Séminaire Lotharingien de Combinatoire **93** (2025), Article B93b

FACTORIAL EULER–SEIDEL MATRIX: REFINEMENT AND q-ANALOGUE

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Abstract. We determine the Riordan matrix associated to Eulerian numbers and establish a formula defining Eulerian polynomials as well. We refine the factorial Euler–Seidel matrix considering the excedance distribution over permutations and derangements. We introduce the so called (n, k)-permutations. Combining with the number of cycles, we establish exponential generating functions and their q-analogues as well.

1. Reminder and introduction

Clarke et al. [2], Dumont and Randrianarivony [4], and Rakotondrajao [9] studied Euler's difference table $(d_n^k)_{n,k\geq 0}$, also called difference factorial numbers, which are given by

$$\begin{cases} d_n^n = n!, \\ d_n^k = d_n^{k+1} - d_{n-1}^k, \quad \text{for } 1 \leq k \leq n-1. \end{cases} \tag{1.1}$$

Their matrix is presented in Table 1.

Table 1. Euler's difference matrix $(d_n^k)_{n,k\geq 0}$

	d_n^k					
$n\backslash k$	0	1	2	3	4	$\cdots$
0	0!					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2020 *Mathematics Subject Classification.* 05A05; 05A15.
Key words and phrases. Excedance distribution; Euler–Seidel matrix; Eulerian polynomial; Riordan matrix; q-analogue; generating function.

Dumont [5] reintroduced the Euler–Seidel matrix associated to a given sequence $(a_n)_{n\geq 0}$. It is an infinite matrix and is determined recursively by the formula

$$\begin{cases} a_n^0 = a_n, & \text{for } n \geq 0, \\ a_n^k = a_n^{k-1} + a_{n+1}^{k-1}, & \text{for } n \geq 0,\ k \geq 1. \end{cases} \tag{1.2}$$

The sequence $(a_n^0)_{n\geq 0}$ is called the *initial sequence*, and $(a_0^n)_{n\geq 0}$ is called the *final sequence.*

Theorem 1.1 (Dumont [5]). *The relation between any entry and the initial sequence is determined by*

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0, \quad \text{for } n \geq 0,\ k \geq 0. \tag{1.3}$$

Firengiz and Dil [6] reconsidered the Euler–Seidel matrix, by introducing the parameters p and q as follows:

$$\begin{aligned} a_n^k &= a_n^k(p,q) \\ &= p\ a_n^{k-1} + q\ a_{n+1}^{k-1}, \quad \text{for } n \geq 0,\ k \geq 1. \end{aligned} \tag{1.4}$$

Theorem 1.2 (Firengiz and Dil [6]). *The relation between any entry and the initial sequence is determined by*

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} p^{k-i} q^i a_{n+i}^0. \tag{1.5}$$

Theorem 1.3 (Firengiz and Dil [6]). *The exponential generating function of* $\left(a_n^k\right)_{k,n\geq 0}$ *has the closed form*

$$\sum_{n\geq 0}\sum_{k\geq 0} a_n^k \frac{u^k}{k!}\ \frac{t^n}{n!} = \exp(pu)\, A(t+qu), \ \text{where } A(t) = \sum_{n\geq 0} a_n^0 \frac{t^n}{n!}. \tag{1.6}$$

We study the factorial Euler–Seidel matrix deduced from Euler's difference table as follows. The initial sequence is $(n!)_{n\geq 0}$, and the infinite matrix is determined recursively by

$$\begin{cases} a_n^0 = n!, \\ a_n^k = -a_n^{k-1} + a_{n+1}^{k-1}, & \text{for } n \geq 0,\ k \geq 1. \end{cases} \tag{1.7}$$

Definition 1.1 (Shapiro [12], Sprugnoli [14]). Consider an infinite matrix $R = (r_{n,k})_{n,k\geq 0}$ with complex coefficients and two formal power series $g(x) = \sum_{k=0}^{\infty} g_k x^k$, $f(x) = \sum_{k=1}^{\infty} f_k x^k$ with $g_0 = 1,\ f_1 \neq 0$. Let $R_k(x) = \sum_{n=0}^{\infty} r_{n,k} x^n$ be the formal power series of the k-th column of R. Then R is called a Riordan matrix if

$$R_k(x) = g(x)\big[f(x)\big]^k. \tag{1.8}$$

R is an infinite lower-triangular matrix which we denote by a pair $R = \Big(g(x), f(x)\Big)$. Note that the identity $I = (1, x)$ is a Riordan matrix.

Remark 1.1. If we have $g(x) = \sum_{k=0}^{\infty} g_k \frac{x^k}{k!}$ and $f(x) = \sum_{k=1}^{\infty} f_k \frac{x^k}{k!}$, with $g_0 = 1$ and $f_1 \neq 0$, then $R_k(x) = g(x) \frac{\left[f(x)\right]^k}{k!}$. We denote the matrix by $R = \Big[g(x), f(x)\Big]$.

Theorem 1.4 (Shapiro [12], Shapiro et al. [13]). *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences. Set $a(x)=\sum_{k=0}^{\infty} a_k x^k$ and $b(x)=\sum_{k=0}^{\infty} b_k x^k$. Then we have*

$$\Big(g(x), f(x)\Big) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} \iff g(x)a\big(f(x)\big) = b(x). \tag{1.9}$$

Theorem 1.5 (Shapiro [12], Shapiro et al. [13]). *For any Riordan matrix $\Big(g(x), f(x)\Big)$, its inverse R^{-1} is given by*

$$\begin{aligned} R^{-1} &= \Big(g(x), f(x)\Big)^{-1} \\ &= \left(\frac{1}{g\Big(\bar{f}(x)\Big)}, \bar{f}(x)\right), \end{aligned} \tag{1.10}$$

where $\bar{f}\Big(f(x)\Big) = f\Big(\bar{f}(x)\Big) = x$.

Let us write $[n]=\{1,2,\ldots,n\}$. We say that an integer $i\in[n]$ is a fixed point for a permutation σ if $\sigma(i)=i$. We denote by $\mathrm{FIX}(\sigma)$ the set of fixed points of σ. A derangement is a permutation without fixed points, that is, a permutation σ such that $\mathrm{FIX}(\sigma)=\emptyset$. We denote by D_n the set of all derangements of the symmetric group S_n. We say that a permutation σ is an (n,k)-permutation if the length of σ is equal to $(n+k)$ and $\mathrm{FIX}(\sigma)\subset[n]$.

This class of (n,k)-permutations should not be confused with the "n-fixed-points-permutations" introduced by Rakotondrajao [10]. An n-fixed-points-permutation is an (n,k)-permutation. For example, $(1)(2,3)(4)(5,6)$ is a $(4,2)$-permutation, but it is not a 4-fixed-points-permutation.

We denote by $\mathcal{E}_n^k$ the set of (n,k)-permutations. Note that the coefficient of the n-th column and k-th row of the classical Euler–Seidel matrix is the cardinality of $\mathcal{E}_n^k$. Since $n+1$ is either a fixed point or not for a permutation $\sigma\in\mathcal{E}_{n+1}^k$: if $n+1$ is, then $\sigma\in\mathcal{E}_n^k$; otherwise $\sigma\in\mathcal{E}_n^{k+1}$.

We say that an integer i is an excedance for σ if $\sigma(i)>i$. We denote the number of excedances by $e(\sigma)$ and the number of cycles of σ by $c(\sigma)$.
The numbers $E_{n,k}$ count the number of permutations of size n having k excedances. They are given by the formula

$$E_{n,k} = \begin{cases} 1, & \text{if } n=0,\ k=0, \\ (n-k)E_{n-1,k-1}+(k+1)E_{n-1,k}, & \text{if } 0<k\leq n, \\ 0, & \textit{otherwise}. \end{cases} \tag{1.11}$$

The Eulerian numbers $E_{n,k}$ are classical numbers in mathematics. They can be found in many places in the literature (Brenti [1], Comtet [3], Foata and Schützenberger [7], Roselle [11], Stanley [15], Worpitzky [16]).

We are interested in refining the factorial Euler–Seidel matrix using the excedance distribution over permutations. The Eulerian polynomials $E_n(x)=\sum_{i=0}^{n}E_{n,i}x^i=\sum_{i\geq 0}E_{n,i}x^i$ become the initial sequence. We have

$$\begin{cases} e_n^0(x) = E_n(x), & \text{for } n\geq 0, \\ e_n^k(x) = -e_n^{k-1}(x)+e_{n+1}^{k-1}(x), & \text{for } n\geq 0,\ k\geq 1, \end{cases} \tag{1.12}$$

and their matrix is presented in Table 2.

Table 2. Refined factorial Euler–Seidel matrix

	$e_n^k(x)$				
$k\backslash n$	0	1	2	3	$\cdots$
0	1	1	$1+x$	$1+4x+x^2$	$\cdots$
1	0	x	$3x+x^2$	$\cdots$	$\cdots$
2	x	$2x+x^2$	$\cdots$	$\cdots$	$\cdots$
3	$x+x^2$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The Hurwitz generating function of the classical Eulerian polynomials [7] is

$$\begin{aligned} E(x,t) &= \sum_{n\geq 0}\sum_{k\geq 0} E_{n,k}x^k\frac{t^n}{n!} \\ &= \frac{x-1}{x-\exp\left(t(x-1)\right)}. \end{aligned} \tag{1.13}$$

We define the q-analogue of the refined factorial Euler–Seidel matrix combining the excedance distribution and the number of cycles. We write

$$e_n^k(x) = \sum_{\sigma\in\mathcal{E}_n^k} x^{e(\sigma)}, \tag{1.14}$$

$$e_n^k(x,q) = \sum_{\sigma\in\mathcal{E}_n^k} x^{e(\sigma)}\ q^{c(\sigma)}. \tag{1.15}$$

Mantaci and Rakotondrajao [8] extensively studied the excedance distribution over derangements. The number $a_{n,k}$ counts derangements over $[n]$ having k excedances,

$$a_{n,k} = |\{\delta\in D_n \mid \delta \text{ has } k \text{ excedances}\}|. \tag{1.16}$$

Let us write $A_k(x)=\sum_{i=0}^{k} a_{k,i}x^i$, and let $A_k(x,q)$ be its q-analogue.

Remark 1.2. Note that a derangement of D_n is a $(0,n)$-permutation, and a permutation of S_n is a $(n,0)$-permutation.

Let us denote by $E_n(x,q)$ the q-analogue of the Eulerian polynomials $E_n(x)=\sum_{i=0}^{n} E_{n,i}x^i$ given by

$$E_n(x,q) = \sum_{\sigma\in S_n} x^{e(\sigma)}q^{c(\sigma)}. \tag{1.17}$$

Theorem 1.6 (Brenti [1]). *The polynomials $E_n(x,q)$ satisfy the recursion relation*

$$E_n(x,q) = \left(x(n-1)+q\right)E_{n-1}(x,q) + x(1-x)\frac{d}{dx}E_{n-1}(x,q) \tag{1.18}$$

with initial condition $E_0(x,q)=1$.

Theorem 1.7 (Brenti [1]). *The exponential generating function of $E_n(x,q)$ is given by*

$$E(x,t,q) = \sum_{n\geq 0}\frac{E_n(x,q)}{n!}t^n = \left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q,\ E_0(x,q)=1. \tag{1.19}$$

2. Riordan matrix representation of Worpitzky identity

For a fixed integer n with $n \geq 0$, let us consider the infinite matrix $R_n = \left(r_{k,\ell}^{(n)}\right)_{k,\ell \geq 0}$ given by

$$R_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2^n & 1 & 0 & 0 & 0 & \cdots \\ 3^n & 2^n & 1 & 0 & 0 & \cdots \\ 4^n & 3^n & 2^n & 1 & 0 & \cdots \\ 5^n & 4^n & 3^n & 2^n & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1}$$

Theorem 2.1. *R_n is a Riordan matrix and*

$$R_n = \left(\frac{E_n(x)}{(1-x)^{n+1}}, x\right). \tag{2.2}$$

Its inverse R_n^{-1} is given by

$$R_n^{-1} = \left(\frac{(1-x)^{n+1}}{E_n(x)}, x\right). \tag{2.3}$$

Proof. We have

$$\frac{E_n(x)}{(1-x)^{n+1}} = \sum_{k \geq 0} (k+1)^n x^k.$$

Let $U = R_n$. If $U = (g(x), f(x))$, from Definition 1.1, we have Equation (2.2), that is

$$\begin{cases} U_0(x) = g(x) = \frac{E_n(x)}{(1-x)^{n+1}}, \\ U_1(x) = g(x)f(x) = x\frac{E_n(x)}{(1-x)^{n+1}}, \\ \vdots \\ U_i(x) = g(x)\left(f(x)\right)^i = x^i \frac{E_n(x)}{(1-x)^{n+1}}, \\ \vdots \end{cases}$$

Then $U^{-1} = \left(\frac{(1-x)^{n+1}}{E_n(x)}, x\right) = R_n^{-1}$ (cf. Theorem 1.5). $\square$

For a fixed integer n with $n \geq 0$, let us consider the Eulerian number vector E_n and the signed binomial coefficient vector B_n given by

$$E_n = \begin{bmatrix} E_{n,0} \\ E_{n,1} \\ \vdots \\ E_{n,n+1} \end{bmatrix} \quad \text{and} \quad B_n = \begin{bmatrix} 1 \\ -\binom{n+1}{1} \\ \vdots \\ (-1)^i \binom{n+1}{i} \\ \vdots \\ (-1)^{n+1}\binom{n+1}{n+1} \end{bmatrix}.$$

Theorem 2.2. *For a fixed integer n with $n \geq 0$, we have*

$$\begin{bmatrix} E_{n,0} \\ E_{n,1} \\ E_{n,2} \\ E_{n,3} \\ E_{n,4} \\ \vdots \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2^n & 1 & 0 & 0 & 0 & \cdots \\ 3^n & 2^n & 1 & 0 & 0 & \cdots \\ 4^n & 3^n & 2^n & 1 & 0 & \cdots \\ 5^n & 4^n & 3^n & 2^n & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} 1 \\ -\binom{n+1}{1} \\ \binom{n+1}{2} \\ -\binom{n+1}{3} \\ \binom{n+1}{4} \\ \vdots \end{bmatrix}. \tag{2.4}$$

Proof. Equation (2.4) is a matrix representation of the Worpitzky [16] formula

$$E_{n,k} = \sum_{i=0}^{k} (-1)^i \, (k+1-i)^n \binom{n+1}{i}. \tag{2.5}$$

□

Theorem 2.3. *Let us consider the infinite matrix $R = (r_{k,\ell})_{k,\ell \geq 0}$ given by*

$$r_{k,\ell} = \begin{cases} (k-\ell+1)^p, & \text{if } k \geq \ell \text{ and } \frac{2+(5+p)p}{2} \geq k \geq \frac{(3+p)p}{2}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{with } p = \left\lfloor \frac{-3+\sqrt{9+8\ell}}{2} \right\rfloor.$$

We have

$$\begin{pmatrix} E_0 & & & & \\ & E_1 & & 0 & \\ & & E_2 & & \\ & 0 & & E_3 & \\ & & & & \ddots \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} B_0 & & & & \\ & B_1 & & 0 & \\ & & B_2 & & \\ & 0 & & B_3 & \\ & & & & \ddots \end{pmatrix}. \tag{2.6}$$

Proof. For a fixed integer n with $n \geq 0$, let us consider the matrix $\bar{R}_n = \left(\bar{r}_{k,\ell}^{(n)}\right)_{k,\ell \geq 0}$ of the first $(n+2)$ rows and $(n+2)$ columns extracted from R_n in Equation (2.1), that is,

$$\bar{R}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2^n & 1 & 0 & \cdots & 0 \\ 3^n & 2^n & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+2)^n & (n+1)^n & n^n & \cdots & 1 \end{pmatrix}.$$

From Theorem 2.2, we have $E_n = \bar{R}_n \, B_n$. Hence, we have

$$\begin{pmatrix} E_0 & & & & \\ & E_1 & & 0 & \\ & & E_2 & & \\ & 0 & & E_3 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} \bar{R}_0 & & & & \\ & \bar{R}_1 & & 0 & \\ & & \bar{R}_2 & & \\ & 0 & & \bar{R}_3 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} B_0 & & & & \\ & B_1 & & 0 & \\ & & B_2 & & \\ & 0 & & B_3 & \\ & & & & \ddots \end{pmatrix}.$$

That is,

$$\begin{pmatrix} \boxed{\begin{matrix} E_{0,0} \\ 0 \end{matrix}} & 0 & \cdots \\ 0 & \boxed{\begin{matrix} E_{1,0} \\ E_{1,1} \\ 0 \end{matrix}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} & 0 & \cdots \\ 0 & \boxed{\begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{matrix}} & \cdots \\ \vdots & \vdots & \vdots \quad \ddots \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} \binom{1}{0} \\ -\binom{1}{1} \end{matrix}} & 0 & \cdots \\ 0 & \boxed{\begin{matrix} \binom{2}{0} \\ -\binom{2}{1} \\ \binom{2}{2} \end{matrix}} & \cdots \\ \vdots & \vdots & \vdots \quad \ddots \end{pmatrix}.$$

Identifying the coefficients, we get

$$r_{k,\ell} = \begin{cases} (k-\ell+1)^p, & \text{if } k \geq \ell \text{ and } \frac{2+(5+p)p}{2} \geq k \geq \frac{(3+p)p}{2}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{with } p = \left\lfloor \frac{-3+\sqrt{9+8\ell}}{2} \right\rfloor. \qquad \square$$

3. Refined factorial Euler–Seidel matrix and q-analogue

Let us consider the refined factorial Euler–Seidel matrix coefficient $e_n^k(x)$.

Theorem 3.1. *For any integers $n \geq 0$ and $k \geq 0$, we have*

$$e_n^k(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E_{n+i}(x). \tag{3.1}$$

Proof. Setting $p = -1$ and $q = 1$ in Theorem 1.2, we get

$$\begin{aligned} e_n^k(x) &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} e_{n+i}^0(x) \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E_{n+i}(x). \end{aligned} \qquad \square$$

Corollary 3.1. *For any integer $k \geq 0$, we have*

$$e_0^k(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E_i(x). \tag{3.2}$$

Proof. Set $n = 0$ in Theorem 3.1. $\square$

Remark 3.1. We have $e_0^k(x) = A_k(x) = \sum_{i=0}^{k} a_{k,i} x^i$.

Theorem 3.2. *The polynomial $e_n^k(x)$ is the polynomial of the excedance distribution over $\mathcal{E}_n^k$.*

Proof. From the construction of set $\mathcal{E}_{n+1}^{k-1}$, we have

$$e_{n+1}^{k-1}(x) = e_n^k(x) + e_n^{k-1}(x).$$

This follows from the interpretation of $e_n^k(x)$ as the excedance distribution over $\mathcal{E}_{n+1}^{k-1}$ where $n+1$ is not a fixed point, and $e_n^{k-1}(x)$ as the excedance distribution over $\mathcal{E}_{n+1}^{k-1}$ where $n+1$ is a fixed point (Equation (1.14)). $\square$

Theorem 3.3. *The exponential generating function of $e_n^k(x)$ has the closed form*

$$F(x,u,t) = \sum_{n\geq 0} \sum_{k\geq 0} e_n^k(x) \frac{u^k}{k!} \frac{t^n}{n!} = \frac{(x-1)\exp(-u)}{x - \exp\left((t+u)(x-1)\right)}. \tag{3.3}$$

Proof. From Equation (1.12) and Theorem 1.3, we have

$$\sum_{n\geq 0}\sum_{k\geq 0} e_n^k(x)\frac{u^k}{k!}\frac{t^n}{n!} = exp(-u)\sum_{n\geq 0} E_n(x)\frac{(t+u)^n}{n!}.$$

From Equation (1.13), we have

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{x-1}{x-\exp\left(t(x-1)\right)}.$$

Hence,

$$\sum_{n\geq 0}\sum_{k\geq 0} e_n^k(x)\frac{u^k}{k!}\frac{t^n}{n!} = \frac{(x-1)\exp(-u)}{x-\exp\left((t+u)(x-1)\right)}.$$

□

Corollary 3.2. *The exponential generating function of $A_k(x)$ has the closed form*

$$A(x,u) = F(x,u,0) = \frac{(x-1)\exp(-u)}{x-\exp\left(u(x-1)\right)}. \tag{3.4}$$

Moreover, we can express the initial sequence $(e_n^0(x))_{n\geq 0}$ as an infinite matrix. The coefficient $[x^\ell]e_n^0(x)$ is the classical Eulerian number $E_{n,\ell}$, and Equation (1.12) becomes

$$[x^\ell]e_n^k(x) = -[x^\ell]e_n^{k-1}(x) + [x^\ell]e_{n+1}^{k-1}(x).$$

That is,

$$\begin{cases} [x^\ell]e_n^0(x) = E_{n,\ell}, & \text{for } n\geq 0,\\ [x^\ell]e_n^k(x) = -[x^\ell]e_n^{k-1}(x) + [x^\ell]e_{n+1}^{k-1}(x), & \text{for } n\geq 0,\ k\geq 1. \end{cases} \tag{3.5}$$

Theorem 3.4. *For fixed integers k and ℓ, the coefficient of x^ℓ in $e_0^k(x)$ is*

$$[x^\ell]e_0^k(x) = a_{k,\ell} = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}E_{i,\ell}, \quad \textit{for } k\geq 0 \textit{ and } l\geq 0. \tag{3.6}$$

Proof. We have

$$\begin{aligned} a_{k,\ell} &= [x^\ell]e_0^k(x)\\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left([x^\ell]e_i^0(x)\right),\\ &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}E(i,\ell). \end{aligned}$$

Here, we obtained the second line by setting $p=-1$ and $q=1$ in Equation (1.5). □

Theorem 3.5. *We have*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 0 & 0 & \cdots\\ 0 & 1 & 0 & 0 & 0 & \cdots\\ 0 & 1 & 1 & 0 & 0 & \cdots\\ 0 & 1 & 7 & 1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\\ -1 & 1 & 0 & 0 & 0 & \cdots\\ 1 & -2 & 1 & 0 & 0 & \cdots\\ -1 & 3 & -3 & 1 & 0 & \cdots\\ 1 & -4 & 6 & -4 & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\\ 1 & 0 & 0 & 0 & 0 & \cdots\\ 1 & 1 & 0 & 0 & 0 & \cdots\\ 1 & 4 & 1 & 0 & 0 & \cdots\\ 1 & 11 & 11 & 1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.7}$$

That is,

$$(a_{k,\ell})_{k,\ell\geq 0} = \Big[\exp(-t), t\Big]\ (E_{n,\ell})_{n,\ell\geq 0}\,, \tag{3.8}$$

with $S = \Big[\exp(-t), t\Big]$ *the Riordan matrix associated to the refined factorial Euler–Seidel matrix.*

Theorem 3.6. *For all integers $k \geq 0$ and $l \geq 0$, we have*

$$a_{k,\ell} = \sum_{i=0}^{k}\sum_{j=0}^{\ell}(-1)^{k-i+j}\binom{k}{i}\binom{i+1}{j}(\ell+1-j)^i, \quad \textit{for } k\geq 0,\ \ell\geq 0. \tag{3.9}$$

Proof. Using the Worpitzky identity in Equation (2.5) with $n = i$ and $k = \ell$ and Theorem 3.4, we have

$$\begin{aligned} a_{k,\ell} &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{j=0}^{\ell}\left((-1)^j(\ell+1-j)^i\right)\binom{i+1}{j} \\ &= \sum_{i=0}^{k}\sum_{j=0}^{\ell}(-1)^{k-i+j}\binom{k}{i}\binom{i+1}{j}(\ell+1-j)^i. \end{aligned}$$

□

Theorem 3.7. *We have*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.10}$$

That is,

$$(E_{k,\ell})_{k,\ell\geq 0} = \Big[\exp(t), t\Big]\ (a_{n,\ell})_{n,\ell\geq 0}\,,\ \textit{with } \Big[\exp(t), t\Big] = S^{-1}. \tag{3.11}$$

Proof. This is obtained from Equation (3.8) and Theorem 1.5. □

Corollary 3.3. *We have*

$$(E_{k,\ell})_{k,\ell\geq 0} = \left(\binom{k}{n}\right)_{k,n\geq 0}\ (a_{n,\ell})_{n,\ell\geq 0}\,. \tag{3.12}$$

Corollary 3.4. *For a fixed integer $\ell \geq 0$, the sequence $(E(n,\ell))_{n\geq 0}$ of a row of the Eulerian number matrix is given by*

$$\begin{bmatrix} E_{0,\ell} \\ E_{1,\ell} \\ E_{2,\ell} \\ E_{3,\ell} \\ E_{4,\ell} \\ \vdots \end{bmatrix} = \Big[\exp(t), t\Big] \begin{bmatrix} a_{0,\ell} \\ a_{1,\ell} \\ a_{2,\ell} \\ a_{3,\ell} \\ a_{4,\ell} \\ \vdots \end{bmatrix}. \tag{3.13}$$

Theorem 3.8. *The Euler–Seidel matrix preserves the excedance distribution.*

Proof. This process is derived from the construction of the $(n+1,k-1)$-permutation. Consider first the case where $n+1$ is a fixed point. By removing $n+1$ from the permutation and reordering the sequence, that is, decreasing by 1 all integers greater than $n+1$ in the remaining permutation, we obtain an $(n,k-1)$-permutation. Conversly, if $n+1$ is not a fixed point, then the $(n+1,k-1)$-permutation is, by definition, an (n,k)-permutation. This construction leads directly to the recurrence relation

$$e_n^k(x) = -e_n^{k-1}(x) + e_{n+1}^{k-1}(x). \qquad \square$$

Theorem 3.9. *For a fixed integer $\ell \geq 0$, the exponential generating function of $[x^\ell]e_n^k(x)$, that is, the exponential generating function of all (n,k)-permutations with ℓ excedances, is given by*

$$\begin{aligned} e_\ell(t,u) &= \sum_{n\geq 0}\sum_{k\geq 0}[x^\ell]e_n^k(x)\frac{u^k}{k!}\frac{t^n}{n!} \\ &= \sum_{i=0}^{\ell}\frac{(-1)^i}{i!}\Big((\ell-i+1)(t+u)\Big)^{i-1}\Big((\ell-i+1)(t+u)+i\Big)\exp\Big(t+(\ell-i)(t+u)\Big). \end{aligned} \tag{3.14}$$

Proof. For fixed integers ℓ and n, from the Worpitzky identity in Equation (2.5), we consider the initial term of the sequence

$$E_{n,\ell} = \sum_{i=0}^{\ell}(-1)^i(\ell+1-i)^n\binom{n+1}{i}.$$

Hence,

$$\begin{aligned} \sum_{n\geq 0}E_{n,\ell}\frac{t^n}{n!} &= \sum_{n\geq i-1}\sum_{i=0}^{\ell}(-1)^i(\ell+1-i)^n\frac{(n+1)t^n}{(n-i+1)!\,i!} \\ &= \sum_{i=0}^{\ell}\Bigg[\sum_{n\geq i}\frac{(-1)^i}{i!}\frac{((\ell+1-i)\ t)^{n-i+1}}{(n-i)!} \\ &\qquad + \sum_{n\geq i-1}\frac{(-1)^i\,i}{i!}\frac{((\ell+1-i)\ t)^{n-i+1}}{(n-i+1)!}\Bigg]\Big(t(\ell+1-i)\Big)^{i-1} \\ &= \sum_{i=0}^{\ell}\frac{(-1)^i}{i!}\Big(t(\ell+1-i)\Big)^{i-1}\Big(t\ (\ell+1-i)+i\Big)\exp\Big(t(\ell+1-i)\Big) \\ &= \sum_{i=0}^{\ell}\frac{(-1)^i}{i!}\Big(t(\ell+1-i)\Big)^{i-1}\Big(t\ (\ell+1)+i(1-t)\Big)\exp\Big(t\ (\ell+1-i)\Big). \end{aligned} \tag{3.15}$$

From Equations (1.6) and (3.5), setting $p=-1$ and $q=1$, we obtain

$$\sum_{n\geq 0}\sum_{k\geq 0}[x^\ell]e_n^k(x)\frac{u^k}{k!}\frac{t^n}{n!} = \exp(-u)\sum_{n\geq 0}E_{n,\ell}\frac{(t+u)^n}{n!}.$$

By Equation (3.15), we have

$$\begin{aligned} \sum_{n\geq 0}\sum_{k\geq 0}[x^\ell]e_n^k(x)\frac{u^k}{k!}\frac{t^n}{n!} &= \exp(-u)\sum_{i=0}^{\ell}\frac{(-1)^i}{i!}\Big((\ell+1-i)(t+u)\Big)^{i-1} \\ &\quad \cdot\Big((t+u)(\ell+1)+i(1-t-u)\Big)\exp\Big((\ell+1-i)(t+u)\Big) \end{aligned}$$

$$= \sum_{i=0}^{\ell} \frac{(-1)^i}{i!} \Big((\ell+1-i)(t+u)\Big)^{i-1} \Big((t+u)(\ell+1)+i(1-t-u)\Big) \cdot \exp\Big(t(\ell+1-i)+u(\ell-i)\Big). \qquad \square$$

Corollary 3.5. *For a fixed integer $\ell \geq 0$, the exponential generating function $a_\ell(u) = \sum_{k\geq 0} a_{k,\ell}\frac{u^k}{k!}$ of derangements with ℓ excedances has the closed form*

$$a_\ell(u) = \sum_{i=0}^{\ell} \frac{(-1)^i}{i!} \Big(u(\ell-i+1)\Big)^{i-1} \Big(u(\ell-i+1)+i\Big) \exp\Big(u(\ell-i)\Big). \tag{3.16}$$

Proof. Setting $t = 0$ in Equation (3.14), we get $a_\ell(u) = e_\ell(0, u)$. $\square$

Corollary 3.6. *For a fixed integer $\ell \geq 0$, the exponential generating function $e_\ell(t) = \sum_{n\geq 0} E_{n,\ell}\frac{t^n}{n!}$ of permutations with ℓ excedances has the closed form*

$$e_\ell(t) = \sum_{i=0}^{\ell} \frac{(-1)^i}{i!} \Big(t(\ell-i+1)\Big)^{i-1} \Big(t(\ell-i+1)+i\Big) \exp\Big(t(\ell-i+1)\Big). \tag{3.17}$$

Proof. Setting $u = 0$ in Equation (3.14), we get $e_\ell(t) = e_\ell(t, 0)$. $\square$

Theorem 3.10. *The q-analogue of the refined factorial Euler–Seidel matrix $\big(e_n^k(x)\big)_{k,n\geq 0}$ is determined by the formula*

$$e_n^k(x,q) = -q\ e_n^{k-1}(x,q) + e_{n+1}^{k-1}(x,q). \tag{3.18}$$

Proof. From the construction of $(n+1, k+1)$-permutations, we distinguish whether the integer $(n+1)$ is a fixed point or not. By doing the corresponding case analysis, we get the result. $\square$

Theorem 3.11. *We have*

$$e_n^k(x,q) = \sum_{i=0}^{k} (-q)^{k-i}\binom{k}{i} E_{n+i}(x,q). \tag{3.19}$$

Proof. From Equation (3.18) and Theorem 1.2, we get the result. $\square$

Theorem 3.12. *We have*

$$\begin{bmatrix} e_n^0(x,q) \\ e_n^1(x,q) \\ e_n^2(x,q) \\ e_n^3(x,q) \\ e_n^4(x,q) \\ \vdots \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -q & 1 & 0 & 0 & 0 & \cdots \\ q^2 & -2q & 1 & 0 & 0 & \cdots \\ -q^3 & 3q^2 & -3q & 1 & 0 & \cdots \\ q^4 & -4q^3 & 6q^2 & -4q & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} E_n(x,q) \\ E_{n+1}(x,q) \\ E_{n+2}(x,q) \\ E_{n+3}(x,q) \\ E_{n+4}(x,q) \\ \vdots \end{bmatrix}. \tag{3.20}$$

That is,

$$\left[e_n^k(x,q)\right]_{k\geq 0} = [\exp(-qt), t]\ \ \left[E_{n+k}(x,q)\right]_{k\geq 0}. \tag{3.21}$$

Proof. Equation (3.20) follows from Equation (3.19). $\square$

Theorem 3.13. *The matrix T given by*

$$T = [\exp(-qt), t] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -q & 1 & 0 & 0 & 0 & \cdots \\ q^2 & -2q & 1 & 0 & 0 & \cdots \\ -q^3 & 3q^2 & -3q & 1 & 0 & \cdots \\ q^4 & -4q^3 & 6q^2 & -4q & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.22}$$

is the q-analogue of $[\exp(-t), t]$.

Proof. We have

$$\sum_{\ell \geq 0} (-q)^\ell \frac{t^\ell}{\ell!} = \exp(-qt).$$

If $T = [g(t), f(t)]$, from Definition 1.1, we have Equation (3.22), that is,

$$\begin{cases} T_0(t) = g(t) = \exp(-qt), \\ T_1(t) = g(t)f(t) = tg(t), \\ \vdots \\ T_i(t) = g(t)\,(f(t))^i = t^i g(t), \\ \vdots \end{cases}$$

□

Remark 3.2. $T^{-1} = [\exp(qt), t]$ is the q-analogue of the inverse Riordan matrix of T.

Theorem 3.14. *We have*

$$\begin{bmatrix} e_0^k(x,q) \\ e_1^k(x,q) \\ e_2^k(x,q) \\ e_3^k(x,q) \\ e_4^k(x,q) \\ \vdots \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 1 & 0 & 0 & 0 & \cdots \\ q^2 & 2q & 1 & 0 & 0 & \cdots \\ q^3 & 3q^2 & 3q & 1 & 0 & \cdots \\ q^4 & 4q^3 & 6q^2 & 4q & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} A_k(x,q) \\ A_{k+1}(x,q) \\ A_{k+2}(x,q) \\ A_{k+3}(x,q) \\ A_{k+4}(x,q) \\ \vdots \end{bmatrix}. \tag{3.23}$$

That is,

$$\left[e_n^k(x,q)\right]_{n\geq 0} = [\exp(qt), t] \; \left[A_{k+n}(x,q)\right]_{n \geq 0}, \; \textit{with} \; [\exp(qt), t] = T^{-1}. \tag{3.24}$$

Proof. From Equation (3.18), we get

$$e_{n+1}^{k-1}(x,q) = q\; e_n^{k-1}(x,q) + e_n^k(x,q).$$

Let us consider $b_k^n(x,q) = e_{n+1}^{k-1}(x,q)$. Then

$$b_k^n(x,q) = q\; b_k^{n-1}(x,q) + b_{k+1}^{n-1}(x,q) \text{ and } b_k^0 = A_k(x,q).$$

Combining this with Theorem 1.2, we get the result. □

Theorem 3.15. *We have*

$$\begin{bmatrix} E_0(x,q) \\ E_1(x,q) \\ E_2(x,q) \\ E_3(x,q) \\ E_4(x,q) \\ \vdots \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q & 1 & 0 & 0 & 0 & \cdots \\ q^2 & 2q & 1 & 0 & 0 & \cdots \\ q^3 & 3q^2 & 3q & 1 & 0 & \cdots \\ q^4 & 4q^3 & 6q^2 & 4q & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{bmatrix} A_0(x,q) \\ A_1(x,q) \\ A_2(x,q) \\ A_3(x,q) \\ A_4(x,q) \\ \vdots \end{bmatrix}. \tag{3.25}$$

That is,

$$[E_n(x,q)]_n = [\exp(qt),t]\ \ [A_n(x,q)]_n\,. \tag{3.26}$$

Proof. Setting $k=0$ in Equation (3.23), we get the result. ☐

Theorem 3.16. *For a fixed integer n with $n \geq 0$, the exponential generating function $A(x,t,q) = \sum_{n\geq 0} A_n(x,q)\frac{t^n}{n!}$ has the closed form*

$$A(x,t,q) = \exp(-qt)\left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q. \tag{3.27}$$

Proof. Setting $n=0$ in Equation (3.21), we get

$$\left[e_0^k(x,q)\right]_{k\geq 0} = [\exp(-qt),t]\ [E_k(x,q)]_{k\geq 0}\,.$$

Hence

$$\sum_{k\geq 0}\frac{E_k(x,q)}{k!}t^k = \left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q \text{(cf. Brenti [1])},$$

and from Theorem 1.4 we get the result. ☐

Corollary 3.7. *We have*

$$A(x,t,q) = \left(A(x,t)\right)^q. \tag{3.28}$$

Proof. We have

$$\begin{aligned} A(x,t) &= \frac{(1-x)\exp(-xt)}{1-x\exp\left((1-x)t\right)} \text{ (cf. Mantaci and Rakotondrajao [8])} \\ &= \frac{(x-1)\exp(-t)}{x-\exp\left(t(x-1)\right)}. \end{aligned}$$

Hence

$$\begin{aligned} A(x,t,q) &= \exp(-qt)\left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q \\ &= \left(\frac{(x-1)\exp(-t)}{x-\exp\left(t(x-1)\right)}\right)^q. \end{aligned}$$

It follows that $A(x,t,q) = (A(x,t))^q$. ☐

Theorem 3.17. *For a fixed integer n with $n \geq 0$, the exponential generating function $F_n(x,t,q) = \sum_{k\geq 0} e_n^k(x,q)\frac{t^k}{k!}$ of (n,k)-permutations has the closed form*

$$F_n(x,t,q) = \exp(-qt)\frac{d^n}{dt^n}\left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q. \tag{3.29}$$

Proof. From Equation (3.21) and Theorem 1.4, we get

$$\begin{aligned} F_n(x,t,q) &= \exp(-qt)\ \sum_{k\geq 0} E_{n+k}(x,q)\frac{t^k}{k!} \\ &= \exp(-qt)\frac{d^n}{dt^n}\left(\sum_{k\geq 0} E_k(x,q)\frac{t^k}{k!}\right) \\ &= \exp(-qt)\frac{d^n}{dt^n}\left(\frac{x-1}{x-\exp\left(t(x-1)\right)}\right)^q. \end{aligned}$$

☐

Corollary 3.8. *We have*

$$F_n(x,t,q) = \exp(-qt)\ \frac{d^n}{dt^n}E(x,t,q). \tag{3.30}$$

Theorem 3.18. *For a fixed integer k with $k \geq 0$, the exponential generating function $F_k(x,t,q) = \sum_{n\geq 0} e_n^k(x,q)\frac{t^n}{n!}$ of (n,k)-permutations has the closed form*

$$F_k(x,t,q) = \exp(qt)\frac{d^k}{dt^k}\left(\frac{(x-1)\exp(-t)}{x-\exp\left(t(x-1)\right)}\right)^q. \tag{3.31}$$

Proof. From Equation (3.24) and Theorem 1.4, we get

$$\begin{aligned} F_k(x,t,q) &= \exp(qt)\ \sum_{k\geq 0} A_{k+n}(x,q)\frac{t^k}{k!} \\ &= \exp(qt)\frac{d^k}{dt^k}\left(\sum_{n\geq 0} A_n(x,q)\frac{t^n}{n!}\right) \\ &= \exp(qt)\frac{d^k}{dt^k}\left(\frac{(x-1)\exp(-t)}{x-\exp\left(t(x-1)\right)}\right)^q. \end{aligned}$$

□

Corollary 3.9. *We have*

$$F_k(x,t,q) = \exp(qt)\ \frac{d^k}{dt^k}A(x,t,q). \tag{3.32}$$

Theorem 3.19. *The exponential generating function*

$$F(x,t,u,q) = \sum_{n\geq 0}\sum_{k\geq 0} e_n^k(x,q)\frac{u^k}{k!}\frac{t^n}{n!}$$

of (n,k)-permutations has the closed form

$$F(x,t,u,q) = \exp(-qu)\left(\frac{x-1}{x-\exp\left((x-1)(u+t)\right)}\right)^q. \tag{3.33}$$

Proof. From Equation (3.18) and Theorem 1.3, we have

$$\begin{aligned} F(x,t,u,q) &= \sum_{n\geq 0}\sum_{k\geq 0} e_n^k(x,q)\frac{u^k}{k!}\frac{t^n}{n!} \\ &= \exp(-qu)E(x,t+u,q) \\ &= \exp(-qu)\left(\frac{x-1}{x-\exp\left((x-1)(u+t)\right)}\right)^q. \end{aligned}$$

Here, the last line follows from Equation (1.19). □

Corollary 3.10. *We have*

$$F(x,t,u,q) = \left(F(x,t,u)\right)^q. \tag{3.34}$$

Proof. We have

$$\begin{aligned} \left(F(x,t,u)\right)^q &= \left(\frac{(x-1)\exp(-u)}{x-\exp\left((x-1)(u+t)\right)}\right)^q \\ &= \exp(-qu)\left(\frac{x-1}{x-\exp\left((x-1)(u+t)\right)}\right)^q. \end{aligned}$$

□

Acknowledgements

One of the authors, Ratovilebamboavison Dimbiniaina, would like to thank the European Union for the financial support through the project "Appui à la Formation, à la Recherche et à l'Innovation pour le Développement Intra-Afrique (AFRIDI)".

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