Séminaire Lotharingien de Combinatoire **92** (2025), Article B92a

COMBINATORICS OF THE IRREDUCIBLE COMPONENTS OF $\mathcal{H}_n^\Gamma$ IN TYPE D AND E

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ABSTRACT. We give a combinatorial model (in terms of symmetric cores) of the indexing set of the irreducible components of $\mathcal{H}_n^\Gamma$ (the Γ-fixed points of the Hilbert scheme of n points in the plane) containing a monomial ideal, whenever Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to the binary dihedral group. Moreover, we show that, if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to the binary tetrahedral group, to the binary octahedral group or to the binary icosahedral group, then the Γ-fixed points of $\mathcal{H}_n$ which are also fixed under the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$ are in fact $\mathrm{SL}_2(\mathbb{C})$-fixed points. Finally, we prove that in this case the irreducible components of $\mathcal{H}_n^\Gamma$ containing a monomial ideal are zero-dimensional.

1. INTRODUCTION

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and, for $n \in \mathbb{Z}_{\geq 0}$, let $\mathcal{H}_n$ be the Hilbert scheme of n points in $\mathbb{C}^2$. The natural action of Γ on $\mathbb{C}^2$ induces a Γ-action on $\mathbb{C}[x,y]$ and thus on $\mathcal{H}_n$. In this article, we are interested in the combinatorics of the parametrization set of the irreducible components of $\mathcal{H}_n^\Gamma$. When Γ is equal to the cyclic subgroup of the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$, a combinatorial model using partitions has already been constructed by Iain Gordon [Gor08, Lemma 7.8] and by Cédric Bonnafé and Ruslan Maksimau [BM21, Lemma 4.9]. We will therefore only consider the groups of type D and E. Type D corresponds to the class of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ that are isomorphic to the binary dihedral subgroups. In the second section, we introduce important notation concerning affine root systems and partitions of integers. In the third section, we then define the binary dihedral group and give its character table and its McKay graph. We then present a folding of that Dynkin diagram which will be of use in the subsequent section. In section four, we define and give the main properties of a generalization of the residue to type D. In the fifth section, we prove the first theorem, which can be stated as follows.

Theorem 1. *Let ℓ be an integer greater than or equal to 2, and let Γ be a binary dihedral subgroup of $\mathrm{SL}_2(\mathbb{C})$ of order 4ℓ. Then the set of all irreducible components of $\mathcal{H}_n^\Gamma$ containing a monomial ideal is in bijection with the set of all symmetric 2ℓ-cores λ such that $|\lambda| \equiv n \pmod{2\ell}$ and $|\lambda| \leq n$. Moreover, for all symmetric partitions μ_1, μ_2 of n, the monomial ideals attached to μ_1 and μ_2 are in the same irreducible component of $\mathcal{H}_n^\Gamma$ if and only if the 2ℓ-cores of μ_1 and μ_2 are equal.*

In section six, we start by giving a presentation of the binary tetrahedral group, its character table and its McKay graph. Moreover, we prove that, if Γ is isomorphic to the binary tetrahedral group, then the points in $\mathcal{H}_n$ that are fixed under Γ and the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$ are exactly the $\mathrm{SL}_2(\mathbb{C})$-fixed points. Since the binary octahedral group and the binary icosahedral group contain a subgroup isomorphic to the binary tetrahedral group, the previous result generalizes to these two isomorphism classes of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Finally, in section seven, we prove the following theorem.

Theorem 2. *If* Γ *is a finite subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *of type E, then, for* $I \in \mathcal{H}_n^{\mathrm{SL}_2(\mathbb{C})}$, *the irreducible component of* $\mathcal{H}_n^\Gamma$ *containing I is zero-dimensional.*

Acknowledgement. The author would like to thank Cédric Bonnafé for suggesting the study of the combinatorics of the irreducible components of the Γ-fixed points in the punctual Hilbert scheme of $\mathbb{C}^2$, when Γ is of type D and E, and the referee for valuable comments, suggestions and improvements.

2. STARTING POINT

Fix a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. In this subsection we recall the general description of the indexing set of the irreducible components of $\mathcal{H}_n^\Gamma$ in terms of roots that has been obtained in [Pae1]. Denote by I_Γ the set of all irreducible characters of Γ and let $\chi_0 \in I_\Gamma$ denote the trivial character of Γ. Let $\Delta_\Gamma^+ (\subset \Delta_\Gamma)$ be the free monoid (respectively free abelian group) associated with I_Γ. Let $\tilde{T}_\Gamma$ be the type of the McKay graph seen as an affine Dynkin diagram. One can then associate with Γ a realization (cf. [Kac, §1.1])

$$\left(\mathfrak{h}_\Gamma, \Pi_\Gamma := \{\alpha_\chi \mid \chi \in I_\Gamma\}, \Pi_\Gamma^\vee := \left\{\alpha_\chi^\vee \mid \chi \in I_\Gamma\right\}\right)$$

of the generalized Cartan matrix of type $\tilde{T}_\Gamma$. Denote by $Q(\tilde{T}_\Gamma)$ and $W(\tilde{T}_\Gamma)$ the root lattice and Weyl group, respectively, associated with the previously mentioned realization. From now on, we will identify $Q(\tilde{T}_\Gamma)$ with Δ_Γ. Let δ^Γ denote the null root.

For $d \in \Delta_\Gamma$, let $(d_\chi) \in \mathbb{Z}^{|I_\Gamma|}$ be such that $d = \sum_{\chi \in I_\Gamma} d_\chi \chi$. For $d \in \Delta_\Gamma^+$, let $|d|_\Gamma := \sum_{\chi \in I_\Gamma} d_\chi \delta_\chi^\Gamma \in \mathbb{Z}_{\geq 0}$. Finally, a new statistic on Δ_Γ has been defined in [Pae1, Definition 4.8]. The group $W(\tilde{T}_\Gamma)$ naturally acts by reflections on $\mathfrak{h}_\Gamma^*$. This action will be denoted by $*$. Define a new action of $W(\tilde{T}_\Gamma)$ on Δ_Γ denoted by . such that

$$\omega * (\Lambda_{\chi_0} - d) = \Lambda_{\chi_0} - \omega.d, \qquad \text{for all } (\omega, d) \in W(\tilde{T}_\Gamma) \times \Delta_\Gamma,$$

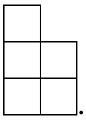
where Λ_{χ_0} is the fundamental weight associated with χ_0, the trivial character of Γ.

One can then prove that, for $d \in \Delta_\Gamma$, there exists a unique integer r such that d and $r\delta^\Gamma$ are in the $W(\tilde{T}_\Gamma)$-orbit for the . action, see [Pae1, Lemma 4.7]. We denote this integer r by $\mathrm{wt}(d)$.

Recall the result of [Pae1, Theorem 4.10], which will be our starting point. For a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ we have indexed the irreducible components of $\mathcal{H}_n^\Gamma$ by the set

$$\mathcal{A}_\Gamma^n := \left\{d \in \Delta_\Gamma^+ \mid |d|_\Gamma = n \text{ and } \mathrm{wt}(d) \geq 0\right\}.$$

Before diving into the type D study, let us introduce a bit more notation. A partition λ of n is a tuple $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ of integers, such that $|\lambda| := \sum_{i=1}^r \lambda_i$ is equal to n. Denote by $\mathcal{P}_n$ the set of all partitions of n and by $\mathcal{P}$ the set of all partitions of integers. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}$, denote by $\mathcal{Y}(\lambda) := \{(i,j) \in \mathbb{Z}_{\geq 0}^2 \mid i < \lambda_1, j < r\}$ its associated Young diagram. The conjugate partition of a partition λ of n, denoted by λ^*, is the partition associated with the reflection of $\mathcal{Y}(\lambda)$ in the diagonal (which is again a Young diagram of a partition of n). We will draw Young diagrams upright and the box that is lowest and furthest to the left will have index $(0,0)$. Let i and j denote the row and column indices, respectively. For example, consider $\lambda = (2,2,1)$. Its associated Young diagram is



In that case $\lambda^* = (3,2)$. A partition λ will be called symmetric if it is equal to its conjugate. We denote the set of all symmetric partitions by $\mathcal{P}^s$, and we define $\mathcal{P}_n^s := \mathcal{P}^s \cap \mathcal{P}_n$.

A hook of a partition λ in position $(i,j) \in \mathcal{Y}(\lambda)$, denoted by $H_{(i,j)}(\lambda)$, is

$$\{(a,b) \in \mathcal{Y}(\lambda) \mid (a = i \text{ and } b \geq j) \text{ or } (a > i \text{ and } b = j)\}.$$

Define the length of a hook $H_{(i,j)}(\lambda)$ to be its cardinality.

Definition 2.1. For a given integer $r \geq 1$, a partition λ is said to be an r-core if $\mathcal{Y}(\lambda)$ does not contain any hook of length r. The set of all r-cores is denoted by $\mathfrak{C}_r$, and we define $\mathfrak{C}_r^s := \mathfrak{C}_r \cap \mathcal{P}^s$.

3. FROM TYPE D TO TYPE C

Fix $\ell \geq 2$. Let μ_ℓ denote the cyclic subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by the diagonal matrix $\mathrm{diag}(\zeta_\ell, \zeta_\ell^{-1})$, where $\zeta_\ell = e^{\frac{2i\pi}{\ell}}$. We will work with the following model of the binary dihedral group in $\mathrm{SL}_2(\mathbb{C})$. Let $\mathrm{BD}_{2\ell} := \langle \omega_{2\ell}, s\rangle$, where

$$\omega_{2\ell} := \begin{pmatrix} \zeta_{2\ell} & 0 \\ 0 & \zeta_{2\ell}^{-1} \end{pmatrix}, \qquad s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group $\mathrm{BD}_{2\ell}$ is of order 4ℓ. Note that BD_4 is isomorphic to the quaternion group (cf. [CM13, §1.7]). Let $\tau_{2\ell}$ be the character of $\mu_{2\ell}$ that maps $\omega_{2\ell}$ to $\zeta_{2\ell}$. For $i \in \mathbb{Z}$, let

$$\chi_i := \mathrm{Ind}_{\mu_{2\ell}}^{\mathrm{BD}_{2\ell}}\left(\tau_{2\ell}^i\right).$$

Note that χ_i is irreducible if and only if i is not congruent to 0 or ℓ modulo 2ℓ. If ℓ is even, the character table of $\mathrm{BD}_{2\ell}$ is

cardinality	1	1	2	ℓ	ℓ
classes	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	${\omega_{2\ell}}^p (0 < p < \ell)$	s	$s\omega_{2\ell}$
χ_{0^+}	1	1	1	1	1
χ_{0^-}	1	1	1	-1	-1
χ_{ℓ^+}	1	1	$(-1)^p$	-1	1
χ_{ℓ^-}	1	1	$(-1)^p$	1	-1
χ_k $(0 < k < \ell)$	2	$(-1)^k 2$	$2\cos\left(\frac{kp\pi}{\ell}\right)$	0	0

and if ℓ is odd, the character table of $\mathrm{BD}_{2\ell}$ is

cardinality	1	1	2	ℓ	ℓ
classes	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	${\omega_{2\ell}}^p (0 < p < \ell)$	s	$s\omega_{2\ell}$
χ_{0^+}	1	1	1	1	1
χ_{0^-}	1	1	1	-1	-1
χ_{ℓ^+}	1	-1	$(-1)^p$	ζ_4	$-\zeta_4$
χ_{ℓ^-}	1	-1	$(-1)^p$	$-\zeta_4$	ζ_4
χ_k $(0 < k < \ell)$	2	$(-1)^k 2$	$2\cos\left(\frac{kp\pi}{\ell}\right)$	0	0

.

The McKay graph of $\mathrm{BD}_{2\ell}$ is a Dynkin diagram of affine type $\tilde{D}_{\ell+2}$

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The irreducible characters of $\mathrm{BD}_{2\ell}$ are labeled by their index in the McKay graph.

We want to give a combinatorial description of $\mathcal{A}^n_{\mathrm{BD}_{2\ell}}$. Let $\mathbb{T}_1$ denote the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$. In what follows, using symmetric partitions, we will give a combinatorial description of the irreducible components of $\mathcal{H}_n^{\mathrm{BD}_{2\ell}}$ containing a monomial ideal. To do so, restrict $\mathcal{A}^n_{\mathrm{BD}_{2\ell}}$ to the irreducible components of $\mathcal{H}_n^{\mathrm{BD}_{2\ell}}$ containing a $\mathbb{T}_1$-fixed point. Let us denote this subset of $\mathcal{A}^n_{\mathrm{BD}_{2\ell}}$ by $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$. Note also that, in this context, the coefficients of the null root in the basis of simple roots are

$$\delta^{\mathrm{BD}_{2\ell}}_{\chi_i} := \begin{cases} 1, & \text{if } i = 0^+,\ 0^-,\ \ell^+,\ \ell^-, \\ 2, & \text{otherwise.} \end{cases}$$

The central object of study will be the affine root lattice of type $\tilde{D}_{\ell+2}$ (which is the same object as the coroot lattice of type $\tilde{D}_{\ell+2}$ since it is a simply laced type) $Q(\tilde{D}_{\ell+2}) \subset \mathfrak{h}^*_{\mathrm{BD}_{2\ell}}$. Let $\tau_\ell := \alpha_{\chi_{0^-}} + \alpha_{\chi_{\ell^+}} + \alpha_{\chi_{\ell^-}} + \sum_{i=1}^{\ell-1} 2\alpha_{\chi_i}$ be the highest root of the finite root system of type $D_{\ell+2}$.

Definition 3.1. Define a bijection from the set $I_{\mathrm{BD}_{2\ell}}$ to itself by

$$\sigma_{0^-} : \begin{array}{ccc} I_{\mathrm{BD}_{2\ell}} & \to & I_{\mathrm{BD}_{2\ell}}, \\ \chi & \mapsto & \chi_{0^-}\cdot\chi, \end{array}$$

and define also an automorphism of the Dynkin diagram of type $\tilde{D}_{\ell+2}$ by

$$\sigma : \begin{array}{ccc} \Pi_{\mathrm{BD}_{2\ell}} & \to & \Pi_{\mathrm{BD}_{2\ell}}, \\ \alpha_\chi & \mapsto & \alpha_{\sigma_{0^-}(\chi)}. \end{array}$$

This automorphism swaps the first two vertices (the ones with labels 0^+ and 0^-) and the last two (with the labels ℓ^+ and ℓ^-) and fixes all the others.

We can apply Stembridge's construction [Stem] to the root system of type $\tilde{D}_{\ell+2}$ and to the automorphism σ. Denote the simple roots by $(\beta_i)_{i\in[\![0,\ell]\!]}$ and the simple coroots associated with the root system $\Phi(\tilde{D}^\sigma_{\ell+2})$ by $(\beta_i^\vee)_{i\in[\![0,\ell]\!]}$. By construction, we have

- $\beta_0 = \alpha_{\chi_{0^+}} + \alpha_{\chi_{0^-}}$,
- for all $i \in [\![1,\ell-1]\!]$, $\beta_i = \alpha_{\chi_i}$,
- $\beta_\ell = \alpha_{\chi_{\ell^+}} + \alpha_{\chi_{\ell^-}}$,
- $\beta_0^\vee = \frac{\alpha^\vee_{\chi_{0^+}} + \alpha^\vee_{\chi_{0^-}}}{2}$,
- for all $i \in [\![1,\ell-1]\!]$, $\beta_i^\vee = \alpha^\vee_{\chi_i}$,
- $\beta_\ell^\vee = \frac{\alpha^\vee_{\chi_{\ell^+}} + \alpha^\vee_{\chi_{\ell^-}}}{2}$.

Let $A = (a_{ij})$ be a generalized Cartan matrix. Recall that, if two vertices (i,j) are connected by more than one edge in the associated Dynkin diagram, then these edges are equipped with an arrow pointing towards i if $|a_{ij}| > 1$. With these conventions, the root system $\Phi(\tilde{D}^\sigma_{\ell+2})$ has the following Dynkin diagram:

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Proposition 3.2. *The set $\Phi(\tilde{D}^\sigma_{\ell+2})$ is a crystallographic root system of type $\tilde{C}_\ell$.*

Definition 3.3. Let $Q(\tilde{D}^\sigma_{\ell+2})[0^+] := Q(\tilde{D}^\sigma_{\ell+2}) \sqcup \left(Q(\tilde{D}^\sigma_{\ell+2}) + \alpha_{\chi_{0^+}}\right) \subset Q(\tilde{D}_{\ell+2})$. Written more explicitly,

$$Q(\tilde{D}^\sigma_{\ell+2})[0^+] = \left\{ \sum_{\chi\in I_{\mathrm{BD}_{2\ell}}} a_\chi \alpha_\chi \in Q(\tilde{D}_{\ell+2}) \;\middle|\; 0 \le a_{\chi_0^+} - a_{\chi_0^-} \le 1 \text{ and } a_{\chi_{\ell^+}} = a_{\chi_{\ell^-}} \right\}.$$

Definition 3.4. Define the map

$$\mathcal{T} : Q(\tilde{D}^\sigma_{\ell+2})[0^+] \to Q^\vee(\tilde{D}^\sigma_{\ell+2}) = Q^\vee(\tilde{C}_\ell)$$

given by

$$\sum_{i=0}^{\ell} a_i\beta_i + q\alpha_{\chi_{0^+}} \mapsto (2a_0+q)\beta_0^\vee + \sum_{i=1}^{\ell-1} a_i\beta_i^\vee + 2a_\ell\beta_\ell^\vee,$$

with $q \in \{0,1\}$.

In type $\tilde{C}_\ell$, the null root is $\delta(\tilde{C}_\ell) := \beta_0 + \sum_{i=1}^{\ell-1} 2\beta_i + \beta_\ell \in Q(\tilde{C}_\ell)$, and the null coroot is $\delta^\vee(\tilde{C}_\ell) := \sum_{i=0}^{\ell}\beta_i^\vee$. For $\chi \in I_{\mathrm{BD}_{2\ell}}$, let $s_\chi \in W(\tilde{D}_{\ell+2})$ denote the simple reflection associated with α_χ.

Definition 3.5. For $\chi \in I_{\mathrm{BD}_{2\ell}}$, let $\sigma.s_\chi := s_{\sigma_{0^-}(\chi)}$ and extend this action to $W(\tilde{D}_{\ell+2})$, the Weyl group of type $\tilde{D}_{\ell+2}$. Let $W(\tilde{D}_{\ell+2})^\sigma := \{\omega \in W(\tilde{D}_{\ell+2}) \mid \sigma.\omega = \omega\}$, which is a subgroup of $W(\tilde{D}_{\ell+2})$.

Remark 3.6. The set $\{s_0 := s_{\chi_{0^+}}s_{\chi_{0^-}}, s_1 := s_{\chi_1}, \dots, s_{\ell-1} := s_{\chi_{\ell-1}}, s_\ell := s_{\chi_{\ell^+}}s_{\chi_{\ell^-}}\}$ is a set of generators of $W(\tilde{D}_{\ell+2})^\sigma$. Applying [Stem, Claim 3] to our situation, we obtain a group isomorphism from $W(\tilde{D}^\sigma_{\ell+2})$ to $W(\tilde{D}_{\ell+2})^\sigma$. Let us, from now on, identify these two groups and refer to them as $W(\tilde{C}_\ell)$. This group acts naturally by reflections on $Q(\tilde{D}^\sigma_{\ell+2})[0^+]$ and $Q^\vee(\tilde{C}_\ell)$. Denote this action by $*$.

Definition 3.7. Define a $W(\tilde{C}_\ell)$-action on $Q(\tilde{D}^\sigma_{\ell+2})[0^+]$ by

$$s_i \blacklozenge \alpha := s_i * \alpha + \delta_i^0\alpha_{\chi_{0^+}}, \qquad \text{for all } i \in [\![0,\ell]\!] \text{ and } \alpha \in Q(\tilde{D}^\sigma_{\ell+2})[0^+].$$

Similarly, define a $W(\tilde{C}_\ell)$-action on the coroot lattice $Q^\vee(\tilde{D}^\sigma_{\ell+2})$ by

$$s_i \blacklozenge \beta^\vee := s_i * \beta^\vee + \delta_i^0\beta_0^\vee, \qquad \text{for all } i \in [\![0,\ell]\!] \text{ and } \beta^\vee \in Q^\vee(\tilde{C}_\ell).$$

A simple computation shows the equivariance of $\mathcal{T}$ with respect to the former defined actions.

Proposition 3.8. *The map $\mathcal{T}$ is $W(\tilde{C}_\ell)$-equivariant.*

Remark 3.9. Note also that $\mathcal{T}$ preserves sizes,

$$|\alpha|_{\tilde{D}_{\ell+2}} = |\mathcal{T}(\alpha)|_{\tilde{C}_\ell}, \quad \text{for all } \alpha \in Q(\tilde{D}^\sigma_{\ell+2})[0^+].$$

Let G be an abstract group acting on a set X. For $x \in X$, we denote the orbit of x under the action of G by $\overline{x}^G$. The following lemma will be used later on when proving the first theorem.

Lemma 3.10. *If $\beta^\vee \in \overline{0}^{W(\tilde{C}_\ell)} \subset Q^\vee(\tilde{C}_\ell)$ and $k \in \mathbb{Z}$, then $(\beta^\vee + k\delta^\vee(\tilde{C}_\ell)) \in \overline{k\delta^\vee(\tilde{C}_\ell)}^{W(\tilde{C}_\ell)}$.*

Proof. It is enough to check this on the set of generators $\{s_i \mid i \in [\![0,\ell]\!]\}$. If $i \in [\![1,\ell]\!]$, the action is by reflections. It is then linear and s_i stabilizes $k\delta^\vee(\tilde{C}_\ell)$. For $i = 0$, we can combine the fact

$$s_0 \blacklozenge (\beta_1^\vee + \beta_2^\vee) = s_0 \blacklozenge \beta_1^\vee + s_0 \blacklozenge \beta_2^\vee - \beta_0^\vee, \qquad \text{for all } \beta_1^\vee, \beta_2^\vee \in Q^\vee(\tilde{C}_\ell)$$

with the fact that $s_0 \blacklozenge \delta^\vee(\tilde{C}_\ell) = \delta^\vee(\tilde{C}_\ell) + \beta_0^\vee$ to conclude that

$$s_0 \blacklozenge (\beta^\vee + k\delta^\vee(\tilde{C}_\ell)) = s_0 \blacklozenge \beta^\vee + k\delta^\vee(\tilde{C}_\ell).$$

□

Finally, let us say a few words about the dual root system of $\Phi(\tilde{D}^{\sigma}_{\ell+2})$. It can be obtained as a folding of type A. This will simplify proofs in the next section. Recall that $\mu_{2\ell}$ denotes the cyclic subgroup of order 2ℓ contained in the maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$ and that $\tau_{2\ell}$ denotes the irreducible character of $\mu_{2\ell}$ mapping the generator $\omega_{2\ell}$ to $\zeta_{2\ell}$. The McKay graph of $\mu_{2\ell}$ is a Dynkin diagram of affine type $\tilde{A}_{2\ell}$ with 2ℓ vertices (since $\mu_{2\ell}$ is abelian). Consider the automorphism of the Dynkin diagram of type $\tilde{A}_{2\ell}$ given by

$$\varsigma : \begin{array}{ccc} \Pi_{\mu_{2\ell}} & \to & \Pi_{\mu_{2\ell}}, \\ \alpha_{\tau^i} & \mapsto & \alpha_{\tau^{-i}}. \end{array}$$

It fixes α_{τ^0} and α_{τ^ℓ}. Applying Stembridge's construction to $(\tilde{A}_{2\ell}, \varsigma)$ and identifying it with the dual root system (cf. [Kac, §3.1]) of $\Phi(\tilde{D}^{\sigma}_{\ell+2})$, we obtain the following result.

Proposition 3.11. *The set* $\Phi(\tilde{A}^{\varsigma}_{2\ell})$ *is the dual root system of* $\Phi(\tilde{D}^{\sigma}_{\ell+2})$.

4. $\mathrm{BD}_{2\ell}$-RESIDUE

The $\mathbb{T}_1$-fixed points in $\mathcal{H}_n$ are the ideals I_λ generated by $\{x^i y^j \mid (i,j) \in \mathbb{N}^2 \setminus \mathcal{Y}(\lambda)\}$ for λ a partition of n. These ideals are called monomial ideals. Among these ideals, the ideals fixed by $s \in \mathrm{BD}_{2\ell}$ are exactly the monomial ideals parametrized by symmetric partitions of n. This implies that $\mathbb{C}[x,y]/I_\lambda$ is a $\mathrm{BD}_{2\ell}$-module whenever λ is symmetric. In this section, our goal is to generalize the residue "of type A", i.e., the usual residue of partitions, to a residue of type D. Recall that we identify the root lattice constructed out of Γ with the Grothendieck ring of Γ. The property from the residue that we want to generalize is that the residue of a partition λ is equal to the character of the representation $\mathbb{C}[x,y]/I_\lambda$. Thus, we want to construct a map Res_D from $\mathcal{P}^s_n$ to $Q(\tilde{D}_{\ell+2})$. To do so, let us first define the functions $d_k : \mathcal{P}^s_n \to \mathbb{Z}_{\geq 0}$, for $k \in [\![0, \ell]\!]$.

Let $\mathcal{Y}(\lambda)_k := \{(i,j) \in \mathcal{Y}(\lambda) \mid i - j \equiv k \pmod{2\ell}\}$ for $k \in [\![0, 2\ell-1]\!]$.

Definition 4.1. For $k \in [\![1, \ell]\!]$ define $d_k(\lambda) := \#\big(\mathcal{Y}(\lambda)_k \cup \mathcal{Y}(\lambda)_{2\ell-k}\big)$. For $k = 0$, we put $\tilde{d}_0(\lambda) := \#\{(i,j) \in \mathcal{Y}(\lambda) \mid i = j\}$ and $d_0(\lambda) := \#\mathcal{Y}(\lambda)_0 - \tilde{d}_0(\lambda)$.

Furthermore, we introduce the notations

- $d'_0(\lambda) := \frac{d_0(\lambda)}{2} + \tilde{d}_0(\lambda) - \left\lfloor \frac{\tilde{d}_0(\lambda)}{2} \right\rfloor$,
- $d''_0(\lambda) := \frac{d_0(\lambda)}{2} + \left\lfloor \frac{\tilde{d}_0(\lambda)}{2} \right\rfloor$.

We are now able to define the residue in type D.

Definition 4.2. The residue of type D is

$$\mathrm{Res}_D : \begin{array}{ccc} \mathcal{P}^s_n & \to & Q(\tilde{D}_{\ell+2}), \\ \lambda & \mapsto & d'_0(\lambda)\alpha_{\chi_{0^+}} + d''_0(\lambda)\alpha_{\chi_{0^-}} + \sum_{i=1}^{\ell-1} \frac{d_i(\lambda)}{2}\alpha_{\chi_i} + \frac{d_\ell(\lambda)}{2}(\alpha_{\chi_{\ell^+}} + \alpha_{\chi_{\ell^-}}). \end{array}$$

Remark 4.3. Using the fact that the partition is symmetric, it is easy to see that the image of Res_D is indeed in the $\mathbb{Z}$-span of the roots $\{\alpha_\chi \mid \chi \in I_{\mathrm{BD}_{2\ell}}\}$. Note moreover, that

$$\text{for all } \lambda \in \mathcal{P}^s_n,\ |\mathrm{Res}_D(\lambda)|_{\tilde{D}_{\ell+2}} = |\lambda| = n.$$

Example 4.4. Take $\ell = 2$ and consider $\lambda = (4,4,3,2)$, which is symmetric and has Young diagram

1	2		
2	1	0^+	
1	0^-	1	2
0^+	1	2	1

.

Hence, $\mathrm{Res}_D(\lambda) = 2\alpha_{\chi_{0+}} + \alpha_{\chi_{0-}} + 3\alpha_{\chi_1} + 2\alpha_{\chi_{2+}} + 2\alpha_{\chi_{2-}}$.

Proposition 4.5. *For $\lambda \in \mathcal{P}_n^s$, $\mathrm{Res}_D(\lambda)$ is the character of the $\mathrm{BD}_{2\ell}$-representation $\mathbb{C}[x,y]/I_\lambda$.*

Proof. Consider a basis $(\overline{x^iy^j})_{(i,j)\in\mathcal{Y}(\lambda)}$ of the representation $\mathbb{C}[x,y]/I_\lambda$. Since λ is symmetric, restrict the attention to $\mathcal{Y}^-(\lambda) := \{(i,j) \in \mathcal{Y}(\lambda) \mid i > j\}$ and to the diagonal $\{(i,j) \in \mathcal{Y}(\lambda) \mid i = j\}$. Take first $(i,j) \in \mathcal{Y}^-(\lambda)$ and consider $V_{(i,j)} = \mathrm{Vect}(\overline{x^iy^j}, \overline{x^jy^i})$, a subspace of $\mathbb{C}[x,y]/I_\lambda$. Let k be an element of $[\![1, \ell-1]\!]$. For $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i - j \equiv k \pmod{2\ell}$, we have $V_{(i,j)} \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_k}$ (recall that X_{χ_k} is an irreducible representation of $\mathrm{BD}_{2\ell}$ with character equal to χ_k). Moreover when $i - j \equiv 2\ell - k \pmod{2\ell}$, we have $V_{(i,j)} \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_k}$. If $k = \ell$, then, for each pair $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i - j \equiv \ell \pmod{2\ell}$, we have $V_{(i,j)} \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_{\ell+}} \oplus X_{\chi_{\ell-}}$. In the same way, if $(i,j) \in \mathcal{Y}^-(\lambda)$ such that $i \equiv j \pmod{2\ell}$, we have $V_{(i,j)} \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_{0+}} \oplus X_{\chi_{0-}}$. It remains to understand the action of $\mathrm{BD}_{2\ell}$ on the diagonal. For $i \in \mathbb{Z}_{\geq 0}$, we have $\omega_{2\ell}.\overline{x^iy^i} = \overline{x^iy^i}$ and $s.\overline{x^iy^i} = (-1)^i\,\overline{x^iy^i}$. These two computations show that, if $i \equiv 0 \pmod 2$, then $V_i := V_{(i,i)} \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_{0+}}$ and that, if $i \equiv 1 \pmod 2$, then $V_i \simeq_{\mathrm{BD}_{2\ell}} X_{\chi_{0-}}$. To sum it all up, the character of $\mathbb{C}[x,y]/I_\lambda$ is $\mathrm{Res}_D(\lambda)$. □

By construction Res_D factors through $Q(\tilde{D}^{\sigma}_{\ell+2})[0^+]$. For $a, b \in \mathbb{Z}$, let $\mathrm{rem}(a,b) \in [\![0, b-1]\!]$ denote the remainder of the Euclidean division of a by b. By the work of Christopher R. H. Hanusa and Brant C. Jones [HJ12, Theoreom 5.8], we can endow the set $\mathfrak{C}^s_{2\ell}$ of symmetric 2ℓ-cores with a $W(\tilde{C}_\ell)$-action. Let us quickly recall how this action is constructed.

Definition 4.6. For a symmetric 2ℓ-core λ define the C-residue of a box positioned in row i and column j in the Young diagram of λ as

$$\begin{cases} \mathrm{rem}(j-i, 2\ell), & \text{if } 0 \leq \mathrm{rem}(j-i,2\ell) \leq \ell, \\ 2\ell - \mathrm{rem}(j-i,2\ell), & \text{if } l < \mathrm{rem}(j-i,2\ell) < 2\ell. \end{cases}$$

Example 4.7. Take $\ell = 2$ and the same symmetric 4-core $(4,4,3,2)$. The Young diagram filled with the C-residue of each box is

1	2		
2	1	0	
1	0	1	2
0	1	2	1

Remark 4.8. Note that for each symmetric 2ℓ-core λ, the C-residue of each box of λ is always an integer between 0 and ℓ.

Definition 4.9. The action of $W(\tilde{C}_\ell)$ on $\mathfrak{C}^s_{2\ell}$ is defined on generators. Take $s_i \in W(\tilde{C}_\ell)$ and $\lambda \in \mathfrak{C}^s_{2\ell}$. There are three disjoint cases. Either we can add boxes with C-residue i, or we can remove such boxes, or there are no such boxes. Define $s_i.\lambda$ as the partition obtained from λ in either adding all boxes of $\mathcal{Y}(\lambda)$ with C-residue i so that $s_i.\lambda$ remains a partition or removing all boxes of $\mathcal{Y}(\lambda)$ with C-residue i so that $s_i.\lambda$ remains a partition.

Definition 4.10. The C-region of index $k \in \mathbb{Z}$ of a symmetric 2ℓ-core is the following subset of $\mathcal{Y}(\lambda)$:

$$\mathcal{R}_k := \{(i,j) \in \mathcal{Y}(\lambda) \mid (i-j) \in \{2k\ell, \ldots, 2(k+1)\ell - 1\}\}.$$

More generally, we can define a shifted C-region. Let $(k,h) \in \mathbb{Z}^2$ and define the h-shifted C-region of index k by

$$\mathcal{R}_{k,h} := \{(i,j) \in \mathcal{Y}(\lambda) \mid (i-j) \in \{2k\ell + h, \dots, 2(k+1)l - 1 + h\}\}.$$

Proposition 4.11. $\mathrm{Res}_D : \mathfrak{C}^s_{2\ell} \to Q(\tilde{D}^\sigma_{\ell+2})[0^+]$ *is* $W(\tilde{C}_\ell)$*-equivariant.*

Proof. By Proposition 3.11, we have $Q(\tilde{D}^\sigma_{\ell+2})[0^+] \subset Q^\vee(\tilde{A}_{2\ell})$. Moreover, the type $\tilde{A}_{2\ell}$ is simply laced. We can thus identify $Q^\vee(\tilde{A}_{2\ell})$ with $Q(\tilde{A}_{2\ell})$. Using Proposition 3.11, we can also identify $W(\tilde{A}^\varsigma_{2\ell})$ with $W(\tilde{C}_\ell)$. Now, by [BJV09, Proposition 3.2.5], the usual residue map, $\mathrm{Res} : \mathfrak{C}_{2\ell} \to Q(\tilde{A}_{2\ell})$, is $W(\tilde{A}_{2\ell})$-equivariant. Finally, the restriction of this map to $\mathfrak{C}^s_{2\ell}$ gives $\mathrm{Res}_D : \mathfrak{C}^s_{2\ell} \to Q(\tilde{D}^\sigma_{\ell+2})[0^+]$. Indeed, using Proposition 4.5 and the definition of the irreducible characters of $\mathrm{BD}_{2\ell}$ we see that it is already true that Res_D is the restriction of the usual residue map to symmetric partitions. We can thus conclude that $\mathrm{Res}_D : \mathfrak{C}^s_{2\ell} \to Q(\tilde{D}^\sigma_{\ell+2})[0^+]$ is $W(\tilde{A}^\varsigma_{2\ell})$-equivariant. ☐

Proposition 4.12. $\mathcal{T} \circ \mathrm{Res}_D : \mathfrak{C}^s_{2\ell} \to \overline{0}^{W(\tilde{C}_\ell)} \subset Q^\vee(\tilde{C}_\ell)$ *is a bijection.*

Proof. By definition, we have $\mathcal{T}(\mathrm{Res}_D(\emptyset)) = 0$ and the stabilizer of $\emptyset \in \mathfrak{C}^s_{2\ell}$ in $W(\tilde{C}_\ell)$ is equal to $W(C_\ell)$, the Weyl group of the finite type C_ℓ, which is equal to the stabilizer of $0 \in \overline{0}^{W(\tilde{C}_\ell)}$ in $W(\tilde{C}_\ell)$. Moreover, using Propositions 3.8 and 4.11, we know that $\mathcal{T} \circ \mathrm{Res}_D$ is $W(\tilde{C}_\ell)$-equivariant. To conclude, it is enough to show that the $W(\tilde{C}_\ell)$-action defined on $\mathfrak{C}^s_{2\ell}$ (Definition 4.9) is transitive. This has been proven in [HJ12, Proposition 6.2]. ☐

Remark 4.13. Note that Proposition 4.12 can also be deduced from [BM21, Proposition 4.4] and Proposition 3.11.

Proposition 4.14. *The composition of maps*

$$\varphi : \mathfrak{C}^s_{2\ell} \xrightarrow{\mathcal{T}\circ\mathrm{Res}_D} Q^\vee(\tilde{C}_\ell) \overset{\pi}{\twoheadrightarrow} Q^\vee(\tilde{C}_\ell)/\mathbb{Z}\delta^\vee(\tilde{C}_\ell)$$

is a bijection.

Proof. Consider the bijection $\overline{0}^{W(\tilde{C}_\ell)} \xrightarrow{\sim} Q^\vee(C_\ell)$ which is the composition of the two bijections

$$\overline{0}^{W(\tilde{C}_\ell)} \xrightarrow{\sim} W(\tilde{C}_\ell)/W(C_\ell) \xrightarrow{\sim} Q^\vee(C_\ell).$$

The second bijection boils down to the choice of a representative with coordinate 0 along $\beta_0^\vee$. Moreover, consider the bijection

$$\begin{array}{ccc} Q^\vee(\tilde{C}_\ell)/\mathbb{Z}\delta^\vee(\tilde{C}_\ell) & \xrightarrow{\sim} & Q^\vee(C_\ell), \\ \beta^\vee & \mapsto & \beta^\vee - \beta_0^\vee\delta^\vee(\tilde{C}_\ell). \end{array}$$

We then have the following commutative diagram:

$$\begin{tikzcd}
\overline{0}^{W(\tilde{C}_\ell)} \arrow[rr, two heads, "\pi"] \arrow[dr, "\sim"'] & & Q^\vee(\tilde{C}_\ell)/\mathbb{Z}\delta^\vee(\tilde{C}_\ell) \arrow[dl, "\sim"] \\
& Q^\vee(C_\ell) &
\end{tikzcd}.$$

From there, we can use Proposition 4.12 to prove that φ is a bijection. ☐

5. Combinatorial description in type D

We now have everything needed to give a combinatorial description of the set $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$. Note that, by Proposition 4.5, $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}} \subset \mathcal{A}^{n}_{\mathrm{BD}_{2\ell}} \cap Q(\tilde{D}^\sigma_{\ell+2})[0^+]$.

Consider the map

$$\begin{array}{rcl} \epsilon : Q(\tilde{D}^\sigma_{\ell+2})[0^+] & \to & \mathfrak{C}^s_{2\ell}, \\ d & \mapsto & (\varphi^{-1} \circ \pi \circ \mathcal{T})(d). \end{array}$$

Theorem 5.1. *The map ϵ defines a bijection between $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$ and the symmetric 2ℓ-cores λ, with the property that $|\lambda| \equiv n \pmod{2\ell}$ and $|\lambda| \leq n$. Moreover, for $\mu_1, \mu_2 \in \mathcal{P}^s_n$, I_{μ_1} and I_{μ_2} are in the same irreducible component of $\mathcal{H}^{\mathrm{BD}_{2\ell}}_n$ if and only if the 2ℓ-cores of μ_1 and μ_2 are equal.*

Proof. We show first that, if $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$, then $|\epsilon(d)| \equiv n \pmod{2\ell}$. Put $\lambda := \epsilon(d)$. Then

$$\text{there exists } k \in \mathbb{Z} \text{ such that } \mathcal{T}(d) = \mathcal{T}(\mathrm{Res}_D(\lambda)) + k\delta^\vee(\tilde{C}_\ell).$$

In particular $|\mathcal{T}(d)|_{\tilde{C}_\ell} = |\mathcal{T}(\mathrm{Res}_D(\lambda))|_{\tilde{C}_\ell} + 2k\ell$. Now, since $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$, we have $|d|_{\tilde{D}_{\ell+2}} = n$. Furthermore, using Remark 3.9, we have $n = |\lambda| + 2k\ell$. Moreover, we claim that if $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$, then $|\epsilon(d)| \leq n$. Indeed, by Lemma 3.10 and the fact that $\mathcal{T}(\mathrm{Res}_D(\lambda)) \in \overline{0}^{W(\tilde{C}_\ell)}$, we have $\mathcal{T}(d) \in \overline{k\delta^\vee(\tilde{C}_\ell)}^{W(\tilde{C}_\ell)}$. Since $\mathrm{wt}(d) \geq 0$, there exists $k' \in \mathbb{Z}_{\geq 0}$ such that $d \in \overline{k'\delta(\tilde{D}_{\ell+2})}^{W(\tilde{D}_{\ell+2})}$. In fact $d \in \overline{k'\delta(\tilde{D}_{\ell+2})}^{W(\tilde{C}_\ell)}$ since $d \in Q(\tilde{D}^\sigma_{\ell+2})[0^+]$. The map $\mathcal{T}$ sends $\delta(\tilde{D}_{\ell+2})$ to $2\delta^\vee(\tilde{C}_\ell)$, which then implies that $\mathcal{T}(d) \in \overline{2k'\delta^\vee(\tilde{C}_\ell)}^{W(\tilde{C}_\ell)}$. Hence, we obtain $k = 2k'$ by construction of $\mathrm{wt}(d)$ (cf. [Pae1, Lemma 4.7]). Since $n = |\lambda| + 2k\ell$, we have that $k \geq 0$ if and only if $|\lambda| \leq n$. The map $\epsilon : \mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}} \to \{\lambda \in \mathfrak{C}^s_{2\ell} \mid |\lambda| \equiv n \pmod{2\ell},\ |\lambda| \leq n\}$ has now been proven to be well defined. By construction, ϵ is the inverse map of Res_D and establishes a bijection between $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$ and $\{\lambda \in \mathfrak{C}^s_{2\ell} \mid |\lambda| \equiv n \pmod{2\ell},\ |\lambda| \leq n\}$.

Concerning the second assertion, we observe that I_{μ_1} and I_{μ_2} are in the same irreducible component of $\mathcal{H}^{\mathrm{BD}_{2\ell}}_n$ if and only if the character of $\mathbb{C}[x,y]/I_{\mu_1}$ is equal to the character of $\mathbb{C}[x,y]/I_{\mu_2}$, by [Pae1, Corollary 4.3]. Now using Proposition 4.5, we know that this is the case if and only if $\mathrm{Res}_D(\mu_1) = \mathrm{Res}_D(\mu_2)$. By construction, for $i \in \{1,2\}$, $\epsilon(\mathrm{Res}_D(\mu_i))$ is the 2ℓ-core of μ_i which then yields the result. □

Remark 5.2. Take $d \in \mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$ and $\lambda \in \mathcal{P}^s_n$ such that I_λ is in the irreducible component parametrized by d. Let $\gamma_{2\ell}$ denote the 2ℓ-core of λ. We have, as a by-product of the proof of Theorem 5.1, that $\frac{n-|\gamma_{2\ell}|}{2\ell}$, which is the number of 2ℓ-hooks that we need to remove from λ to obtain its 2ℓ-core, is equal to $2\mathrm{wt}(d)$.

Example 5.3. The set $\mathcal{A}^{n,\mathbb{T}_1}_{\mathrm{BD}_{2\ell}}$ is a proper subset of $\mathcal{A}^{n}_{\mathrm{BD}_{2\ell}}$. If $\ell = 2$, for $r \in \mathbb{Z}_{>0}$ we can find an irreducible component of $\mathcal{H}^{\mathrm{BD}_4}_{8r+4}$ of dimension $2r$ that is parametrized by an element of $\mathcal{A}^{8r+4}_{\mathrm{BD}_4} \setminus \mathcal{A}^{8r+4,\mathbb{T}_1}_{\mathrm{BD}_4}$. Let $\omega = s_{\chi_{2+}} s_{\chi_1} s_{\chi_{0+}} \in W(\tilde{T}_{\mathrm{BD}_4})$, and consider $\omega.r\delta^{\mathrm{BD}_4}$. We have $(\omega.r\delta^{\mathrm{BD}_4})_{\chi_{2+}} = (\omega.r\delta^{\mathrm{BD}_4})_{\chi_{2-}} + 1$, which implies that this element is not in $\mathcal{A}^{8r+4,\mathbb{T}_1}_{\mathrm{BD}_4}$ due to Proposition 4.5.

6. ABSENCE OF COMBINATORICS IN TYPE E

The binary tetrahedral group $\tilde{A}_4$ is a central extension of A_4, the alternating group on 4 elements [CM13, §6.5]. It has order 24 and has the following presentation

$$\langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle.$$

We set $z := abc$, which is a central element of $\tilde{A}_4$. Note that z has order 2. The group $\tilde{A}_4$ has the following character table:

cardinality	1	1	6	4	4	4	4
classes	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	z	a	b	c	b^2	c^2
χ_0	1	1	1	1	1	1	1
ψ	1	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3
ψ^2	1	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2
X	3	3	-1	0	0	0	0
χ_{std}	2	-2	0	1	1	-1	-1
$\psi\chi_{\text{std}}$	2	-2	0	ζ_3	ζ_3^2	$-\zeta_3^2$	$-\zeta_3$
$\psi^2\chi_{\text{std}}$	2	-2	0	ζ_3^2	ζ_3	$-\zeta_3$	$-\zeta_3^2$

The McKay graph of any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to $\tilde{A}_4$ is of affine type $\tilde{E}_6$

.

The goal here is to study the combinatorics of the irreducible components of $\mathcal{H}_n^\Gamma$ when Γ is of type $\tilde{E}_6$ (meaning that Γ is isomorphic to $\tilde{A}_4$). We claim that the irreducible components containing a monomial ideal are fixed under $\mathrm{SL}_2(\mathbb{C})$. Indeed, let X_{std} denote the standard representation of $\mathrm{SL}_2(\mathbb{C})$ with its canonical basis (e_1, e_2), and denote by B_1 and B_2 the stabilizers of e_1 and e_2 in $\mathrm{SL}_2(\mathbb{C})$, respectively. The subgroups B_1 and B_2 are the two Borel subgroups of $\mathrm{SL}_2(\mathbb{C})$ containing $\mathbb{T}_1$. Let us fix Γ a finite subgroup of type $\tilde{E}_6$ in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 6.1. *The group Γ is not conjugate to any subgroup of the normalizer of $\mathbb{T}_1$ in $\mathrm{SL}_2(\mathbb{C})$, which we denote by $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$. Furthermore, the group Γ is neither conjugate to a subgroup of B_1 nor of B_2.*

Proof. The representation $X_{\text{std}} \otimes X_{\text{std}}^*$ is isomorphic to the direct sum of the trivial representation (generated by $e_1 \otimes e_1^* + e_2 \otimes e_2^*$) and the adjoint representation of $\mathrm{SL}_2(\mathbb{C})$. On the one hand, note that for the character χ_{std} of Γ, we have $\langle (\chi_{\text{std}})^2, (\chi_{\text{std}})^2 \rangle = 2$, which implies that the restriction of the adjoint representation to Γ is irreducible. On the other hand, the restriction of the adjoint representation to $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$ is not irreducible since the one-dimensional subspace of $X_{\text{std}} \otimes X_{\text{std}}^*$ generated by $e_1 \otimes e_1^* - e_2 \otimes e_2^*$ is $N_{\mathrm{SL}_2(\mathbb{C})}(\mathbb{T}_1)$-stable. Moreover, the one-dimensional subspace of X_{std} generated by e_1 is B_1-stable and the one-dimensional subspace of X_{std} generated by e_2 is B_2-stable. □

Proposition 6.2. *The subgroup G of $\mathrm{SL}_2(\mathbb{C})$ generated by $\mathbb{T}_1$ and Γ is $\mathrm{SL}_2(\mathbb{C})$.*

Proof. By Lemma 6.1, there exists $x \in \Gamma$ such that $\mathbb{T}_1 \neq x\mathbb{T}_1x^{-1}$. The two subgroups $\mathbb{T}_1$ and $x\mathbb{T}_1x^{-1}$ are both irreducible and connected subgroups of $\mathrm{SL}_2(\mathbb{C})$. Let us denote the subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by these two one-dimensional tori by H. From [Hump, Section 7.5], we know that H is a closed connected subgroup of $\mathrm{SL}_2(\mathbb{C})$. Since H is not equal to $\mathrm{SL}_2(\mathbb{C})$, and is of dimension at least two, H is of dimension 2. Using [Bor12, Corollary 11.6] we know that H is solvable. The algebraic group H is then a Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$ containing $\mathbb{T}_1$ and contained in G. Moreover, by the Bruhat decomposition [Bor12, Theorem 14.12], we know that $\mathrm{SL}_2(\mathbb{C}) = B_1 s B_1 \sqcup B_1 = B_2 s B_2 \sqcup B_2$. Combining the Bruhat decomposition with Lemma 6.1, we see that $s \in G$. By [Bor12, Proposition 11.19], we know that all Borel subgroups containing $\mathbb{T}_1$ are conjugated by the Weyl group of $\mathbb{T}_1$, which we denote by $W(\mathbb{T}_1)$. It is by construction the group generated by $\bar{s} \in W(\mathbb{T}_1)$. We infer that all Borel subgroups containing $\mathbb{T}_1$ are in G. Finally, using [Bor12, Proposition 13.7], we have $G = \mathrm{SL}_2(\mathbb{C})$. □

Definition 6.3. A partition is called a staircase partition if there exists a certain integer m such that the partition is equal to $\lambda_m := (m, m-1, \dots, 1) \vdash \frac{m(m+1)}{2}$. Note that $\mathfrak{C}_2$ is equal to the set of all staircase partitions.

Proposition 6.4. *The only* $\mathrm{SL}_2(\mathbb{C})$ *fixed points of* $\mathcal{H}_n$ *are the monomial ideals associated with staircase partitions of size* n.

Proof. We already know that $\mathbb{T}_1$-fixed points are exactly monomial ideals. Moreover, by [KT, Lemma 12], the fixed points under the subgroup $\mathbb{B}_2$ of $\mathrm{GL}_2(\mathbb{C})$ consisting of all upper triangular matrices are parametrized by staircase partitions. Let $\mathbb{T}_2$ be the maximal diagonal torus of $\mathrm{GL}_2(\mathbb{C})$. Since $\mathbb{B}_2 = \mathbb{T}_2 B_1$, we get that B_1-fixed points of $\mathcal{H}_n$ are also parametrized by staircase partitions, and the result follows. □

Finally, the binary octahedral group (type $\tilde{E}_7$) and the binary icosahedral group (type $\tilde{E}_8$) both contain a subgroup isomorphic to $\tilde{A}_4$, which then implies that the combinatorics of fixed points which are also $\mathbb{T}_1$-fixed is the same as the one of $\mathrm{SL}_2(\mathbb{C})$. We thus have proved the following result.

Proposition 6.5. *If* Γ *is a finite subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *of type* $\tilde{E}_6$, $\tilde{E}_7$ *or* $\tilde{E}_8$, *then, for* $n \in \mathbb{Z}_{\geq 1}$, *there is at most one irreducible component of* $\mathcal{H}_n^\Gamma$ *containing a* $\mathbb{T}_1$-*fixed point, and it is indexed by the staircase partition of size* n *(when it exists).*

7. DIMENSION OF THE IRREDUCIBLE COMPONENTS CONTAINING A $\mathbb{T}_1$-FIXED POINT

In this section we show that each irreducible component of $\mathcal{H}_n^\Gamma$ containing a $\mathbb{T}_1$-fixed point is zero-dimensional whenever Γ is of type $\tilde{E}_6$ in $\mathrm{SL}_2(\mathbb{C})$. By Proposition 6.5, we know that it is enough to compute the dimensions of the irreducible components of $\mathcal{H}_n^\Gamma$ which contain a $\mathbb{T}_1$-fixed point indexed by a staircase partition. The results of this section will not depend on the choice of Γ but only on the McKay graph. Since we need to make explicit computations, we choose to work with the following model of the binary tetrahedral group. Let $t \in \mathrm{SL}_2(\mathbb{C})$ be the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8 \\ \zeta_8^3 & \zeta_8^{-1} \end{pmatrix}.$$

Consider the subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by ω_4, s and t. Let us denote this group by BT. By setting $a = s\omega_4$, $b = t$ and $c = st^2$, one can show that BT has the desired presentation (namely the one of $\tilde{A}_4$). Note also that $\mathrm{BT} = \mathrm{BD}_4 \rtimes \langle t^2 \rangle$.

7.1. **$\tilde{E}_6$-Residue.** The irreducible components of $\mathcal{H}_n^\Gamma$ are isomorphic to quiver varieties over the doubled, framed McKay quiver (cf. [Pae1, Proposition 3.19]). We are interested in the irreducible components containing a $\mathbb{T}_1$-fixed point. Take $m \in \mathbb{Z}_{\geq 1}$. We know that the dimension parameter of this quiver variety is equal to the character of BT of the representation $\mathbb{C}[x,y]/I_{\lambda_m}$. In this subsection, we will then construct a map $\mathrm{Res}_{\tilde{E}_6} : \mathfrak{C}_2 \to Q(\tilde{E}_6)$ which computes the decomposition into irreducible characters of the character of the representations $\mathbb{C}[x,y]/I_{\lambda_m}$ for $\lambda_m \in \mathfrak{C}_2$. Before doing that, we need to introduce some notation. If $m = 2k \in \mathbb{Z}_{\geq 0}$, define

$$d_m^0 := \begin{cases} 1 + \left\lfloor \frac{k-2}{3} \right\rfloor, & \text{if } m \equiv 0 \pmod 3, \\ \left\lfloor \frac{k}{3} \right\rfloor, & \text{if } m \equiv 1 \pmod 3, \\ 1 + \left\lfloor \frac{k-1}{3} \right\rfloor, & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Let $d_m := \frac{m - 2d_m^0}{4}$. The fact that $d_m \in \mathbb{Z}_{\geq 0}$ results from the definition of d_m^0. If now $m = 2k+1 \in \mathbb{Z}_{\geq 0}$, define

$$a_m := 1 + \left\lfloor \frac{k-1}{2} \right\rfloor.$$

Let $b_m := m - 3a_m$. Moreover define

$$e_m^0 := \begin{cases} \left\lfloor \frac{b_m+1}{3} \right\rfloor, & \text{if } m \equiv 0 \pmod 3, \\ 1 + \left\lfloor \frac{b_m}{3} \right\rfloor, & \text{if } m \equiv 1 \pmod 3, \\ \left\lfloor \frac{b_m}{3} \right\rfloor, & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Let $e_m := \frac{b_m - e_m^0}{2}$. The fact that $e_m \in \mathbb{Z}_{\geq 0}$ results from the definition of e_m^0. For the sake of clarity, we introduce

$$\beta_m := \begin{cases} d_m^0 \alpha_{\chi_{\text{std}}} + d_m \alpha_{\psi\chi_{\text{std}}} + d_m \alpha_{\psi^2\chi_{\text{std}}}, & \text{if } m \text{ is even}, \\ a_m \alpha_{\mathrm{X}} + e_m^0 \alpha_{\chi_0} + e_m \alpha_{\psi} + e_m \alpha_{\psi^2}, & \text{otherwise}, \end{cases} \qquad \text{for all } m \in \mathbb{Z}_{\geq 0}.$$

We define $\mathrm{Res}_{\tilde{E}_6}$ such that the difference between $\mathrm{Res}_{\tilde{E}_6}(\lambda_m)$ and $\mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1})$ is exactly the element β_m of the $\tilde{E}_6$-root lattice.

Definition 7.1. Define the map $\tilde{E}_6$-Residue in the following way

$$\mathrm{Res}_{\tilde{E}_6} : \begin{array}{ccc} \mathfrak{C}_2 & \to & Q(\tilde{E}_6), \\ \lambda_0 & \mapsto & 0, \\ \lambda_m & \mapsto & \mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1}) + \beta_m. \end{array}$$

Example 7.2. Take $\lambda_3 = (3,2,1)$. Its Young diagram is filled as follows:

$$\begin{array}{|c|c|c|} \cline{1-1} \mathrm{X} & \multicolumn{2}{c}{} \\ \cline{1-2} \chi_{\text{std}} & \mathrm{X} & \multicolumn{1}{c}{} \\ \hline \chi_0 & \chi_{\text{std}} & \mathrm{X} \\ \hline \end{array}\,.$$

This gives $\mathrm{Res}_{\tilde{E}_6}(\lambda_3) = \alpha_{\chi_0} + \alpha_{\chi} + \alpha_{\mathrm{X}}$.

The following proposition assures us that $\mathrm{Res}_{\tilde{E}_6}$ fulfills its purpose.

Proposition 7.3. *For $\lambda_m \in \mathfrak{C}_2$, $\mathrm{Res}_{\tilde{E}_6}(\lambda_m)$ is the character of the* BT*-representation $\mathbb{C}[x,y]/I_{\lambda_m}$.*

Proof. To decompose the character of the $\mathrm{SL}_2(\mathbb{C})$-representation $\mathbb{C}[x,y]/I_{\lambda_m}$ along the irreducible characters of BT, we will use the fact that $\mathrm{BT} = \mathrm{BD}_4 \rtimes \langle t^2 \rangle$. The group $\langle t^2 \rangle$ is conjugate to μ_3 in $\mathrm{SL}_2(\mathbb{C})$. Moreover, we deduce from the character tables and Clifford theory (see [I11, Theorem 6.2]) that $\mathrm{X} = \mathrm{Ind}_{\mathrm{BD}_4}^{\mathrm{BT}}(\chi_{2^+})$. By Proposition 4.5, we infer that the recursive Definition 7.1 is the character of the BT-representation $\mathbb{C}[x,y]/I_{\lambda_m}$. □

Now that we have computed the decomposition of the character of $\mathbb{C}[x,y]/I_{\lambda_m}$ for $\lambda_m \in \mathfrak{C}_2$, we need to define the Euler form to compute the dimension of the irreducible component of $\mathcal{H}^{\mathrm{BT}}_{\frac{m(m+1)}{2}}$ containing I_{λ_m}. To define this form, one needs to choose an orientation of the McKay quiver. Let us work with the following orientation:

Let $E_{\tilde{E}_6}$ be the set of oriented arrows of the McKay quiver $\tilde{E}_6$. For an arrow $h \in E_{\tilde{E}_6}$, we denote by h' and h'' the source and target of h, respectively.

7.2. **Zero-dimensional irreducible components.**

Definition 7.4. The Euler form is a bilinear form defined on the root lattice (which is identified with the lattice of dimension parameters) in the following way:

$$\langle v, w\rangle := \sum_{\chi \in I_{\tilde{E}_6}} v_\chi w_\chi - \sum_{h \in E_{\tilde{E}_6}} v_{h'} w_{h''}, \quad \text{for all } (v,w) \in Q(\tilde{E}_6)^2.$$

Remark 7.5. Our results will only involve the Tits form (which is the associated quadratic form). Thus they will not depend on the choice of an orientation. Only the intermediate computations will use the Euler form.

Theorem 7.6. *For $m \in \mathbb{Z}_{\geq 0}$, the irreducible component of $\mathcal{H}^{\mathrm{BT}}_{\frac{m(m+1)}{2}}$ containing I_{λ_m} is of dimension* 0.

Proof. Combining [Pae1, Proposition 3.19] and Proposition 7.3, we see that the irreducible component of $\mathcal{H}^{\mathrm{BT}}_{\frac{m(m+1)}{2}}$ containing I_{λ_m} is isomorphic to the quiver variety on the McKay quiver with dimension parameter $\mathrm{Res}_{\tilde{E}_6}(\lambda_m)$. By [Nak98, Corollary 3.12], the dimension of this quiver variety is equal to

$$2\left(\mathrm{Res}_{\tilde{E}_6}(\lambda_m)_{\chi_0} - \langle \mathrm{Res}_{\tilde{E}_6}(\lambda_m), \mathrm{Res}_{\tilde{E}_6}(\lambda_m)\rangle\right).$$

It remains to prove that this integer is equal to zero. To improve readability, we prove the remaining equality in Proposition 7.9. ☐

Before being able to finish the proof of Theorem 7.6, we need to prove a technical lemma.

Lemma 7.7. *For $k \in \mathbb{Z}_{\geq 1}$, we have*

$$d^0_{2(k+1)} + d^0_{2k} = e^0_{2k+1} + a_{2k+1}, \tag{1}$$

$$e^0_{2k-1} + e^0_{2k+1} = d^0_{2k}, \tag{2}$$

$$k+1 = a_{2k+1} + a_{2k+3}. \tag{3}$$

Proof. To prove relation (1), we consider the following cases:

- If $2k-1 \equiv 0 \pmod 3$, then $d^0_{2(k+1)} + d^0_{2k} = 1 + \left\lfloor \frac{k-1}{3} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor$ and $e^0_{2k+1} + a_{2k+1} = \left\lfloor \frac{2k+1}{3} \right\rfloor$. In that case $d^0_{2(k+1)} + d^0_{2k} = e^0_{2k+1} + a_{2k+1}$.
- If $2k-1 \equiv 1 \pmod 3$, then $d^0_{2(k+1)} + d^0_{2k} = \left\lfloor \frac{k+1}{3} \right\rfloor + 1 + \left\lfloor \frac{k-1}{3} \right\rfloor$ and $e^0_{2k+1} + a_{2k+1} = \left\lfloor \frac{2k+2}{3} \right\rfloor$. In that case $d^0_{2(k+1)} + d^0_{2k} = e^0_{2k+1} + a_{2k+1}$.

- If $2k-1 \equiv 2 \pmod 3$, then $d^0_{2(k+1)} + d^0_{2k} = 1 + \left\lfloor \frac{k}{3} \right\rfloor + 1 + \left\lfloor \frac{k-1}{3} \right\rfloor$ and $e^0_{2k+1} + a_{2k+1} = 1 + \left\lfloor \frac{2k+1}{3} \right\rfloor$. In that case $d^0_{2(k+1)} + d^0_{2k} = e^0_{2k+1} + a_{2k+1}$.

The same can be done to prove relation (2). Relation (3) is a direct consequence of the definition of a_{2k+1}. □

Definition 7.8. Take $m \in \mathbb{Z}_{\geq 1}$. For $k \in [\![1, m]\!]$, define the slice k of λ_m to be the subset of the Young diagram of λ_m given by $\{(i,j) \in \mathcal{Y}(\lambda_m) \mid i + j = k-1\}$. Note that, if we remove the slice m from λ_m, we obtain λ_{m-1}.

We are now able to finish the proof of Theorem 7.6.

Proposition 7.9. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\langle \mathrm{Res}_{\tilde{E}_6}(\lambda_m), \mathrm{Res}_{\tilde{E}_6}(\lambda_m) \rangle = \mathrm{Res}_{\tilde{E}_6}(\lambda_m)_{\chi_0}.$$

Proof. We proceed by induction on m. The proposition is obviously true for $m = 0$. Due to the definition of $\mathrm{Res}_{\tilde{E}_6}$, we need to distinguish two cases depending on the parity of m.

- If $m = 2k$, then by Definition 7.1 we have

$$\begin{aligned}\langle \mathrm{Res}_{\tilde{E}_6}(\lambda_m), \mathrm{Res}_{\tilde{E}_6}(\lambda_m) \rangle = {} & \langle \mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1}), \mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1}) \rangle \\ & + \langle \mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1}), \beta_m \rangle \\ & + \langle \beta_m, \mathrm{Res}_{\tilde{E}_6}(\lambda_{m-1}) \rangle \\ & + (d^0_m)^2 + 2(d_m)^2.\end{aligned}$$

In the interest of readability, we abbreviate $\mathrm{Res}_{\tilde{E}_6}(\lambda_m)$ by $\mathrm{R}(m)$. By the induction hypothesis, we have $\langle \mathrm{R}(m-1), \mathrm{R}(m-1) \rangle = \mathrm{R}(m-1)_{\chi_0}$. By Definition 7.1, we know that $\mathrm{R}(m-1)_{\chi_0} = \mathrm{R}(m)_{\chi_0}$, since m is even. It is then enough to prove that

$$\langle \mathrm{R}(m-1), \beta_m \rangle + \langle \beta_m, \mathrm{R}(m-1) \rangle + (d^0_m)^2 + 2(d_m)^2 = 0. \tag{4}$$

By Definition 7.1, we have

$$\begin{aligned}\mathrm{R}(m-1)_{\chi_0} &= \sum_{j=0}^{k-1} e^0_{2j+1}, \\ \mathrm{R}(m-1)_{\mathrm{X}} &= \sum_{j=0}^{k-1} a_{2j+1}, \\ \mathrm{R}(m-1)_{\chi_{\mathrm{std}}} &= \sum_{j=0}^{k-1} d^0_{2j}.\end{aligned}$$

By (1), we obtain

$$\sum_{j=0}^{k-1} d^0_{2(j+1)} + \sum_{j=0}^{k-1} d^0_{2j} = \sum_{j=0}^{k-1} e^0_{2j+1} + \sum_{j=0}^{k-1} a_{2j+1}.$$

Since $d^0_j = 0$ for $j = 0$, the previous equality yields

$$d^0_m = \mathrm{R}(m-1)_{\chi_0} + \mathrm{R}(m-1)_{\mathrm{X}} - 2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}}. \tag{5}$$

Moreover, by Proposition 7.3, the number of boxes that lie in the odd slices between slice 1 and slice $m-1$ is equal to

$$\mathrm{R}(m-1)_{\chi_0} + 2\mathrm{R}(m-1)_{\psi} + 3\mathrm{R}(m-1)_{\mathrm{X}}.$$

Thus we have

$$\mathrm{R}(m-1)_{\chi_0} + 2\mathrm{R}(m-1)_{\psi} + 3\mathrm{R}(m-1)_{\mathrm{X}} = k^2.$$

In the same way, the number of boxes that lie in the even slices between slice 1 and slice $m-1$ is equal to $2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}} + 4\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}}$. Hence we have

$$2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}} + 4\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} = k(k-1).$$

From there we have the following two relations:

$$\begin{aligned} d_m^0 &= \mathrm{R}(m-1)_{\chi_0} + \mathrm{R}(m-1)_{\mathrm{X}} - 2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}}, \\ k &= \mathrm{R}(m-1)_{\chi_0} + 2\mathrm{R}(m-1)_{\psi} + 3\mathrm{R}(m-1)_{\mathrm{X}} - \left(2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}} + 4\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}}\right). \end{aligned}$$

They imply that

$$k - d_m^0 = 2\left(\mathrm{R}(m-1)_{\psi} + \mathrm{R}(m-1)_{\mathrm{X}} - 2\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}}\right).$$

Now, since $m = 2k$ and $d_m = \frac{m-2d_m^0}{4}$, we obtain

$$d_m = \mathrm{R}(m-1)_{\psi} + \mathrm{R}(m-1)_{\mathrm{X}} - 2\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}}. \tag{6}$$

Recall that, since m is even, $\beta_m = d_m^0\alpha_{\chi_{\mathrm{std}}} + d_m\alpha_{\psi\chi_{\mathrm{std}}} + d_m\alpha_{\psi^2\chi_{\mathrm{std}}}$. By construction of the Euler form, we have

$$\langle \mathrm{R}(m-1), \beta_m\rangle + \langle \beta_m, \mathrm{R}(m-1)\rangle = (\mathrm{R}(m-1), \beta_m),$$

where $(\ ,\ )$ denotes the nondegenerate bilinear form on $\mathfrak{h}_\Gamma^*$ (cf. [Kac, §2.1]). Using the McKay graph of type $\tilde{E}_6$, we deduce that

$$\begin{aligned} (\mathrm{R}(m-1), \beta_m) = {} & 2d_m^0\mathrm{R}(m-1)_{\chi_{\mathrm{std}}} + 4d_m\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} \\ & - d_m^0\left(\mathrm{R}(m-1)_{\chi_0} + \mathrm{R}(m-1)_{\mathrm{X}}\right) - 2d_m\left(\mathrm{R}(m-1)_{\mathrm{X}} + \mathrm{R}(m-1)_{\psi}\right). \end{aligned}$$

Rearranging the right-hand side of the previous equation, we may rewrite this right-hand side as

$$\begin{aligned} & d_m^0\Big(2\mathrm{R}(m-1)_{\chi_{\mathrm{std}}} - \mathrm{R}(m-1)_{\chi_0} - \mathrm{R}(m-1)_{\mathrm{X}}\Big) \\ & \qquad + 2d_m\Big(2\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} - \mathrm{R}(m-1)_{\mathrm{X}} - \mathrm{R}(m-1)_{\psi}\Big). \end{aligned}$$

We recognize the expressions of d_m^0 and d_m obtained in (5) and (6). This leads to the equation

$$(\mathrm{R}(m-1), \beta_m) = -(d_m^0)^2 - 2(d_m)^2.$$

This gives the desired equality (4) and concludes the proof when m is even.

- Let us suppose now that $m = 2k+1$ is odd. It is then enough to prove that

$$\langle \mathrm{R}(m-1), \beta_m\rangle + \langle \beta_m, \mathrm{R}(m-1)\rangle + {e_m^0}^2 + 2{e_m}^2 + {a_m}^2 = e_m^0.$$

Due to relations (2) and (3), we first have

$$2\mathrm{R}(m-1)_{\chi_0} + e_m^0 - \mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} = 1. \tag{7}$$

Secondly, we have

$$4\mathrm{R}(m-1)_{\psi} + 2e_m - 2\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} = 0. \tag{8}$$

Thirdly, we have

$$2\mathrm{R}(m-1)_{\mathrm{X}} + a_m - \mathrm{R}(m-1)_{\chi_{\mathrm{std}}} - 2\mathrm{R}(m-1)_{\psi\chi_{\mathrm{std}}} = 0. \tag{9}$$

Equations (7), (8) and (9) give the desired result when m is odd.

This concludes the proof of the proposition and also of Theorem 7.6. ☐

Remark 7.10. Theorem 7.6 implies that, if Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to the binary octahedral group (of type $\tilde{E}_7$), or if Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ isomorphic to the binary icosahedral group (of type $\tilde{E}_8$), then all irreducible components of $\mathcal{H}_n^{\Gamma}$ containing a $\mathbb{T}_1$-fixed point are of dimension 0 since these two finite groups contain a subgroup of type $\tilde{E}_6$.

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