Smirnov words and the Delta conjectures

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Università di Pisa
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Joint work with P. Nadeau and A. Vanden Wyngaerd

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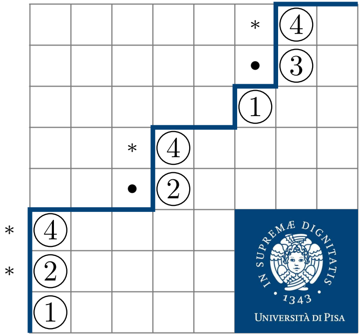

Symmetric functions

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If $\mathbb{K} = \mathbb{Q}(q, t)$, we also have the Macdonald polynomials $\widetilde{H}_\mu(q, t)$.

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Let us define $\nabla \colon \Lambda \to \Lambda$ as $\nabla \widetilde{H}_\mu := e_{|\mu|}[B_\mu] \widetilde{H}_\mu$. We have

$$\mathsf{Frob}_{q,t}(\mathcal{DH}_n) = \nabla e_n,$$

and that Macdonald polynomials are the Frobenius characteristics of the Garsia-Haiman submodules of $\mathcal{DH}_n$.

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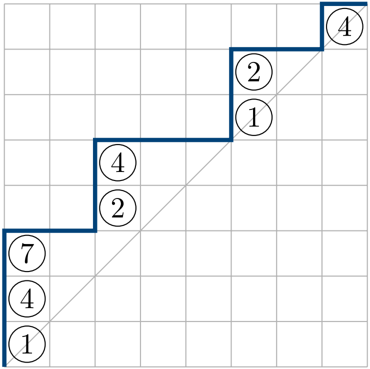

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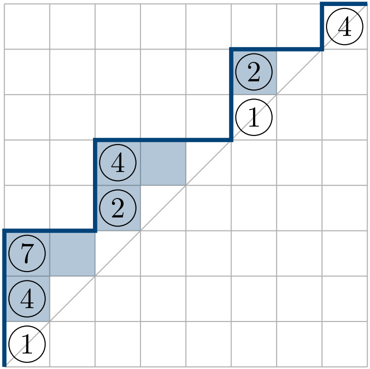

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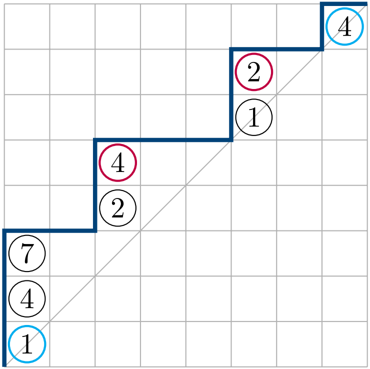

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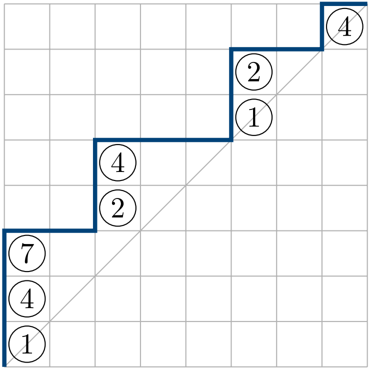

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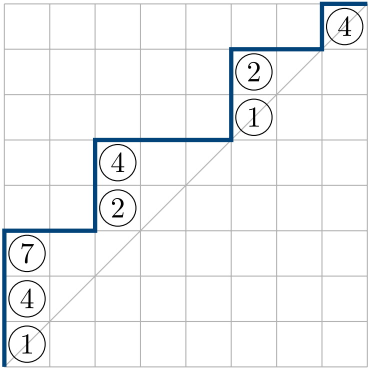

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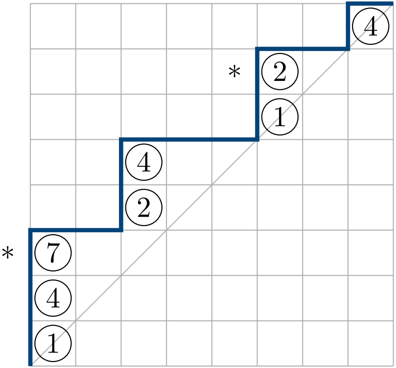

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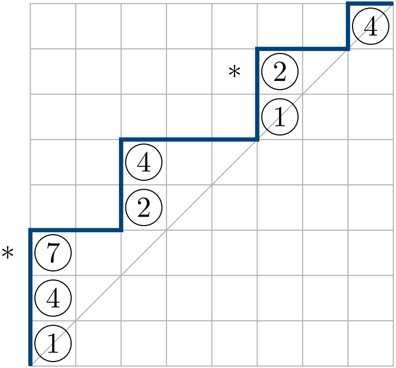

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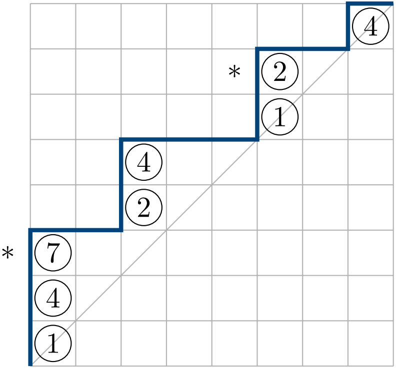

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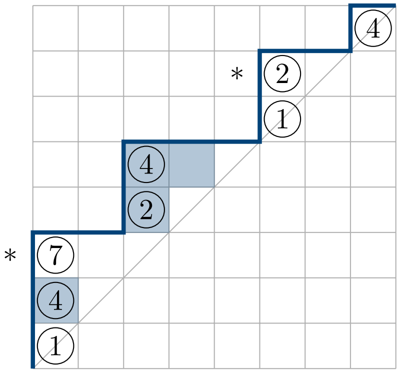

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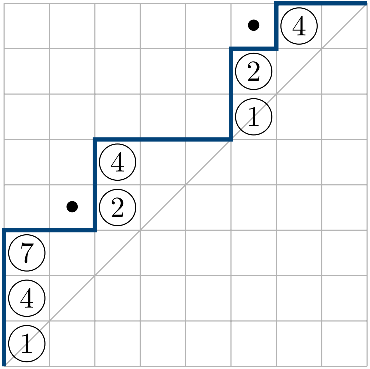

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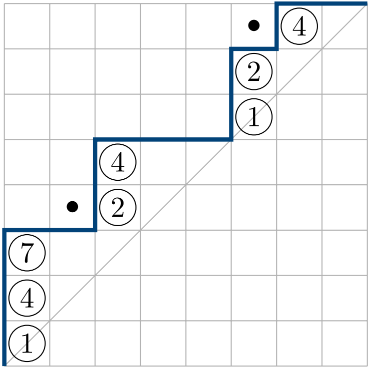

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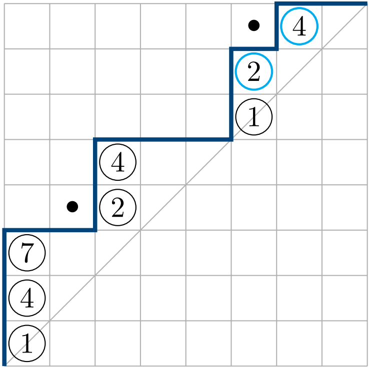

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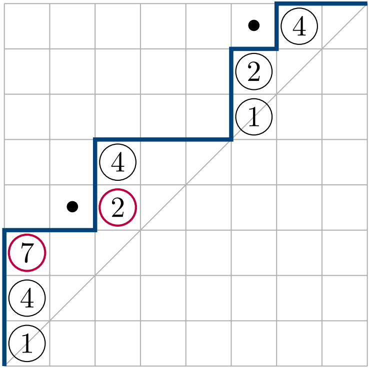

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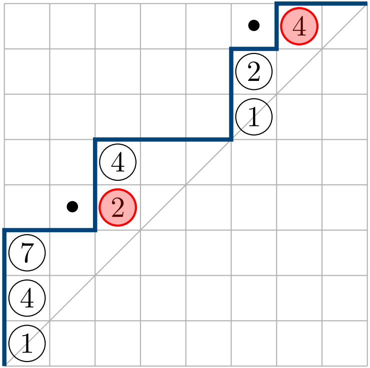

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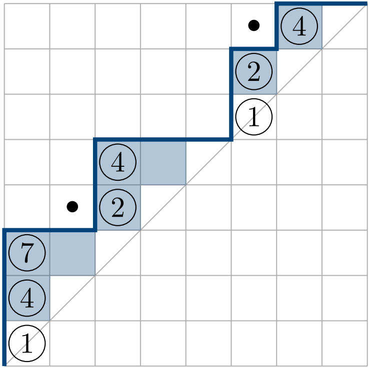

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Super Diagonal Coinvariants

Let $X_i = (x_1^{(i)}, \dots, x_n^{(i)})$ and $\Theta_j = (\theta_1^{(j)}, \dots, \theta_n^{(j)})$ be sets of n variables.

Let $\mathcal{A}_n^{(b,f)} = \mathbb{C}[X_1, \dots, X_b] \otimes \Lambda\{\Theta_1, \dots, \Theta_f\}$ be the tensor product of a symmetric algebra and an exterior algebra, endowed with an action of S_n given by diagonal permutation of the $b + f$ sets of variables.

The representation

$$\mathcal{DH}_n^{(b,f)} = \mathcal{A}_n^{(b,f)} \Big/ ((\mathcal{A}_n^{(b,f)})_+^{S_n})$$

is known as *super diagonal coinvariants*. As before, the action is multi-homogeneous so the representation is multigraded.

When $b = 2$ and $f = 0$, we get back the usual diagonal coinvariants. For other small values of b and f, we get results in the same fashion as the shuffle theorem (e.g. $(2, 1)$ gives the Delta conjecture).

State of the art

(b, f)	**Symmetric function**	**Combinatorics**

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Smirnov words

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$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}\big|_{t=0} = \sum_{w\in\mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)} x_w$$

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$x_w (= x_1^6 x_2^7 x_3^5 x_4^8)$ is the monomial associated to w.

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$$w = 4\,|\,4{\color{red}2}3\,|\,1{\color{red}2}14\,|\,43{\color{red}2}414\,|\,3\,|\,1{\color{red}2}3141{\color{red}2}\,|\,4\,|\,{\color{red}2}3{\color{red}2}$$

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$$w = \color{red}{4} \,|\, \color{red}{4}23 \,|\, 121\color{red}{4} \,|\, \color{red}{4}32\color{red}{4}1\color{red}{4} \,|\, 3 \,|\, 1231\color{red}{4}12 \,|\, \color{red}{4} \,|\, 232$$

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Main recurrence

We want to show that

$$\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}\big|_{t=0} = \sum_{w\in\mathsf{SW}(n,k,l)} q^{\mathsf{sminv}(w)} x_w$$

by proving that the recurrence relation

$$h_j^{\perp}\mathsf{SF}(n,k,l) = \sum_{r=0}^{j}\sum_{a=0}^{j}\sum_{i=0}^{j} \begin{bmatrix} n-k-l-(j-r-a)-1 \\ i \end{bmatrix}_q \times q^{\binom{a-i}{2}} \begin{bmatrix} n-k-l-(j-r-a+i) \\ a-i \end{bmatrix}_q \begin{bmatrix} n-k-l \\ j-r-a+i \end{bmatrix}_q \times q^{\binom{r-i}{2}} \begin{bmatrix} n-k-l-(j-r-a+i) \\ r-i \end{bmatrix}_q \mathsf{SF}(n-j,k-r,l-a)$$

with initial conditions $\mathsf{SF}(0,k,l) = \delta_{k,0}\delta_{l,0}$ and $\mathsf{SF}(n,k,l) = 0$ if $n < 0$, is satisfied by both.

Segmented permutations

The recurrence for $\Theta_{e_k}\Theta_{e_l}\nabla e_{n-k-l}|_{t=0}$ is a result by D'Adderio and Romero (2020). We proved the combinatorial one, and show here the case $j=1$, corresponding to segmented permutations.

Let $\mathsf{SP}(n,k,l)$ be the set of segmented permutations with k ascents and l descents, and let

$$\mathsf{SP}_q(n,k,l) = \sum_{\sigma \in \mathsf{SP}(n,k,l)} q^{\mathsf{sminv}(\sigma)}.$$

We have

$$\begin{aligned} \mathsf{SP}_q(n,k,l) = [n-k-l]_q \left(\mathsf{SP}_q(n-1,k,l) + \mathsf{SP}_q(n-1,k-1,l) \right. \\ \left. + \mathsf{SP}_q(n-1,k,l-1) + \mathsf{SP}_q(n-1,k-1,l-1)\right). \end{aligned}$$

with initial conditions $\mathsf{SP}_q(0,k,l) = \delta_{k,0}\delta_{l,0}$.

An example

We want to show that $\mathsf{SP}_q(9,3,2)$ is equal to

$$[4]_q(\mathsf{SP}_q(8,3,2) + \mathsf{SP}_q(8,2,2) + \mathsf{SP}_q(8,3,1) + \mathsf{SP}_q(8,2,1)).$$

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Let $\sigma \in \mathsf{SP}(9,3,2)$. The four summands corresponds to the possibilities for the maximal entry 9; the q-binomial counts the sminversions in which it is the middle entry of the $2-31$ pattern.

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If it is a singleton block, we remove it, together with its block separator.

$715|4|2386$

$715\|4\|2386\|9$	$715\|4\|9\|2386$	$715\|9\|4\|2386$	$9\|715\|4\|2386$
q^0	q^1	q^2	q^3

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If it is an ascent but not a descent, we remove it.

$$715|4|23|86$$

$715\|4\|23\|869$	$715\|4\|239\|86$	$715\|49\|23\|86$	$7159\|4\|23\|86$
q^0	q^1	q^2	q^3

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Let $\sigma \in \mathsf{SP}(9,3,2)$. The four summands corresponds to the possibilities for the maximal entry 9; the q-binomial counts the sminversions in which it is the middle entry of the $2-31$ pattern.

If it is a descent but not an ascent, we remove it.

$715|4|238|6$

$715\|4\|238\|96$	$715\|4\|9238\|6$	$715\|94\|238\|6$	$9715\|4\|238\|6$
q^0	q^1	q^2	q^3

An example

We want to show that $\mathsf{SP}_q(9,3,2)$ is equal to

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Let $\sigma \in \mathsf{SP}(9,3,2)$. The four summands corresponds to the possibilities for the maximal entry 9; the q-binomial counts the sminversions in which it is the middle entry of the $2-31$ pattern.

If it is both an ascent and a descent, we replace it with a block separator.

$$7|15|4|23|86$$

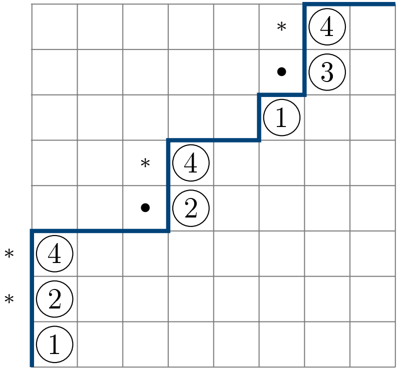
$7\|15\|4\|23986$	$7\|15\|4923\|86$	$7\|1594\|23\|86$	$7915\|4\|23\|86$
q^0	q^1	q^2	q^3

A unified Delta conjecture

There is a bijection

$$\phi\colon \mathsf{SW}(n,k,l) \leftrightarrow \{\pi \in \mathsf{LD}(n)^{*k,\bullet l} \mid \mathsf{area}(\pi) = 0\}$$

such that $\mathsf{sdinv}(w) = \mathsf{dinv}(\phi(w))$ when $k = 0$ or $l = 0$.

$12424|143 \longleftrightarrow$ 

References

Erik Carlsson and Anton Mellit.
A proof of the shuffle conjecture.
J. Amer. Math. Soc., 31(3):661–697, 2018.

Michele D'Adderio, Alessandro Iraci, and Anna Vanden Wyngaerd.
Theta operators, refined Delta conjectures, and coinvariants.
Advances in Mathematics, 376:107447, January 2021.

Michele D'Adderio and Anton Mellit.
A proof of the compositional Delta conjecture.
Advances in Mathematics, 402:108342, June 2022.

James Haglund, Jeffrey B. Remmel, and Andrew T. Wilson.
The Delta Conjecture.
Trans. Amer. Math. Soc., 370(6):4029–4057, 2018.

Alessandro Iraci, Philippe Nadeau, and Anna Vanden Wyngaerd.
Smirnov words and the Delta Conjectures.
arXiv e-prints, page arXiv:2312.03956, December 2023.

Bonus slides!

Bases of Λ

The bases of $\Lambda^{(n)}$ are indexed by $\lambda \vdash n$.

$$e_\lambda = \prod e_{\lambda_i}, \qquad e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \qquad \text{(elementary)}$$

$$h_\lambda = \prod h_{\lambda_i}, \qquad h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} \qquad \text{(homogeneous)}$$

$$p_\lambda = \prod p_{\lambda_i}, \qquad p_k = \sum_{i \geq 1} {x_i}^k \qquad \text{(power symmetric)}$$

$$m_\lambda = \sum_{i_1, \ldots, i_{\ell(\lambda)}} x_{i_1}^{\lambda_1} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}} \qquad \text{(monomial)}$$

Bases of Λ

The bases of $\Lambda^{(n)}$ are indexed by $\lambda \vdash n$.

$$
\begin{aligned}
e_{(2,1)} &= (x_1x_2 + x_1x_3 + x_2x_3 + \dots)(x_1 + x_2 + x_3 + \dots) \\
h_{(2,1)} &= ({x_1}^2 + x_1x_2 + {x_2}^2 + x_1x_3 + \dots)(x_1 + x_2 + x_3 + \dots) \\
p_{(2,1)} &= ({x_1}^2 + {x_2}^2 + {x_3}^2 + \dots)(x_1 + x_2 + x_3 + \dots) \\
m_{(2,1)} &= ({x_1}^2x_2 + x_1{x_2}^2 + {x_1}^2x_3 + {x_2}^2x_3 + \dots)
\end{aligned}
$$

The Schur functions

A *semi-standard Young tableau* of shape $\lambda \vdash n$ is a filling of the Ferrers diagram of λ with positive integer numbers that is weakly increasing along rows and strictly increasing along columns.

1	1	3	7	7
2	3	4	8	
3	7			

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1	1	3	7	7
2	3	4	8	
3	7			

Given a partition $\lambda \vdash n$, we define

$$s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} x^T$$

where $\mathsf{SSYT}(\lambda)$ is the set of semi-standard Young tableaux of shape λ, and x^T denote the products of the variables indexed by the entries of the tableau.

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1	1
2	

1	2
2	

1	2
3	

1	3
2	

2	2
3	

$\ldots$

Given a partition $\lambda \vdash n$, we define

$$s_{(2,1)} = {x_1}^2 x_2 + x_1 {x_2}^2 + 2x_1 x_2 x_3 + {x_2}^2 x_3 + \ldots$$

where $\mathsf{SSYT}(\lambda)$ is the set of semi-standard Young tableaux of shape λ, and x^T denote the products of the variables indexed by the entries of the tableau.

Plethystic notation

Let $A(q,t;x_1,x_2,\dots) \in \mathbb{Q}(q,t)((x_1,x_2,\dots))$, and let

$$f = \sum_\lambda f_\lambda(q,t) p_\lambda \in \Lambda$$

with $f_\lambda(q,t) \in \mathbb{Q}(q,t)$. The *plethystic evaluation* of f in A is

$$f[A] := \sum_\lambda f_\lambda(q,t) \prod_{i=1}^{\ell(\lambda)} A(q^{\lambda_i}, t^{\lambda_i}; {x_1}^{\lambda_i}, {x_2}^{\lambda_i}, \dots) \in \mathbb{Q}(q,t)((x_1,x_2,\dots)).$$

Equivalently, if A has an expression as sum of monomials (in q, t, x_i with coefficient 1), then $f[A]$ is the expression obtained from $f[X]$ by replacing the x_i's with such monomials, where $X = x_1 + x_2 + \dots$.

In this sense, we can interpret a sum of monomials as an alphabet, and a sum of expressions as concatenation of alphabets.

Macdonald polynomials

The (modified) Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$ are defined by the triangularity and normalization axioms

$$\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \geq \mu} a_{\lambda\mu}(q,t) s_\lambda[X]$$

$$\widetilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda \geq \mu'} b_{\lambda\mu}(q,t) s_\lambda[X]$$

$$\langle \widetilde{H}_\mu[X;q,t], s_{(n)}[X] \rangle = 1$$

for suitable coefficients $a_{\lambda\mu}(q,t)$, $b_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$. Here $\leq$ denotes the dominance order on partitions, and the square brackets denote the plethystic evaluation of symmetric functions.

The λ-ring structure

A λ-ring is a ring Λ with a collection of ring homomorphisms $p_n \colon \Lambda \to \Lambda$ satisfying

$$p_0[x] = 1, \qquad p_1[x] = x, \qquad p_m[p_n[x]] = p_{mn}[x]$$

for $m, n \in \mathbb{N}$ and $x \in \Lambda$.

In the case of symmetric functions, the homomorphisms are defined by

$$p_n[f(q,t;x_1,x_2,\dots)] = f(q^n,t^n;x_1^n,x_2^n,\dots),$$

which is also called the plethystic evaluation of p_n in f. This in fact extends to a more general operation which comes in extremely handy when dealing with symmetric functions.

The Hopf algebra structure

A Hopf algebra is a structure that is simultaneously an algebra and a co-algebra such that the structures are compatible, which is also equipped with an anti-automorphism, called antipode, satisfying certain relations.

In the case of symmetric functions, the coproduct is defined by

$$\Delta(f[X]) = f[X+Y] \in \Lambda[X] \otimes \Lambda[Y]$$

and the antipode map by $\omega(s_\lambda) = s_{\lambda'}$.

Note that, since Λ is commutative ω is actually a homomorphism; in fact, $\omega(e_n) = h_n$ and these generate Λ as an algebra. Moreover, since the Schur functions are orthonormal, it is also an isometry.

Delta and Theta operators

The Delta operators are two families of linear operators $\Delta_f, \Delta'_f \colon \Lambda \to \Lambda$ (for $f \in \Lambda$) that extend ∇. These operators are defined as

$$\Delta_f \widetilde{H}_\mu = f[B_\mu]\widetilde{H}_\mu, \qquad \Delta'_f \widetilde{H}_\mu = f[B_\mu - 1]\widetilde{H}_\mu.$$

In particular the Macdonald polynomials are eigenvectors for all these operators, and $\nabla|_{\Lambda^{(n)}} \equiv \Delta_{e_n}|_{\Lambda^{(n)}}$.

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In particular the Macdonald polynomials are eigenvectors for all these operators, and $\nabla|_{\Lambda^{(n)}} \equiv \Delta_{e_n}|_{\Lambda^{(n)}}$.

Theta operators are a family of linear operators $\Theta_f \colon \Lambda \to \Lambda$ (for $f \in \Lambda$), defined as

$$\Theta_f(g) = \mathbf{\Pi} f \left[\frac{X}{(1-q)(1-t)} \right] \mathbf{\Pi}^{-1} g,$$

where $\mathbf{\Pi} = \sum_{k \in \mathbb{N}} (-1)^k \Delta'_{e_k}$, and the square brackets denote plethysm.

The eigenvalues of ∇ and Δ_{e_k}

Let $\mu \vdash n$. We define $B_\mu(q,t) := \sum_{c\in\lambda} q^{a'(c)}t^{\ell'(c)}$, where a' and ℓ' denote the *coarm* and the *coleg* of a cell.

For $\mu = (5,4,2)$, we have the diagram

1	q	q^2	q^3	q^4
t	qt	q^2t	q^3t	
t^2	qt^2			

and taking the sum of the entries we get

$$B_\mu(q,t) = 1 + q + t + q^2 + qt + t^2 + q^3 + q^2t + qt^2 + q^4 + q^3t.$$

The plethystic evaluation of e_k in B_μ is the expression $e_k[B_\mu]$ given by the sum over all the choices of k different monomials, among the ones appearing in B_μ, of the product of the chosen monomials.

The bigraded Frobenius characteristic

Let $\mathcal{M}$ be a (x,y)-graded vector space, with a bi-homogeneous action of the symmetric group. Recall that irreducible representations of S_n are indexed by partitions of n, and denote by $\lambda(V)$ the partition indexing an irreducible S_n-module V.

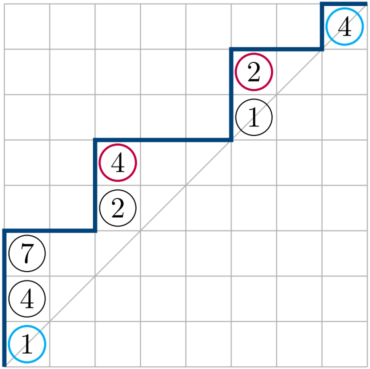
We define

$$\mathsf{Frob}_{q,t}(\mathcal{M}) := \sum_{\substack{V \subseteq \mathcal{M} \\ V \text{ irreducible}}} q^{\deg_x(V)} t^{\deg_y(V)} s_{\lambda(V)}$$

which is an element of the symmetric functions algebra Λ over $\mathbb{Q}(q,t)$.

Diagonal inversions

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^{\pi}$$



$\mathsf{dinv}(\pi)$ is the total number of **diagonal inversions**.

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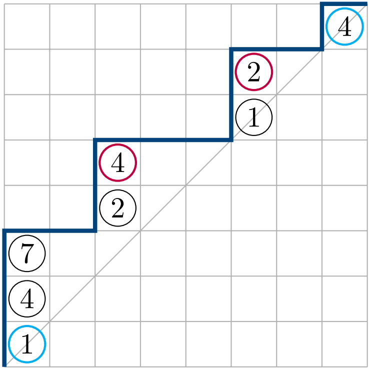

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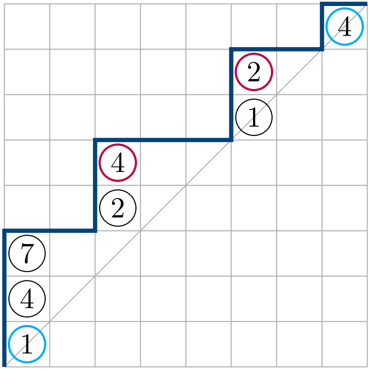

$\mathsf{dinv}(\pi)$ is the total number of **diagonal inversions**.

A **primary** diagonal inversion is a pair of labels in the same diagonal, such that the bottom-most one is smaller.

A **secondary** diagonal inversion is a pair of labels in two consecutive diagonals, such that the bottom-most one is greater and higher.