Séminaire Lotharingien de Combinatoire **93** (2025), Article B93a

AN EASY PROOF OF RAMANUJAN'S FAMOUS CONGRUENCES $p(5m+4) \equiv 0 \equiv \tau(5m+5) \pmod 5$

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ABSTRACT. We present a proof of Ramanujan's congruences

$$p(5n+4) \equiv 0 \pmod 5 \text{ and } \tau(5n+5) \equiv 0 \pmod 5.$$

The proof only requires a limiting case of Jacobi's triple product, a result that Ramanujan knew well, and a technique which Ramanujan used himself to compute values of $\tau(n)$.

1. Introduction

Let $p(n)$ be the number of unordered partitions of a non-negative integer n. They are given by the generating function

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Similarly, Ramanujan's τ function is defined by the generating function

$$\sum_{n=0}^{\infty} \tau(n+1)q^n = \prod_{k=1}^{\infty} (1-q^k)^{24}.$$

The objective of this paper is to give a proof of Ramanujan's congruences

$$p(5m+4) \equiv 0 \pmod 5 \text{ and } \tau(5m+5) \equiv 0 \pmod 5.$$

Ramanujan's proof of the $p(5m+4)$ congruence appears in [10, Paper 25]; for his proof of the second congruence, see Berndt and Ono [6]. Our proof proves both at the same time; even so, it is easier.

We embed these in an infinite family of congruences. Let $P_r(n)$ be defined using the equation

$$P(q)^r = \sum_{n=0}^{\infty} P_r(n)q^n, \tag{1.1}$$

where r is a rational number. In this notation $P_1(n) = p(n)$ and $P_{-24}(n) = \tau(n+1)$. We will show:

$$P_r(5m+4) \equiv 0 \pmod 5, \text{ if } r \equiv 1 \pmod 5, \tag{1.2}$$

where r is a rational number.[1] The cases $r = 1$ and $r = -24$ give Ramanujan's congruences.

2020 *Mathematics Subject Classification.* Primary: 11P83; Secondary: 05A17.

Key words and phrases. Partitions, Ramanujan's τ function, congruences.

[1] In these congruences, a rational number a/b, with a and b relatively prime and b not divisible by 5, is understood as $a \cdot b^{-1}$, where b^{-1} is the multiplicative inverse of b modulo 5.

The main ingredient of our proof is Jacobi's identity, a limiting case of his triple product identity [5, p. 14],

$$P(q)^{-3} = \prod_{k=1}^{\infty}(1-q^k)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}. \tag{1.3}$$

Ramanujan had rediscovered Jacobi's result. In our notation, it can be rewritten as follows:

$$P_{-3}(n) = \begin{cases} (-1)^k(2k+1), & \text{if } n = \frac{k(k+1)}{2}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

We mention that the result (1.2), and this proof, appears in the authors' previous paper [1]. The objective of this note is to present the bare essentials of this proof.

2. The proof

We first prove a recurrence relation satisfied by the coefficients of powers of any generating function.

Lemma 2.1. *Let $P(q)$ be any power series, r and s be non-zero, real, numbers, and $P_r(n)$ defined by (1.1). Then we have*

$$\sum_{k=0}^{n}\Big(n-\big(r/s+1\big)k\Big)P_r(n-k)P_s(k) = 0. \tag{2.1}$$

Proof. We take the log derivatives on both sides of (1.1), and multiply by q, to obtain

$$r\Big(q\frac{d}{dq}\log P(q)\Big)\sum_{n=0}^{\infty}P_r(n)q^n = \sum_{n=0}^{\infty}nP_r(n)q^n.$$

Similarly, we have

$$s\Big(q\frac{d}{dq}\log P(q)\Big)\sum_{n=0}^{\infty}P_s(n)q^n = \sum_{n=0}^{\infty}nP_s(n)q^n.$$

This gives, on eliminating the common factor,

$$\frac{r}{s}\Big(\sum_{n=0}^{\infty}P_r(n)q^n\Big)\Big(\sum_{n=0}^{\infty}nP_s(n)q^n\Big) = \Big(\sum_{n=0}^{\infty}nP_r(n)q^n\Big)\Big(\sum_{n=0}^{\infty}P_s(n)q^n\Big)$$

The recurrence (2.1) follows by comparing coefficients of q^n on both sides. □

Remark. In this lemma, $P(q)$ can be any generating function, and r and s any non-zero real or complex numbers, as long as $P(q)^r$ and $P(q)^s$ are formal power series. However, we will only apply it when $P(q)$ is the generating function for partitions, and when r and s are rational numbers.

Proof of (1.2). We use an inductive argument using the recurrence relation (2.1), with $s=-3$, in the form

$$nP_r(n) = \sum_{j=1}^{\infty}(-1)^{j+1}(2j+1)\Big(n+\big(r/3-1\big)j(j+1)/2\Big)P_r\big(n-j(j+1)/2\big). \tag{2.2}$$

Here we have used Jacobi's result in the form (1.4) for $P_{-3}(k)$.

Take $n = 5m + 4$. For $m = 0$, (2.2) reduces to

$$4P_r(4) = (9 + r)P_r(3) - 5(r + 1)P_r(2),$$

so $P_r(4) \equiv 0 \pmod 5$ if $r \equiv 1 \pmod 5$.

For $m > 0$, consider the general term

$$(-1)^{j+1}(2j + 1)\,(n + (r/3 - 1)j(j + 1)/2)\,P_r\big(n - j(j + 1)/2\big)$$

in (2.2) for each j. When $j \equiv 1 \pmod 5$ it is of the form

$$(-1)^{j+1}(9 + r)P_r(**) \pmod 5,$$

and, when $j \equiv 3 \pmod 5$, it is of the form

$$(-1)^{j+1}2(2r - 2)P_r(**) \pmod 5.$$

So, when $r \equiv 1$, these terms are $0 \pmod 5$. When $j \equiv 2 \pmod 5$, then the expression is of the form

$$(-1)^{j+1}5(1 + r)P_r(**) \pmod 5;$$

clearly, it is divisible by 5. Finally, when $j \equiv 4, 5 \pmod 5$, this reduces to an expression of the form

$$\big(***\big)P_r(5m + 4 - 5k),$$

for some $k > 0$, and so is $\equiv 0 \pmod 5$, by the induction hypothesis.

Thus each term of the sum on the right-hand side of (2.2) is $0 \pmod 5$. This completes the proof by induction. □

The following companions to (1.2) can be obtained from (2.2) by a similar proof:

$$P_r(5m + 1) \equiv 0 \pmod 5 \text{ if } r \equiv 0 \pmod 5; \tag{2.3}$$

$$P_r(5m + 2) \equiv 0 \pmod 5 \text{ if } r \equiv 2 \pmod 5; \tag{2.4}$$

$$P_r(5m + 3) \equiv 0 \pmod 5 \text{ if } r \equiv 4 \pmod 5. \tag{2.5}$$

For further such congruences modulo 3, and some related results, see [1, 2].

3. Commentary

Taking log derivatives to obtain recurrences from generating functions is a standard procedure in generatingfunctionology, explained in Chapter 1 of Wilf [12]. It was one of Ramanujan's favorite tricks. For example, we find the following recurrence in Ramanujan's work (see [4, p. 108]):

$$\sum_{d=1}^{n} \sigma(d)p(n - d) = np(n).$$

Here $\sigma(n)$ is the sum of divisors of n. This can be obtained by log differentiation of $P(q)$ and comparing coefficients. An extension was given by Ford [7]. See also Entries 12 and 13 in Chapter 10 of Ramanujan's notebooks [3, pages 28–32] for some intricate examples illustrating Ramanujan's use of log differentiation.

Ramanujan's own recurrence for $\tau(n)$ is (2.2) (with $r = -24$); from this he computed 30 values of $\tau(n)$ in his seminal paper [11]. His proof uses log derivatives, and is along the lines of our proof of (2.1). Gould [8] credits an equivalent form of (2.1) to Rothe (1793). Lehmer [9] used Ramanujan's ideas to compute more values of $\tau(n)$.

Interestingly, Ramanujan too considered congruence results for $p(n)$ and $\tau(n)$ in tandem (see [6]); his proof of the $\tau(5n+5)$ congruence uses the $p(5n+4)$ congruence. Here is a relation connecting $\tau(n)$ with $p(n)$, obtained by taking $s=1$ and $r=-24$ in (2.1):

$$\sum_{k=0}^{n}(n+23k)\tau(n-k+1)p(k)=0. \tag{3.1}$$

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