_1^n in Z_F is the number of unlabeled F-structures on n points.

More generally, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots$ in Z_F is the number of "F-structures labeled with the multiset $\{1^{n_1}, 2^{n_2}, \ldots\}$."

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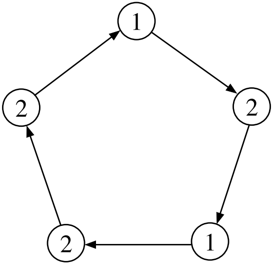
Example: One of the structures counted by the coefficient of $x_1^2 x_2^3$ in Z_{C_5} is



Indirect decompositions

We have seen that the species of set partitions can be expressed as a composition $E \circ E^+$. There are other cases, where we can't easily construct a species directly, but we can find an equation that it satisfies.

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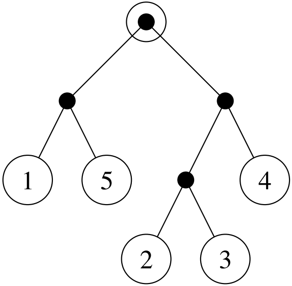
For example, consider the species $\mathcal{G}^c$ of connected graphs. Every graph may be viewed as a set of connected graphs, so the species $\mathcal{G}$ of graphs and the species $\mathcal{G}^c$ of connected graphs are related by $\mathcal{G} = E \circ \mathcal{G}^c$ and so $Z_{\mathcal{G}} = Z_E \circ Z_{\mathcal{G}^c}$. This formula can be inverted to compute $Z_{\mathcal{G}^c}$ and thereby count labeled and unlabeled connected graphs.

Trees

Indirect decompositions also arise in counting trees of various types. For now, I will talk about leaf-labeled (unordered) rooted binary trees, which I'll call simply binary trees.



A binary tree is either a single labeled vertex or an unordered pair of binary trees. So the species R of binary trees satisfies

$$R = X + E_2 \circ R$$

and therefore the cycle index satisfies

$$Z_R = p_1 + h_2 \circ Z_R.$$

For the exponential generating function this reduces to

$$R(x) = x + R(x)^2/2,$$

which can easily be solved to give

$$R(x) = 1 - \sqrt{1-2x} = \sum_{n=1}^{\infty} 1 \cdot 3 \cdots (2n-3) \frac{x^n}{n!}$$

For the cycle index, there is a surprisingly simple formula discovered a few years ago by Sara Billey, Matjaž Konvalinka, and Frederick A. Matsen IV:

$$Z_R = \sum_{\lambda} r_\lambda \frac{p_\lambda}{z_\lambda},$$

where r_λ is zero if λ is not a binary partition (a partition in which every part is a power of 2), and if λ is a binary partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ then

$$r_\lambda = \prod_{i=2}^{k} \bigl(2(\lambda_i + \cdots + \lambda_k) - 1\bigr),$$

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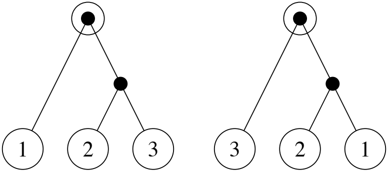
$$r_\lambda = \prod_{i=2}^{k} \bigl(2(\lambda_i + \cdots + \lambda_k) - 1\bigr),$$

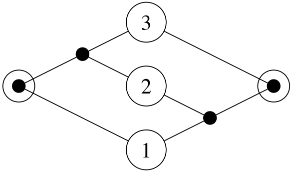
So the number of unlabeled binary trees with n leaves is

$$\sum_{\lambda \vdash n} r_\lambda / z_\lambda.$$

Billey, Konvalinka, and Matsen were interested in tanglegrams, which are ordered pairs of binary trees that share the same leaves. They wanted to count unlabeled tanglegrams.
Here's a tanglegram



which we can also draw as



Since a tanglegram is an ordered pair of trees, the species of tanglegrams is the Cartesian product $R \times R$, so the cycle index for tanglegrams is

$$Z_{R\times R} = Z_R * Z_R = \sum_{\lambda} r_\lambda^2 \frac{p_\lambda}{z_\lambda}.$$

and therefore the number of unlabeled tanglegrams with n leaves is

$$\sum_{\lambda \vdash n} \frac{r_\lambda^2}{z_\lambda}.$$