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COMBINATORICS OF DESCENT ALGEBRAS AND GRAPH COVERINGS

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Abstract. We give a direct combinatorial proof that the product of two descent classes in a symmetric group is a sum of descent classes. The proof is based on the fact that the group product gives a covering map when descent classes are endowed with the graph structure coming from the weak order. The main geometric argument is valid for any Coxeter group, even infinite ones for which the descent algebra does not exist.

1. Introduction

Solomon [2] has proved that the product of two descent classes, in the group algebra of a finite Coxeter group, is a sum of descent classes. Therefore the linear combinations of descent classes form a subalgebra of the group algebra. The proof however does not yield a direct combinatorial explanation. In this paper we give a simple argument, involving covering graphs, which shows that the product of two descent classes is a sum of descent classes. As an outcome we give a refinement of the structure coefficients of the descent algebra taking into account the structure of the covering graphs. For the convenience of readers who are not versed in Coxeter groups we give the proofs for the symmetric groups first and then show how our arguments extend easily to the case of general Coxeter groups. In fact we will see that the geometric argument that we give is also valid in infinite Coxeter groups, even when the descent algebra is not well defined.

This paper is organized as follows: in Section 2 we recall the definition of descents of permutations. In Section 3 we give our main combinatorial result, Proposition 3.3, which is at the core of the proof of the existence of the descent algebra. In Section 4 we recall some standard notions of graph theory concerning coverings of graphs. The existence of the descent algebra follows then easily, as shown in Section 5. In Section 6 we give some explicit examples of our construction and in Section 7 we explore the action of the fundamental group induced by the covering. Finally in Section 8 we show that our main result extends in a simple way to Coxeter groups, by using the exchange property. The existence of the descent algebra follows then by exactly the same reasoning as for the symmetric groups.

2. Descents

For a positive integer n let S_n be the symmetric group acting on $\{1, 2, \dots, n\}$. A permutation $\sigma \in S_n$ with linear representation $\sigma_1\sigma_2 \dots \sigma_n$ has a *descent* at i if $\sigma_i > \sigma_{i+1}$. It has an *inverse descent* at i if σ^{-1} has a descent at i. This happens when $i+1$ appears before i in $\sigma_1\sigma_2 \dots \sigma_n$. Let $Des(\sigma) \subset \{1, 2, \dots, n-1\}$ (respectively $\mathrm{IDes}(\sigma) = Des(\sigma^{-1})$) denote the set of descents (respectively inverse descents) of a permutation. In the following we will consider inverse descents rather than descents for clarity of exposition but it is straightforward to convert our results into statements about descents. For $I \subset \{1, \dots, n-1\}$

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denote by $\mathcal{Y}_I$ the inverse descent class of I, which is the set of all permutations σ such that $\mathrm{IDes}(\sigma) = I$ and let

$$Y_I = \sum_{\sigma \in \mathcal{Y}_I} \sigma,$$

the sum being in the group algebra of S_n (which we consider over the rationals but any field would do). Solomon [2] has proved that there exist nonnegative integers a_{IJK}, indexed by triples of subsets of $\{1, \dots, n-1\}$, such that

$$Y_I Y_J = \sum_K a_{IJK} Y_K. \tag{1}$$

It follows that the linear space generated by the Y_I for $I \subset \{1, 2, \dots, n-1\}$ forms a subalgebra of $\mathbf{Q}S_n$ called the inverse descent algebra of the symmetric group. The proof is indirect, it shows that the sums

$$X_I = \sum_{J \subset I} Y_J$$

satisfy a similar formula for some coefficients b:

$$X_I X_J = \sum_K b_{IJK} X_K$$

then (1) follows since the Y can be obtained from the X by Möbius inversion

$$Y_I = \sum_{J \subset I} (-1)^{|I|-|J|} X_I.$$

Since the sets $\mathcal{Y}_I$ are disjoint the coefficients a_{IJK} are nonnegative integers. The combinatorial significance of the existence of the algebra however remains obscure. In this paper I want to explain some of the combinatorics behind (1) using a simple property of products of permutations, Proposition 3.3 below, and some notions of graph theory.

3. Descent classes

We denote by s_i the simple transposition $(i\, i+1)$ so that $S = \{s_1, s_2, \dots, s_{n-1}\}$ is a generating subset of S_n. The associated right Cayley graph has S_n as vertex set and an edge between π and σ if $\sigma = \pi s_i$ for some i. Since the s_i are involutions the graph is non-oriented. The associated length function $\ell(\sigma)$ is the smallest number of factors in a decomposition $\sigma = s_{i_1} \dots s_{i_k}$, and is also the number of inversions of the permutation σ. The distance function on the graph is $d(\pi, \sigma) = \ell(\pi^{-1}\sigma)$. It allows to define the right weak order $\leq_w$ on S_n: one has $\pi \leq_w \sigma$ if $d(e, \pi) + d(\pi, \sigma) = d(e, \sigma)$ (here e is the identity permutation). The permutation $w_0(i) = n + 1 - i$ is the maximal element for the right weak order. The graph can be realized as the skeleton of a polyhedron of dimension $n-1$, the permutahedron.

Note that σ has an inverse descent at i if and only if $\ell(s_i\sigma) = \ell(\sigma) - 1$. We will endow the sets $\mathcal{Y}_I$ with the graph structure induced from the Cayley graph of S_n. The following is well-known.

Proposition 3.1. *Let $I \subset \{1, \dots, n-1\}$, the set $\mathcal{Y}_I$ has a minimal and a maximal element for the right weak order, called α_I and β_I, moreover*

$$\mathcal{Y}_I = [\alpha_I, \beta_I] = \{\sigma \in S_n \,|\, \alpha_I \leq_w \sigma \leq_w \beta_I\}.$$

The group generated by the s_i, for $i \in I$, is a product of symmetric groups acting on intervals of the form $[k, l]$ and α_I is the product of the maximal elements (for the weak order) of these symmetric groups, for example if $n = 12$ and $I = \{1, 4, 5, 6, 9, 10\}$, then the product is $S_{[1,2]} \times S_{[3]} \times S_{[4,7]} \times S_{[8]} \times S_{[9,11]} \times S_{[12]}$ and $\alpha_I = 21|3|7654|8|\,11\,10\,9|\,12$, where we put bars between the successive descending runs. In order to find β_I note that $\sigma \to \sigma w_0$ is an order reversing bijection between $\mathrm{IDes}(I)$ and $\mathrm{IDes}(I^c)$ therefore $\beta_I = \alpha_{I^c} w_0$. In our example $I^c = \{2, 3, 7, 8, 11\}$ which gives $\beta_I = 11\,12|\,10|\,789|6|5|234|1$, where we put bars between ascending runs. In particular $\mathcal{Y}_I$ is a connected subset of the Cayley graph. A criterion for two neighbouring permutations in the Cayley graph of S_n to belong to the same inverse descent class is the following.

Lemma 3.2. *Let $\sigma \in S_n$ and $i \in \{1, \dots, n-1\}$ then $\mathrm{IDes}(\sigma) = \mathrm{IDes}(\sigma s_i)$ if and only if $|\sigma_i - \sigma_{i+1}| \geq 2$.*

Proof. If $\sigma = \sigma_1 \sigma_2 \dots \sigma_i \sigma_{i+1} \dots \sigma_n$ then $\sigma s_i = \sigma_1 \sigma_2 \dots \sigma_{i+1} \sigma_i \dots \sigma_n$. An inverse descent is created or destroyed by permuting σ_i and σ_{i+1} if and only if $|\sigma_i - \sigma_{i+1}| = 1$. □

The following result is the main combinatorial property of inverse descent classes that we will use.

Proposition 3.3. *Let $\sigma, \pi, \rho \in S_n$ and $i \leq n-1$ be such that*

$$\sigma = \pi\rho \quad \text{and} \quad \mathrm{IDes}(\sigma s_i) = \mathrm{IDes}(\sigma).$$

Then there exists a unique factorization $\sigma s_i = \pi' \rho'$ where either:

1. *$\pi = \pi'$ while $\rho^{-1}\rho' \in S$ and $\mathrm{IDes}(\rho) = \mathrm{IDes}(\rho')$ or*
2. *$\rho = \rho'$ while $\pi^{-1}\pi' \in S$ and $\mathrm{IDes}(\pi) = \mathrm{IDes}(\pi')$.*

Proof. There exist two factorizations $\sigma s_i = \pi' \rho'$ such that either $\pi = \pi'$ or $\rho = \rho'$, namely

- $\pi' = \pi$, $\rho' = \rho s_i$
- $\pi' = \pi(\rho s_i \rho^{-1})$, $\rho' = \rho$

Suppose that $\mathrm{IDes}(\rho) = \mathrm{IDes}(\rho s_i)$. By Lemma 3.2, this equality holds if and only if $|\rho_i - \rho_{i+1}| \geq 2$, then the factorization $\sigma s_i = \pi\rho' = \pi(\rho s_i)$ satisfies (1). In the other factorization, $\sigma s_i = \pi' \rho$, one has $\pi' = \pi t$, where t is the transposition $(\rho_i\, \rho_{i+1})$. Since t is not a simple transposition one has $\pi^{-1}\pi' \notin S$ therefore case (1) of the proposition holds but not (2).

Suppose now that $\mathrm{IDes}(\rho) \neq \mathrm{IDes}(\rho s_i)$. This happens if and only if $|\rho_i - \rho_{i+1}| = 1$. In this case let $j = \min(\rho_i, \rho_{i+1})$ and $j+1 = \max(\rho_i, \rho_{i+1})$ then $\rho s_i \rho^{-1} = s_j$ and one has $|\pi_j - \pi_{j+1}| = |\sigma_i - \sigma_{i+1}| \geq 2$. Therefore, by Lemma 3.2, $\mathrm{IDes}(\pi) = \mathrm{IDes}(\pi s_j)$. Now case (2) holds but not (1).

Finally we see that either (1) or (2) holds and Proposition 3.3 is proved. □

4. Covering of graphs

We consider graphs which are simple, non-oriented and without loops.

Definition 4.1. *Let G and H be graphs with respective vertex sets V, W and edge sets E, F. A map $\Pi : V \to W$ is a covering map for (G, H) if*

- *Π is surjective,*
- *for each edge $uv \in E$ one has $\Pi(u)\Pi(v) \in F$*
- *for each edge $wz \in F$, for each vertex $u \in \Pi^{-1}(w)$ there exists a unique $v \in V$ such that $uv \in E$ and $\Pi(v) = z$.*

The following is standard and we recall the argument.

Lemma 4.2. *Let $\Pi : V \to W$ be a covering map as in Definition 4.1, with H a connected graph, then all fibers of Π have the same cardinality i.e. $|\Pi^{-1}(w)| = |\Pi^{-1}(w')|$ for all $w, w' \in W$.*

Proof. Let $w, z \in W$ such that $wz \in F$. For each $u \in \Pi^{-1}(w)$ let $\mathfrak{v}(u) \in \Pi^{-1}(z)$ be the other end of the edge above wz. Similarly for $v \in \Pi^{-1}(z)$ let $\mathfrak{u}(v) \in \Pi^{-1}(w)$ be the other end of the edge above wz, then $\mathfrak{v} : \Pi^{-1}(w) \to \Pi^{-1}(z)$ and $\mathfrak{u} : \Pi^{-1}(z) \to \Pi^{-1}(w)$ are bijections, inverse to each other. It follows that $|\Pi^{-1}(z)| = |\Pi^{-1}(w)|$. By induction on the distance to w one concludes that the cardinality of the fibers is constant on the connected component of w in H. □

5. The inverse descent algebra

5.1. **The product structure.** Let $I, J \subset S$, consider the product space $\mathcal{Y}_I \times \mathcal{Y}_J$ with the Cartesian product graph structure. Let $K \subset S$ and

$$\mathcal{Z}_{IJK} = \{(\pi, \rho) \in \mathcal{Y}_I \times \mathcal{Y}_J \,|\, \pi\rho \in \mathcal{Y}_K\} \tag{2}$$

with its induced graph structure. It follows from Proposition 3.3 that the product map

$$\mathcal{Z}_{IJK} \to \mathcal{Y}_K : (\pi, \rho) \mapsto \pi\rho \tag{3}$$

is a covering map. Therefore, by Lemma 4.2, for $\sigma \in \mathcal{Y}_K$ the number of pairs π, ρ in $\mathcal{Y}_I \times \mathcal{Y}_J$ such that $\pi\rho = \sigma$ depends only on I, J, K. If a_{IJK} is this number then equation (1) follows and the $\mathcal{Y}_I$ generate an algebra.

5.2. **A refinement of the coefficients of the descent algebra.** It follows from our result that the graph $\mathcal{Z}_{IJK}$ is a union of connected components, each of which maps to $\mathcal{Y}_K$ with some multiplicity. Therefore one can write

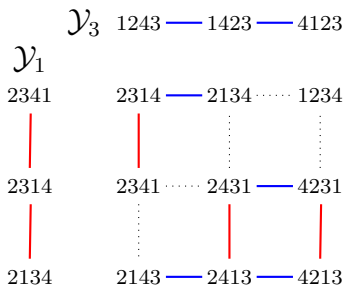
$$a_{IJK} = \sum_l \lambda_l^{IJK}, \tag{4}$$

where λ^{IJK} is the integer partition corresponding to the multiplicities of the connected components of $\mathcal{Z}_{IJK}$. This yields a refinement of the structure coefficients a_{IJK}. A natural problem is to determine which integer partitions can appear in (4). In Section 7 we will investigate some further geometric properties of the covering map.

6. Examples

6.1. **The case of S_4.** We consider the permutations with inverse descent sets $\{1\}$ and $\{3\}$ in S_4, their graphs, ordered from left to right in the weak order, are

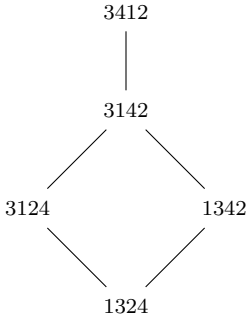
We draw the inverse descent classes inside the product graph, labelling the vertices of the graph by the products of the permutations and colouring the horizontal edges in blue and the vertical ones in red. The other edges of the product graph are dotted.



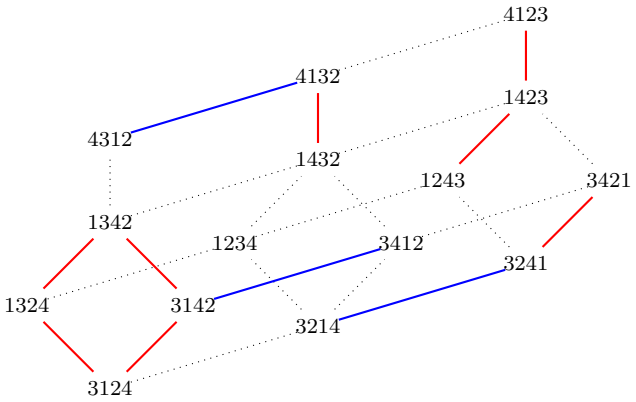
We see that

$$Y_1 Y_3 = Y_\emptyset + Y_1 + Y_{1,3}.$$

Consider now the inverse descent class $\mathcal{Y}_2$, oriented from bottom to top:



We depict the graph of the product $\mathcal{Y}_2\mathcal{Y}_3$ below, with the same colouring convention as above.

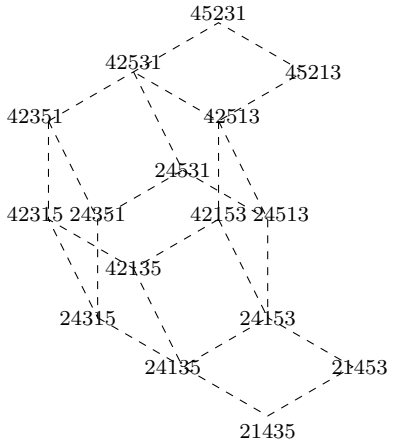


The picture shows that

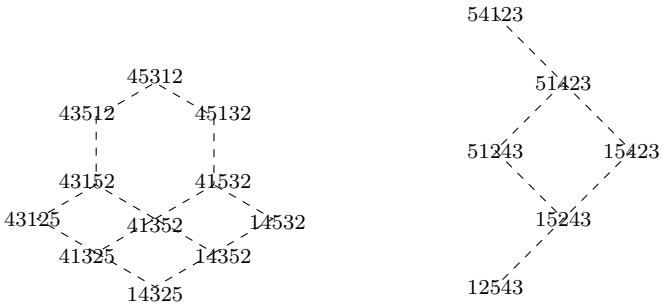
$$Y_2 Y_3 = Y_\emptyset + Y_2 + Y_3 + Y_{2,3} + Y_{1,2}.$$

6.2. **An example in S_5.**

Here is the class $\mathcal{Y}_{1,3}$:



It has multiplicity 2 in the product $Y_{2,3}Y_{3,4}$ whose classes are depicted below



$\mathcal{Y}_{2,3}$ $\mathcal{Y}_{3,4}$

Figure 1 shows $\mathcal{Z}_{2,3/3,4/1,3}$ with the vertices indexed by the products in $\mathcal{Y}_{1,3}$. The covering graph is connected.

7. Action of the fundamental group

7.1. **Fundamental groups and coverings.** We recall a few basic facts about fundamental groups of graphs, see e.g. [3] in the context of algebraic topology. For graphs of course everything can be stated in a purely combinatorial way. For a finite connected graph $G = (V, E)$ a *path* is a sequence of vertices $v_1 - v_2 - \ldots - v_l$ such that each $v_i v_{i+1}$ is an edge. A *loop* is a path with $v_1 = v_l$. The *fundamental group* $\pi_1(G, v)$, based at some point $v \in V$, is the group of homotopy classes of loops, for the concatenation product. For graphs the homotopy equivalence relation between paths is the transitive closure of the relation between two paths of the form $\ldots - x - \ldots$ and $\ldots - x - y - x - \ldots$ obtained by inserting a loop $x - y - x$ into some path. The fundamental group of G is isomorphic to a free group with $|E| - |V| + 1$ generators (in particular it is trivial if G is a tree). For two base points v, v' if $\mu = v - \ldots - v'$ is a path between v and v', with inverse $\mu^{-1} = v' - \ldots - v$, the concatenation map $\lambda \mapsto \mu^{-1}\lambda\mu$ descends to an isomorphism $\pi_1(G, v) \to \pi_1(G, v')$ between the fundamental groups based at v and v'.

7.2. **Action of the fundamental group on the fibers of a covering.** If $\Pi : V \to W$ is a covering map from graph G to H then, for every path $w_1 - w_2 - \ldots - w_l$ in H and every $v_1 \in \Pi^{-1}(w_1) \subset V$, there exists a unique path $v_1 - v_2 - \ldots - v_l$ in the graph G which is mapped to the path $w_1 - w_2 - \ldots - w_l$ by Π. In particular for any vertex $w \in W$ the fundamental group $\pi_1(H, w)$, acts by permutations on the fiber $\Pi^{-1}(w)$: a loop λ

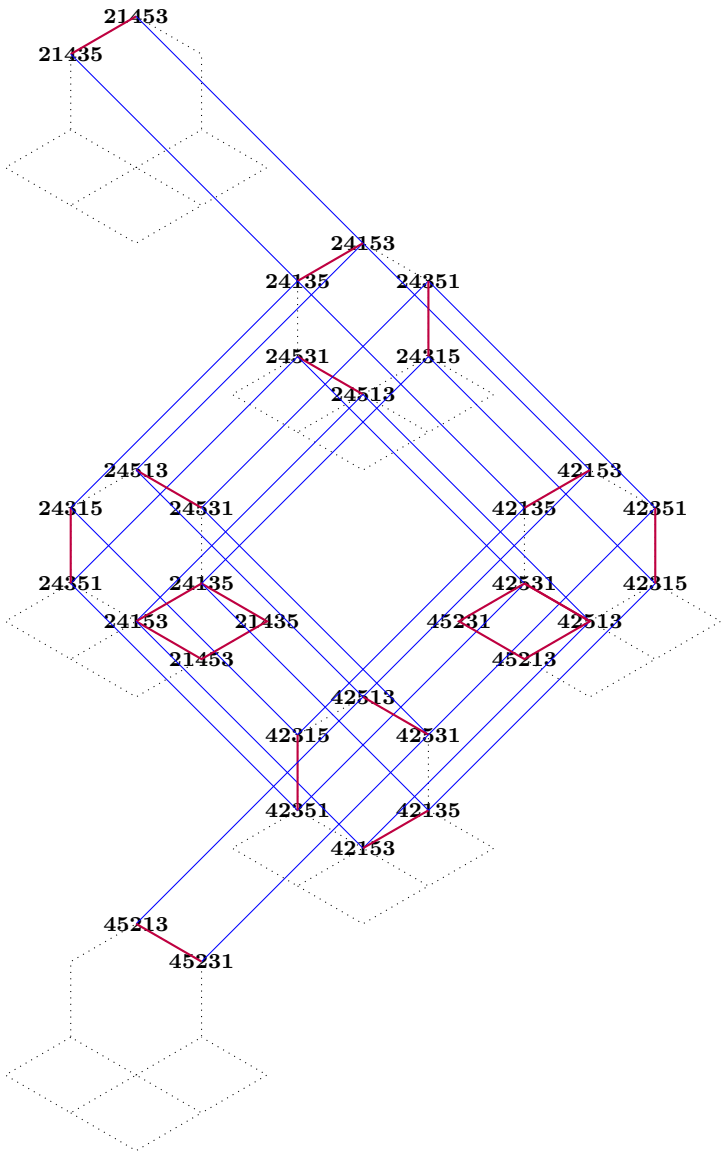


FIGURE 1. $\mathcal{Z}_{2,3/3,4/1,3}$

based at w in H lifts to a path starting at $v \in \Pi^{-1}(w)$. Denote by $[\lambda](v) \in \Pi^{-1}(w)$ its end point, then $[\lambda]$ is a permutation of the set $\Pi^{-1}(w)$ and $\lambda \mapsto [\lambda]$ is a homomorphism from $\pi_1(H, w)$ to the group of permutations of $\Pi^{-1}(w)$. For any two vertices $w, w' \in W$ a path from w to w' gives a bijection between $\Pi^{-1}(w)$ and $\Pi^{-1}(w')$. One obtains thus a conjugation of the actions of $\pi_1(H, w)$ and $\pi_1(H, w')$ on their respective fibers $\Pi^{-1}(w)$ and $\Pi^{-1}(w')$.

7.3. **Fundamental groups of inverse descent classes.** For an inverse descent class $\mathcal{Y}_K$, with the graph structure induced by the weak order, the fundamental group is generated by loops coming from relations in the symmetric group, (which are also boundaries

of two-dimensional cells in the permutahedron). These relations are

$$s_i^2 = e \tag{5}$$

$$s_i s_j = s_j s_i, \qquad \text{for } |i-j| \geq 2, \tag{6}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{7}$$

and the corresponding loops are of the form (provided of course that the permutations involved belong to the same inverse descent class)

$$\sigma - \sigma s_i - \sigma, \qquad \text{relation (5)},$$

$$\sigma - \sigma s_i - \sigma s_i s_j - \sigma s_j - \sigma, \qquad \text{relation (6)},$$

$$\sigma - \sigma s_i - \sigma s_i s_{i+1} - \sigma s_i s_{i+1} s_i - \sigma s_{i+1} s_i - \sigma s_{i+1} - \sigma, \qquad \text{relation (7)},$$

$$\sigma - \sigma s_{i+1} - \sigma s_{i+1} s_i - \sigma s_{i+1} s_i s_{i+1} - \sigma s_i s_{i+1} - \sigma s_i - \sigma, \qquad \text{relation (7)}.$$

The last two loops are inverse of each other.

We have considered loops based at some σ but of course, as recalled above, one can change the base point by using a path from that point to σ.

7.4. **Lifting loops in inverse descent classes.** Let $\mathcal{Y}_I, \mathcal{Y}_J, \mathcal{Y}_K$ be inverse descent classes with $a_{IJK} \geq 1$ and $\Pi : \mathcal{Z}_{IJK} \to \mathcal{Y}_K$, as in (2), the associated projection. We will determine the possible lifts of generating loops of $\mathcal{Y}_K$ in this context and their action on the fibers.

7.4.1. $s_i^2 = e$.

A path $\sigma - \sigma s_i - \sigma s_i s_i = \sigma$ in $\mathcal{Y}_K$ is homotopically trivial and acts by the identity on the fibers.

7.4.2. $s_i s_j = s_j s_i$.

It is easy to see, from the proof of Proposition 3.3, that any loop in $\mathcal{Y}_K$ of the form

$$\sigma - \sigma s_i - \sigma s_i s_j - \sigma s_j - \sigma \qquad |i-j| \geq 2$$

lifts to a loop in $\mathcal{Y}_I \times \mathcal{Y}_J$. Therefore the action of such a loop is trivial on the fiber.

7.4.3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

As we shall see, a loop associated with such a relation acts by a permutation of order 1 or 2 on the fibers. Let $\sigma \in S_n$ then the vertices in the loop

$$\sigma - \sigma s_i - \sigma s_i s_{i+1} - \sigma s_i s_{i+1} s_i - \sigma s_{i+1} s_i - \sigma s_{i+1} - \sigma \tag{8}$$

belong to the same inverse descent class as σ if and only if $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ satisfy

$$|\sigma_i - \sigma_{i+1}| \geq 2,\ |\sigma_{i+1} - \sigma_{i+2}| \geq 2,\ |\sigma_i - \sigma_{i+2}| \geq 2.$$

In the following we assume that σ and i satisfy these inequalities. Let $\sigma = \pi\rho$ be a factorization with $(\pi, \rho) \in \mathcal{Y}_I \times \mathcal{Y}_J$. There are several cases:

- if $|\rho_i - \rho_{i+1}| \geq 2$, $|\rho_{i+1} - \rho_{i+2}| \geq 2$, $|\rho_i - \rho_{i+2}| \geq 2$ then the sequence

  $$\rho - \rho s_i - \rho s_i s_{i+1} - \rho s_i s_{i+1} s_i - \rho s_{i+1} s_i - \rho s_{i+1} - \rho$$

  is a loop in $\mathcal{Y}_J$ and the loop (8) lifts to the loop

  $$(\pi, \rho) - (\pi, \rho s_i) - (\pi, \rho s_i s_{i+1}) - (\pi, \rho s_i s_{i+1} s_i) - (\pi, \rho s_{i+1} s_i) - (\pi, \rho s_{i+1}) - (\pi, \rho)$$

  in $\mathcal{Z}_{IJK}$. It follows that this loop has a trivial action.

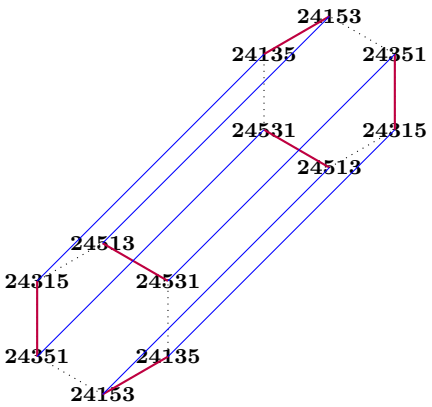


FIGURE 2. Lifting a braid relation

- if exactly two of the values $\rho_i, \rho_{i+1}, \rho_{i+2}$ are adjacent then again (8) lifts to a loop in $\mathcal{Z}_{IJK}$. For example one can check that for $\rho_i = j, \rho_{i+1} = j+1, \rho_{i+2} = k$ with $|j-k|, |j+1-k| \geq 2$, the loop (8) lifts to

$$(\pi,\rho) - (\pi s_j,\rho) - (\pi s_j,\rho s_{i+1}) - (\pi s_j,\rho s_{i+1}s_i) - (\pi,\rho s_{i+1}s_i) - (\pi,\rho s_{i+1}) - (\pi,\rho)$$

  which is a loop in $\mathcal{Z}_{IJK}$ and the action is again trivial. The other cases are similar and left to the reader.
- if $(\rho_i, \rho_{i+1}, \rho_{i+2}) = (j, j+1, j+2)$ or $(j+2, j+1, j)$ then one can check that the loop (8) lifts either to the loop

$$(\pi,\rho) - (\pi s_j,\rho) - (\pi s_j s_{j+1},\rho) - (\pi s_j s_{j+1} s_j,\rho) - (\pi s_{j+1} s_j,\rho) - (\pi s_{j+1},\rho) - (\pi,\rho)$$

  or to its inverse. Again the action is trivial.
- finally if

(9) $(\rho_i, \rho_{i+1}, \rho_{i+2}) = (j+1, j, j+2)$ or $(j+1, j+2, j)$ or $(j, j+2, j+1)$ or $(j+2, j, j+1)$

  then (8) lifts to a path which is not a loop, for example if $(\rho_i, \rho_{i+1}, \rho_{i+2}) = (j+1, j, j+2)$ then, using that $\rho s_{i+1}\rho^{-1} = s_j s_{j+1} s_j$ one checks that the lift is

$$(\pi,\rho) - (\pi s_j,\rho) - (\pi s_j,\rho s_{i+1}) - (\pi s_j s_{j+1},\rho s_{i+1}) \\ - (\pi s_j s_{j+1},\rho) - (\pi s_j s_{j+1} s_j,\rho) - (\pi s_j s_{j+1} s_j,\rho s_{i+1}),$$

  which is not a loop but ends in another factorization of σ. Starting from this new factorization, the lift of the loop (8) gives now

$$(\pi s_j s_{j+1} s_j,\rho s_{i+1}) - (\pi s_{j+1} s_j,\rho s_{i+1}) - (\pi s_{j+1} s_j,\rho) - (\pi s_j,\rho) - (\pi s_j,\rho s_{i+1}) - (\pi,\rho s_{i+1}) - (\pi,\rho),$$

  which gets back to the original factorization. Thus we see that the action of the loop (8) has order 2. One can check that the other values in (9) also yield an action of order 2.

Coming back to Figure 1 we extract from it an explicit example of this action in Figure 2 which shows the loop projecting on the loop

$$24153 - 24135 - 24315 - 24351 - 24531 - 24513 - 24153$$

with multiplicity two.

Note that for any K such that the class $\mathcal{Y}_K$ has no loop of type (3) (e.g. the classes $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$, or $\mathcal{Y}_{3,4}$ in the examples of Section 6) the graph $\Pi^{-1}(\mathcal{Y}_K)$ consists of a_{IJK} connected components, all isomorphic to $\mathcal{Y}_K$.

8. The case of Coxeter groups

We refer to [1] for basic facts about Coxeter groups. Let (W, S) be a Coxeter system, where W is a finite Coxeter group and S a system of simple reflections. The length function with respect to S and the right weak order are defined as in the case of symmetric groups. Recall the fundamental *exchange property* (see [1, Ch. IV, §1, Lemme 3]).

Lemma 8.1. *Let $w \in W$ and $w = s_1 s_2 \dots s_k$ be a reduced decomposition then if $s \in S$ and $\ell(sw) = \ell(w) - 1$ there exists $i \in \{1, 2, \dots, k\}$ such that $sw = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_k$ (this is a reduced decomposition of sw).*

For $w \in W$ its inverse descent set is the set of all $s \in S$ such that $\ell(sw) = \ell(w) - 1$, denoted $\mathrm{IDes}(w)$ (again, descents of w are inverse descent of w^{-1}). For $I \subset S$ we denote again by $\mathcal{Y}_I$ the set of $w \in W$ whose inverse descent set is I and $Y_I = \sum_{w \in \mathcal{Y}_I} w$ is the sum in the group algebra of W. The analogue of Proposition 3.1 holds unchanged while for Lemma 3.2 the analogous result will follow from the following slightly more precise result.

Lemma 8.2. *Let $w \in W$ and $s \in S$ be such that $\ell(ws) = \ell(w) + 1$ then*

1. $\mathrm{IDes}(ws) = \mathrm{IDes}(w)$ *if* $wsw^{-1} \notin S$
2. $\mathrm{IDes}(ws) = \mathrm{IDes}(w) \cup \{wsw^{-1}\} \neq \mathrm{IDes}(w)$ *if* $wsw^{-1} \in S$.

Proof. If $w = s_1 s_2 \dots s_k$ is a reduced decomposition of w then $ws = s_1 s_2 \dots s_k s$ is a reduced decomposition of ws. Suppose that $s' \in \mathrm{IDes}(w)$ then there exists a reduced decomposition of w starting with s' therefore also for ws thus s' is an inverse descent of ws. It follows that $\mathrm{IDes}(w) \subset \mathrm{IDes}(ws)$. Conversely let s' be an inverse descent of ws and $ws = s_1 s_2 \dots s_k s$ a reduced decomposition, by the exchange property, Lemma 8.1, $s'ws$ has a reduced decomposition obtained by deleting a term in $s_1 \dots s_k s$. If the deleted term is s_i with $1 \leq i \leq k$ then $s'w = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_k$ therefore $s' \in \mathrm{IDes}(w)$. If the deleted term is the last one, then $s'ws = w$ and $s' = wsw^{-1}$. In this case $s' \in \mathrm{IDes}(ws)$ but $s' \notin \mathrm{IDes}(w)$ since $s'w = ws$ has length $\ell(w)+1$ thus we have $\mathrm{IDes}(ws) = \mathrm{IDes}(w) \cup \{s'\} \neq \mathrm{IDes}(w)$. This happens if and only if $wsw^{-1} \in S$. The lemma follows. □

Corollary 8.3. *Let $w \in W$ and $s \in S$ then $\mathrm{IDes}(ws) = \mathrm{IDes}(w)$ if and only if $wsw^{-1} \notin S$.*

Finally the analogue of Proposition 3.3 is the following. The proof follows closely the case of the symmetric group, using the above corollary instead of Lemma 3.2 and is left to the reader.

Proposition 8.4. *Let $w, u, v \in W$ and $s \in S$ be such that*

$$w = uv \quad \text{and} \quad \mathrm{IDes}(ws) = \mathrm{IDes}(w)$$

then there exists a unique factorization $ws = u'v'$ where either:

1. *$u = u'$ while $v^{-1}v' \in S$ and $\mathrm{IDes}(v) = \mathrm{IDes}(v')$ or*
2. *$v = v'$ while $u^{-1}u' \in S$ and $\mathrm{IDes}(u) = \mathrm{IDes}(u')$.*

We can now define the sets $\mathcal{Z}_{IJK}$ exactly as in (2) and show that the product maps

$$\mathcal{Z}_{IJK} \to \mathcal{Y}_K$$

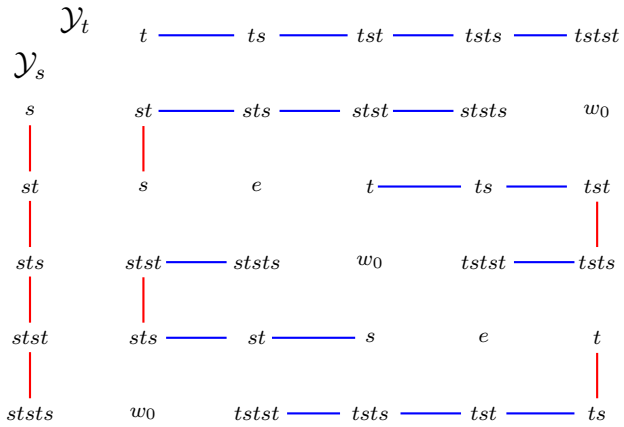
as in (3), are covering graphs. The rest of the argument to prove the existence of the inverse descent algebra is the same.

Remark 8.5. The proofs of Lemmas 8.2 and 8.4 use only the exchange property and not the fact that W is a finite group, therefore they hold also in infinite Coxeter groups. However for infinite Coxeter groups the inverse descent (or descent) algebra in general is not well defined because of infinite multiplicities: some a_{IJK} may be infinite.

8.1. **A final example in a dihedral group.** We consider a finite dihedral group with simple generators s, t satisfying the defining relations $s^2 = t^2 = (st)^n = e$ for some $n \geq 2$. There are four inverse descent classes, namely $\mathcal{Y}_\emptyset = e$, $\mathcal{Y}_s = \{s, st, sts, \ldots\}$ $\mathcal{Y}_t = \{t, ts, tst, \ldots\}$ with $|\mathcal{Y}_s| = |\mathcal{Y}_t| = n - 1$ and $\mathcal{Y}_{s,t} = \{w_0\}$, where w_0 is the longest element:

$$w_0 = (st)^{n/2} = (ts)^{n/2}, \qquad \text{for } n \text{ even},$$
$$w_0 = t(st)^{(n-1)/2} = s(ts)^{(n-1)/2}, \qquad \text{for } n \text{ odd}.$$

We show the graph corresponding to the product $Y_s Y_t$ for $n = 6$ so that $w_0 = ststst = tstsst$.



One has

$$Y_s Y_t = 2Y_\emptyset + 3Y_{s,t} + 2Y_s + 2Y_t.$$

References

[1] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres IV à VI.* Actualités Sci. Indust., No. 1337, Hermann, Paris, 1968.
[2] L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra* **41**(2) (1976), 255–264.
[3] E. H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York–Toronto–London, 1966.

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