Cyclic Sieving of Multisets with Bounded Multiplicity and the Frobenius Coin Problem

Séminaire Lotharingien de Combinatoire 93, Pocinho

Drew Armstrong

March 25, 2025

University of Miami
www.math.miami.edu/~armstrong

1. Generalized Binomial Coefficients

1. Generalized Binomial Coefficients

Two interpretations of binomial coefficients:

$$E(n;t) = \prod_{i=1}^{n}(1+t) = \sum_{k\geq 0}\binom{n}{k}t^k,$$

$$H(n;t) = \prod_{i=1}^{n}(1+t+t^2+\cdots) = \sum_{k\geq 0}\binom{n+k-1}{k}t^k.$$

1. Generalized Binomial Coefficients

Two interpretations of binomial coefficients:

$$E(n;t) = \prod_{i=1}^{n}(1+t) = \sum_{k\geq 0}\binom{n}{k}t^k,$$

$$H(n;t) = \prod_{i=1}^{n}(1+t+t^2+\cdots) = \sum_{k\geq 0}\binom{n+k-1}{k}t^k.$$

We consider the following interpolation:

$$H^{(b)}(n;t) = \prod_{i=1}^{n}(1+t+\cdots+t^{b-1}) = \sum_{k\geq 0}\binom{n}{k}^{(b)}t^k.$$

1. Generalized Binomial Coefficients

Two interpretations of binomial coefficients:

$$E(n;t) = \prod_{i=1}^{n}(1+t) = \sum_{k\geq 0}\binom{n}{k}t^k,$$

$$H(n;t) = \prod_{i=1}^{n}(1+t+t^2+\cdots) = \sum_{k\geq 0}\binom{n+k-1}{k}t^k.$$

We consider the following interpolation:

$$H^{(b)}(n;t) = \prod_{i=1}^{n}(1+t+\cdots+t^{b-1}) = \sum_{k\geq 0}\binom{n}{k}^{(b)}t^k.$$

Note that $\binom{n}{k}^{(2)} = \binom{n}{k}$ and $\binom{n}{k}^{(b)} = \binom{n+k-1}{k}$ when $b > k$. We will write

$$\binom{n}{k}^{(\infty)} = \binom{n+k-1}{k}.$$

Example: $n = 3$ and $b = 4$. The generating function is

$$\begin{aligned} H^{(4)}(3;t) &= (1 + t + t^2 + t^3)^3 \\ &= 1 + 3\,t + 6\,t^2 + 10\,t^3 + 12\,t^4 + 12\,t^5 + 10\,t^6 + 6\,t^7 + 3\,t^8 + t^9. \end{aligned}$$

1. Generalized Binomial Coefficients

Example: $n = 3$ and $b = 4$. The generating function is

$$H^{(4)}(3; t) = (1 + t + t^2 + t^3)^3$$
$$= 1 + 3\,t + 6\,t^2 + 10\,t^3 + 12\,t^4 + 12\,t^5 + 10\,t^6 + 6\,t^7 + 3\,t^8 + t^9.$$

The coefficients are

k	0	1	2	3	4	5	6	7	8	9
$\binom{3}{k}^{(4)}$	1	3	6	10	12	12	10	6	3	1

1. Generalized Binomial Coefficients

Example: $n = 3$ and $b = 4$. The generating function is

$$\begin{aligned} H^{(4)}(3; t) &= (1 + t + t^2 + t^3)^3 \\ &= 1 + 3\,t + 6\,t^2 + 10\,t^3 + 12\,t^4 + 12\,t^5 + 10\,t^6 + 6\,t^7 + 3\,t^8 + t^9. \end{aligned}$$

The coefficients are

k	0	1	2	3	4	5	6	7	8	9
$\binom{3}{k}^{(4)}$	1	3	6	10	12	12	10	6	3	1

In general we have $\binom{n}{k}^{(b)} = 0$ for $k > (b-1)n$ and

$$\binom{n}{0}^{(b)} + \binom{n}{1}^{(b)} + \cdots + \binom{n}{(b-1)n}^{(b)} = b^n.$$

Remark: $\binom{n}{k}^{(b)}/b^n$ is the probability of getting a sum of k in n rolls of a fair b-sided die with sides labeled $\{0, 1, \ldots, b-1\}$.

Remarks:

- The numbers $\binom{n}{k}^{(b)}$ occur often but they don't have a standard name.
- We roughly follow Euler's (1778) notation: $\left(\frac{n}{k}\right)^b$.
- Belbachir and Igueroufa (2020) compiled a historical bibliography.

2. Symmetric Functions and q-Analogues

2. Symmetric Functions and q-Analogues

Recall the generating functions for *elementary* and *complete* symmetric polynomials:

$$E(z_1, \ldots, z_n; t) = \prod_{i=1}^{n} (1 + z_i t) = \sum_{k \geq 0} e_k(z_1, \ldots, z_n) t^k,$$

$$H(z_1, \ldots, z_n; t) = \prod_{i=1}^{n} (1 + z_i t + (z_i t)^2 + \cdots) = \sum_{k \geq 0} h_k(z_1, \ldots, z_n) t^k.$$

Recall the generating functions for *elementary* and *complete* symmetric polynomials:

$$E(z_1, \ldots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t) = \sum_{k\geq 0} e_k(z_1, \ldots, z_n)t^k,$$

$$H(z_1, \ldots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t + (z_i t)^2 + \cdots) = \sum_{k\geq 0} h_k(z_1, \ldots, z_n)t^k.$$

We consider the following interpolation:

$$H^{(b)}(z_1, \ldots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t + \cdots (z_i t)^{b-1}) = \sum_{k\geq 0} h_k^{(b)}(z_1, \ldots, z_n)t^k.$$

Note that $h_k^{(2)} = e_k$ and $h_k^{(b)} = h_k$ when $b > k$. We will write $h_k^{(\infty)} = h_k$.

We can view $h_k^{(b)}(z_1, \ldots, z_n)$ as a generating function for lattice points in a diagonal slice of the integer box $\{0, 1, \ldots, b-1\}^n$:

$$X := \{(x_1, \ldots, x_n) \in \{0, 1 \ldots, b-1\}^n : x_1 + x_2 + \cdots + x_n = k\}.$$

We can view $h_k^{(b)}(z_1, \ldots, z_n)$ as a generating function for lattice points in a diagonal slice of the integer box $\{0, 1, \ldots, b-1\}^n$:

$$X := \{(x_1, \ldots, x_n) \in \{0, 1 \ldots, b-1\}^n : x_1 + x_2 + \cdots + x_n = k\}.$$

Then we have

$$h_k^{(b)}(z_1, \ldots, z_n) = \sum_{\mathbf{x} \in X} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x} \in X} z_1^{x_1} z_2^{x_2} \cdots z_n^{x_n}.$$

We can view $h_k^{(b)}(z_1,\ldots,z_n)$ as a generating function for lattice points in a diagonal slice of the integer box $\{0,1,\ldots,b-1\}^n$:

$$X := \{(x_1,\ldots,x_n) \in \{0,1\ldots,b-1\}^n : x_1+x_2+\cdots+x_n = k\}.$$

Then we have

$$h_k^{(b)}(z_1,\ldots,z_n) = \sum_{\mathbf{x}\in X} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x}\in X} z_1^{x_1} z_2^{x_2} \cdots z_n^{x_n}.$$

We can also view these lattice points as k-multisubsets of $\{1,2,\ldots,n\}$ with multiplicities bounded above by b:

$$(x_1,x_2,\ldots,x_n) \quad \longleftrightarrow \quad \{\underbrace{1,\ldots,1}_{x_1 \text{ times}},\underbrace{2,\ldots,2}_{x_2 \text{ times}}\ldots,\underbrace{n,\ldots,n}_{x_n \text{ times}}\}.$$

$$b=2: \quad k\text{-subsets of } \{1,\ldots,n\},$$

$$b=\infty: \quad k\text{-multisubsets of } \{1,\ldots,n\}.$$

Example: $n = 3$ and $k = 3$ for various values of b:

			030			
		120		021		
	210		111		012	
300		201		102		003

$$h_3^{(2)}(z_1, z_2, z_3) = z_1 z_2 z_3,$$

$$h_3^{(3)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2,$$

$$h_3^{(4)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_2^3,$$

$$h_3^{(5)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_2^3,$$

$$\vdots$$

A natural q-analogue of $\binom{n}{k}^{(b)}$ is given by the principal specialization of $h_k^{(b)}$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} := h_k^{(b)}(1, q, \ldots, q^{n-1}) = \sum_{\mathbf{x} \in X} q^{0x_1 + 1x_2 + 3x_2 + \cdots + (n-1)x_n}.$$

2. Symmetric Functions and q-Analogues

A natural q-analogue of $\binom{n}{k}^{(b)}$ is given by the principal specialization of $h_k^{(b)}$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} := h_k^{(b)}(1, q, \ldots, q^{n-1}) = \sum_{\mathbf{x} \in X} q^{0x_1 + 1x_2 + 3x_2 + \cdots + (n-1)x_n}.$$

This generalizes the standard q-binomial coefficients in the following sense:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(2)} = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(\infty)} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q.$$

Opinion: This is the reason why sometimes we multiply $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by $q^{k(k-1)/2}$ and sometimes we don't.

Example: $n = 3$ and $k = 3$ for various values of b:

0	1	2	3	4	5	6
			030			
		120		021		
	210		111		012	
300		201		102		003

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(2)} = q^3,$$

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(3)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5,$$

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(\infty)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5 + 1 + q^3 + q^6.$$

Remarks:

- Like the numbers $\binom{n}{k}^{(b)}$, the polynomials $h_k^{(b)}(z_1, \ldots, z_n)$ don't have a standard name or notation.

- Doty and Walker (1992) used $h'_k(n)$ and called them *modular complete symmetric polynomials*.

- Fu and Mei (2020) used $h_k^{[b-1]}$ and called them *truncated complete*.

- Grinberg (2022) used $G(b, k)$ and called them *Petrie symmetric functions*. He now regrets this name (personal communication).

- Since the definition is simple I believe that the name should be simple. In the paper I called them *b*-bounded symmetric polynomials.

Remarks:

- Doty and Walker (1992) mention the following generalization of Newton's identities, which they attribute to Macdonald:*

$$h_k^{(b)}(z_1,\ldots,z_n) = \det \begin{pmatrix} p_1^{(b)} & p_2^{(b)} & \cdots & \cdots & p_k^{(b)} \\ -1 & p_1^{(b)} & p_2^{(b)} & & \vdots \\ & -2 & p_1^{(b)} & p_2^{(b)} & \vdots \\ & & \ddots & \ddots & p_2^{(b)} \\ & & & -(k-1) & p_1^{(b)} \end{pmatrix}$$

where

$$p_m^{(b)} = \begin{cases} (1-b)(z_1^m + \cdots + z_n^m) & b \mid m, \\ z_1^m + \cdots + z_n^m & b \nmid m. \end{cases}$$

* They did not express it as a determinant.

Remarks:

- This has an interesting consequence when $z_1 = \cdots = z_n = 1$:

$$\binom{n}{k}^{(b)} = \sum_{\lambda \vdash k} \frac{1}{z_\lambda} (1-b)^{l_b(\lambda)} n^{l(\lambda)},$$

where the sum is over $(\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ with $\sum_i \lambda_i = k$, and

$$l(\lambda) = \#\{i : \lambda_i \neq 0\},$$
$$l_b(\lambda) = \#\{i : b|\lambda_i\},$$
$$m_j = \#\{j : m_j = i\},$$
$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

Remarks:

- In a recent paper (*Lattice points and q-Catalan*, 2024) I proved that

$$\frac{1}{[n+1]_q} \sum_{k=\ell}^{m} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(n+1)} \in \mathbb{Z}[q]$$

  whenever $\gcd(n+1, \ell-1) = \gcd(n+1, m) = 1$, and I conjectured that the coefficients are positive. I called these *q-Catalan germs*.

Remarks:

- In a recent paper (*Lattice points and q-Catalan*, 2024) I proved that

$$\frac{1}{[n+1]_q} \sum_{k=\ell}^{m} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(n+1)} \in \mathbb{Z}[q]$$

  whenever $\gcd(n+1, \ell - 1) = \gcd(n+1, m) = 1$, and I conjectured that the coefficients are positive. I called these *q-Catalan germs*.

- I don't know how this generalizes to $b \neq n+1$.

3. A Bit of Galois Theory

3. A Bit of Galois Theory

Our main theorem will compute

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} \qquad \text{when } q \to \text{roots of unity.}$$

Before stating the theorem, it is worthwhile to mention a very general phenomenon, which follows from some basic Galois theory. This phenomenon is surely well known but I have not seen it written down.

3. A Bit of Galois Theory

Our main theorem will compute

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} \qquad \text{when } q \to \text{roots of unity.}$$

Before stating the theorem, it is worthwhile to mention a very general phenomenon, which follows from some basic Galois theory. This phenomenon is surely well known but I have not seen it written down.

Observation

Let $f(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$ be symmetric polynomial in n variables and let ω be a primitive dth root of unity for some d.

(a) If $d|n$ then $f(1, \omega, \ldots, \omega^{n-1}) = f(\omega, \ldots, \omega^n)$ is an integer.*

(b) If $d|(n-1)$ then $f(1, \omega, \ldots, \omega^{n-1})$ is an integer.

(c) If $d|(n+1)$ then $f(\omega, \ldots, \omega^n)$ is an integer.

* If $\deg(f) = k$ and $d \nmid k$ then this integer is zero.

Proof Sketch: (1) Let ω be a primitive dth root of unity and consider the field extension $\mathbb{Q}(\omega)/\mathbb{Q}$. The Galois group is

$$\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\varphi_r : \gcd(r, d) = 1\},$$

where $\varphi_r : \mathbb{Q}(\omega) \to \mathbb{Q}(\omega)$ is defined by $\varphi_r(\omega) := \omega^r$. If $\alpha \in \mathbb{Z}[\omega]$ satisfies $\varphi_r(\alpha) = \alpha$ for all $\gcd(r, d) = 1$ then Galois theory tells us that $\alpha \in \mathbb{Z}$.

(2) Consider the sequence $\boldsymbol{\omega} := (\omega, \ldots, \omega^{d-1})$. If $\gcd(r, d) = 1$ then φ_r permutes the sequence $\boldsymbol{\omega}$, hence it permutes sequences of the following four types:

$$(1, \boldsymbol{\omega}, \ldots, \boldsymbol{\omega}, 1),$$
$$(\boldsymbol{\omega}, 1, \ldots, \boldsymbol{\omega}, 1),$$
$$(1, \boldsymbol{\omega}, 1, \boldsymbol{\omega}, \ldots, \boldsymbol{\omega}, 1),$$
$$(\boldsymbol{\omega}, 1, \boldsymbol{\omega}, 1, \ldots, 1, \boldsymbol{\omega}).$$

□

Corollary

Let ω be a primitive dth root of unity.

(a) If $d|n$ then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) = h_k^{(b)}(\omega, \ldots, \omega^n) \in \mathbb{Z}.$$

(b) If $d|(n-1)$ then

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) \in \mathbb{Z}.$$

(c) If $d|(n+1)$ then

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(\omega, \ldots, \omega^n) \in \mathbb{Z}.$$

* If $d \nmid k$ then this integer is zero.

Our main theorem will compute these integers.

4. The Main Theorem and Cyclic Sieving

4. The Main Theorem and Cyclic Sieving

Main Theorem (in three parts)

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b,d)=1$.

(a) If $d|n$ then $\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1+t^d+\cdots+(t^d)^{b-1})^{n/d}$, i.e.,

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{n/d}{k/d}^{(b)} \geq 0.$$

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b,d)=1$.

(a) If $d|n$ then $\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1+t^d+\cdots+(t^d)^{b-1})^{n/d}$, i.e.,

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{n/d}{k/d}^{(b)} \geq 0.$$

(b) If $d|(n-1)$ then

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1+t+\cdots+t^{b-1})(1+t^d+\cdots+(t^d)^{b-1})^{(n-1)/d}, \text{ i.e.,}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_\ell \binom{(n-1)/d}{(k-\ell)/d}^{(b)} \geq 0.$$

4. The Main Theorem and Cyclic Sieving

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(c) If $d | (n + 1)$ then

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1 + t^d + \cdots + (t^d)^{b-1})^{(n+1)/d}}{1 + t + \cdots + t^{b-1}} \in \mathbb{Z}[t].$$

These coefficients are sometimes negative and are more difficult to describe. We will give an explicit formula below in terms of the Frobenius Coin Problem.

4. The Main Theorem and Cyclic Sieving

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(c) If $d | (n+1)$ then

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1 + t^d + \cdots + (t^d)^{b-1})^{(n+1)/d}}{1 + t + \cdots + t^{b-1}} \in \mathbb{Z}[t].$$

These coefficients are sometimes negative and are more difficult to describe. We will give an explicit formula below in terms of the Frobenius Coin Problem.

Remark: My paper also gives explicit generating functions for (a),(b),(c) when $\gcd(b, d) \neq 1$, which are more complicated.

4. The Main Theorem and Cyclic Sieving

Parts (a) and (b) have a nice combinatorial interpretation, in terms of cyclic sieving (Reiner-Stanton-White, 2004). Again, consider the set of points in a diagonal slice of the integer box $\{0, 1, \dots, b-1\}^n$:

$$X = \{(x_1, \dots, x_n) \in \{0, 1, \dots, b-1\}^n : x_1 + x_2 + \cdots + x_n = k\}.$$

This set is closed under permutations. Consider the following two permutations:

$$\rho \cdot (x_1, \dots, x_n) := (x_2, \dots, x_n, x_1),$$
$$\tau \cdot (x_1, \dots, x_n) := (x_2, \dots, x_{n-1}, x_1, x_n).$$

Note that $\langle \rho \rangle \cong \mathbb{Z}/n\mathbb{Z}$ and $\langle \tau \rangle \cong \mathbb{Z}/(n-1)\mathbb{Z}$. Recall that we can identify X with k-subsets and k-multisubsets of $\{1, \dots, n\}$ when $b = 2$ and $b = \infty$.

Corollary of Main Theorem

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

Corollary of Main Theorem

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(a) If $d|n$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \rho^{n/d}(\mathbf{x}) = \mathbf{x}\}.$$

Corollary of Main Theorem

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(a) If $d|n$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \rho^{n/d}(\mathbf{x}) = \mathbf{x}\}.$$

(b) If $d|(n-1)$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \tau^{(n-1)/d}(\mathbf{x}) = \mathbf{x}\}.$$

4. The Main Theorem and Cyclic Sieving

Corollary of Main Theorem

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(a) If $d|n$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \rho^{n/d}(\mathbf{x}) = \mathbf{x}\}.$$

(b) If $d|(n-1)$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \tau^{(n-1)/d}(\mathbf{x}) = \mathbf{x}\}.$$

I find the condition $\gcd(b, d) = 1$ surprising!

Remarks:

- This result generalizes the prototypical examples of cyclic sieving (Theorem 1.1 in RSW) for k-subsets (when $b = 2$) and k-multisubsets (when $b = \infty$).
- I find it surprising that it was not already known to the experts.
- Our Main Theorem (a),(b) generalizes Prop 4.2 in RSW, which appears there as a random collection of identities.
- Main Theorem (c) has no analogue in RSW.
- It may be interesting to look at the integers $f(\omega, \ldots, \omega^n) \in \mathbb{Z}$ when $d|(n+1)$ for other classes of symmetric polynomials.

5. The Frobenius Coin Problem

5. The Frobenius Coin Problem

Let ω be a primitive dth root of unity with $d|(n+1)$ and $\gcd(b,d)=1$. Recall that

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} t^k = \frac{(1+t^d+\cdots+(t^d)^{b-1})^{(n+1)/d}}{1+t+\cdots+t^{b-1}} \in \mathbb{Z}[t].$$

The integers $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)}$ are not directly related to cyclic sieving.

5. The Frobenius Coin Problem

Let ω be a primitive dth root of unity with $d|(n+1)$ and $\gcd(b,d)=1$. Recall that

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1+t^d+\cdots+(t^d)^{b-1})^{(n+1)/d}}{1+t+\cdots+t^{b-1}} \in \mathbb{Z}[t].$$

The integers $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ are not directly related to cyclic sieving.

Using the notation $[n]_t = 1+t+\cdots+t^{n-1}$ we can write this as

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{[b]_{t^d}}{[b]_t} [b]_{t^d}^{(n+1)/d-1}.$$

5. The Frobenius Coin Problem

Let ω be a primitive dth root of unity with $d|(n+1)$ and $\gcd(b,d)=1$. Recall that

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1+t^d+\cdots+(t^d)^{b-1})^{(n+1)/d}}{1+t+\cdots+t^{b-1}} \in \mathbb{Z}[t].$$

The integers $\omega^k \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_\omega^{(b)}$ are not directly related to cyclic sieving.

Using the notation $[n]_t = 1+t+\cdots+t^{n-1}$ we can write this as

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{[b]_{t^d}}{[b]_t} [b]_{t^d}^{(n+1)/d-1}.$$

We want to study the coefficients of the polynomial

$$\frac{[b]_{t^d}}{[b]_t} \in \mathbb{Z}[t].$$

It turns out these coefficients are related to the Frobenius Coin Problem.

5. The Frobenius Coin Problem

Given integers $\gcd(b, d) = 1$, consider the function $\nu_{b,d} : \mathbb{N} \to \mathbb{N}$,

$$\nu_{b,d}(n) := \#\{(k, \ell) \in \mathbb{N}^2 : bk + d\ell = n\}.$$

5. The Frobenius Coin Problem

Given integers $\gcd(b, d) = 1$, consider the function $\nu_{b,d} : \mathbb{N} \to \mathbb{N}$,

$$\nu_{b,d}(n) := \#\{(k, \ell) \in \mathbb{N}^2 : bk + d\ell = n\}.$$

The set of non-representable numbers is finite, called the Sylvester set:

$$S_{b,d} = \{n \in \mathbb{N} : \nu_{b,d}(n) = 0\}.$$

For example, $S_{3,5} = \{1, 2, 4, 7\}$. Sylvester (1882) proved that

$$\#S_{b,d} = (b-1)(d-1)/2 \quad \text{and} \quad \max(S_{b,d}) = bd - b - d.$$

5. The Frobenius Coin Problem

Given integers $\gcd(b, d) = 1$, consider the function $\nu_{b,d} : \mathbb{N} \to \mathbb{N}$,

$$\nu_{b,d}(n) := \#\{(k, \ell) \in \mathbb{N}^2 : bk + d\ell = n\}.$$

The set of non-representable numbers is finite, called the Sylvester set:

$$S_{b,d} = \{n \in \mathbb{N} : \nu_{b,d}(n) = 0\}.$$

For example, $S_{3,5} = \{1, 2, 4, 7\}$. Sylvester (1882) proved that

$$\#S_{b,d} = (b-1)(d-1)/2 \quad \text{and} \quad \max(S_{b,d}) = bd - b - d.$$

Let us define the Sylvester polynomial

$$S_{b,d}(t) := \sum_{s \in S_{b,d}} t^s.$$

For example, $S_{3,5}(t) = t + t^2 + t^4 + t^7$.

5. The Frobenius Coin Problem

Brown and Shiue (1993) attribute the following result to Ozluk.

Theorem (Ozluk)

If $\gcd(b, d) = 1$ then we have $[b]_{t^d}/[b]_t = 1 + (t-1)S_{b,d}(t)$, i.e.,

$$S_{b,d}(t) = \frac{t^{bd} - 1}{(1-t^b)(1-t^d)} + \frac{1}{1-t}.$$

5. The Frobenius Coin Problem

Brown and Shiue (1993) attribute the following result to Ozluk.

Theorem (Ozluk)

If $\gcd(b, d) = 1$ then we have $[b]_{t^d}/[b]_t = 1 + (t-1)S_{b,d}(t)$, i.e.,

$$S_{b,d}(t) = \frac{t^{bd} - 1}{(1-t^b)(1-t^d)} + \frac{1}{1-t}.$$

Corollary

If ω is a primitive dth root of unity with $d|(n+1)$, it follows that

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{(n+1)/d-1}{k/d}^{(b)} + \sum_{s\in S_{b,d}} \binom{(n+1)/d-1}{(k-1-s)/d}^{(b)} - \sum_{s\in S_{b,d}} \binom{(n+1)/d-1}{(k-s)/d}^{(b)}.$$

It is not clear from this formula when $\omega^k \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_\omega^{(b)}$ is positive or negative.

Here is a cute formula, which allows us to be much more precise.

Theorem

Let $\gcd(b, d) = 1$. For any $r \in \mathbb{N}$, let $0 \leq \beta_r < b$ and $0 \leq \delta_r < d$ satisfy

$$\beta_r \equiv rd^{-1} \bmod b \quad \text{and} \quad \delta_r \equiv rb^{-1} \bmod d.$$

Then

$$\frac{[b]_{t^d}}{[b]_t} = \frac{[d]_{t^b}}{[d]_t} = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b - \beta_1]_{t^d}[d - \delta_1]_{t^b}.$$

5. The Frobenius Coin Problem

Here is a cute formula, which allows us to be much more precise.

Theorem

Let $\gcd(b, d) = 1$. For any $r \in \mathbb{N}$, let $0 \le \beta_r < b$ and $0 \le \delta_r < d$ satisfy

$$\beta_r \equiv rd^{-1} \bmod b \quad \text{and} \quad \delta_r \equiv rb^{-1} \bmod d.$$

Then

$$\frac{[b]_{t^d}}{[b]_t} = \frac{[d]_{t^b}}{[d]_t} = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b - \beta_1]_{t^d}[d - \delta_1]_{t^b}.$$

Corollary

Let $\gcd(b, d) = 1$. If ω is a primitive dth root of unity and $d|(n+1)$ then

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} \text{ is } \begin{cases} \ge 0 & \text{when } \delta_k < \delta_1, \\ \le 0 & \text{when } \delta_k \ge \delta_1. \end{cases}$$

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

Example: Let $(b, d) = (7, 5)$. Draw an infinite array starting at 0, adding 5 for each right step and subtracting 7 for each down step.

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

Example: Let $(b, d) = (7, 5)$. Draw an infinite array starting at 0, adding 5 for each right step and subtracting 7 for each down step.

The Sylvester set forms a triangle:

$$S_{7,5} = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}.$$

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

In this case we have $(\beta_1, \delta_1) = (3, 3)$, which tells us that the label 1 occurs in position $(\beta_1, \delta_1 - d) = (3, -2)$.

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

The cute formula describes two rectangles with bottom corners at 0 and 1.

$$[b]_{t^d}/[b]_t = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b}$$

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

The cute formula describes two rectangles with bottom corners at 0 and 1:

$$\begin{aligned}[7]_{t^5}/[7]_t &= [3]_{t^5}[3]_{t^7} - t[4]_{t^5}[2]_{t^7} \\ &= 1 + t^5 + t^7 + t^{10} + t^{12} + t^{14} + t^{17} + t^{19} + t^{24} \\ &\quad - (t + t^6 + t^8 + t^{11} + t^{13} + t^{16} + t^{18} + t^{23}).\end{aligned}$$

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

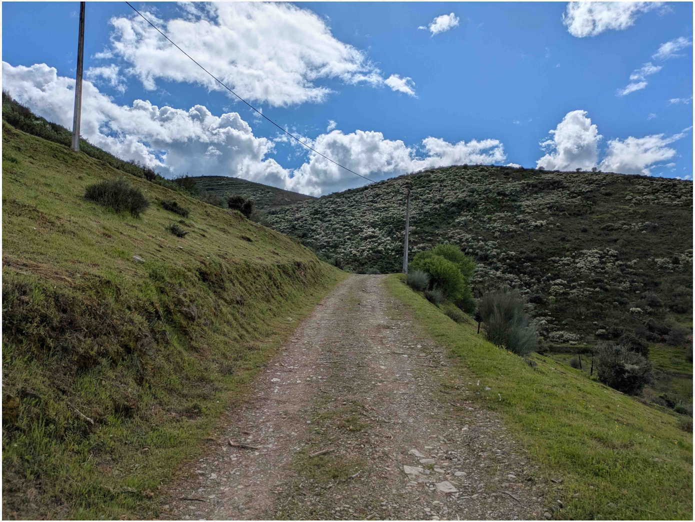
14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
				4	9	14	19	24	.	
					2	7	12	17	.	
						0	5	10	.	

The cute formula describes two rectangles with bottom corners at 0 and 1:

$$\begin{aligned}[7]_{t^5}/[7]_t &= [3]_{t^5}[3]_{t^7} - t[4]_{t^5}[2]_{t^7} \\ &= 1 + t^5 + t^7 + t^{10} + t^{12} + t^{14} + t^{17} + t^{19} + t^{24} \\ &\qquad - (t + t^6 + t^8 + t^{11} + t^{13} + t^{16} + t^{18} + t^{23}).\end{aligned}$$

And this leads to a precise description of $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(7)}$ when $\omega^5 = 1$.

Obrigado!



Thanks to DeepSeek for suggesting the azulejos background image.