Pak-Stanley labeling of hyperplane arrangements

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SLC93
Pocinho, March 26, 2025

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Motivation

In the 1990s Pak and Stanley introduced a labeling of the regions of the m-Shi arrangement of hyperplanes with m-parking functions. It is easy to determine the label assigned to a region but the inverse can be "hard" to find.

We started by extending the Pak-Stanley labeling to other hyperplane arrangements and determined the inverse for them.

Let $n \in \mathbb{N}$. In what follows $[n] := \{1, \ldots, n\}$.

Parking functions

Assume n drivers want to park on a one-way street with n parking spaces.
ith driver prefers space a_i and parks there if it is free. If a_i is occupied, i takes the next available space.
$\mathbf{a} = (a_1, \ldots, a_n)$ is a parking function of length n if all cars can park.

Example

(3,1,1) is a parking function of length 3
$\_\ \_\ \_ \rightarrow \_\ \_\ 1 \rightarrow 2\ \_\ 1 \rightarrow 2\ 3\ 1$
(1,3,3) is not a parking function of length 3
$\_\ \_\ \_ \rightarrow 1\ \_\ \_ \rightarrow 1\ \_\ 2 \rightarrow 1\ \_\ 2$
3 can't park

$$
\begin{aligned}
\mathrm{PF}_n = & \{\text{parking funtions of length } n\} \\
\mathrm{PF}_2 = & \{11, 12, 21\} \\
\mathrm{PF}_3 = & \{111, 112, 113, 121, 122, 123, 131, 132, \\
& 211, 212, 213, 221, 231, \\
& 311, 312, 321\}
\end{aligned}
$$

$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a parking function of length n iff

$$
\begin{aligned}
\mathbf{a}^{-1}([i]) &= \{j \in [n] \mid a_j \leq i\} \\
&= \{\text{drivers who want to park in the first } i \text{ places}\}
\end{aligned}
$$

has at least i elements, for every $i \in [n]$.

$\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is a parking function of length n iff the unique weakly increasing rearrangement $\mathbf{b} = (b_1, \dots, b_n)$ of $\mathbf{a}$ satisfies

$$b_i \leq i, \text{ for every } i \in [n],$$

i.e.,

$$\mathbf{b} \preceq (1, 2, 3, \dots, n).$$

($\preceq$ denotes the product order or componentwise order.)

$\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is an m-parking function of length n iff the unique weakly increasing rearrangement $\mathbf{b} = (b_1, \dots, b_n)$ of $\mathbf{a}$ satisfies

$$b_i \leq m(i-1) + 1, \text{ for every } i \in [n],$$

i.e.,

$$\mathbf{b} \preceq (1, 1+m, 1+2m, \dots, 1+(n-1)m).$$

A parking function $\mathbf{a} \in \mathbb{N}^n$ of length n is prime iff it remains a parking function (of length $n-1$) when we remove a 1.

$$\begin{aligned} \mathrm{PF}'_n =& \{\text{prime parking funtions of length } n\} \\ \mathrm{PF}'_2 =& \{11\} \\ \mathrm{PF}'_3 =& \{111, 112, 121, 211\} \end{aligned}$$

A parking function $\mathbf{a} \in \mathbb{N}^n$ of length n is prime iff the unique weakly increasing rearrangement $\mathbf{b} = (b_1, \ldots, b_n)$ of $\mathbf{a}$ satisfies

$$b_{i+1} \leq i, \text{ for every } i \in [n-1],$$

i.e.,

$$\mathbf{b} \preceq (1, 1, 2, \ldots, n-1).$$

A parking function $\mathbf{a} \in \mathbb{N}^n$ of length n is prime iff

$$\begin{aligned} \mathbf{a}^{-1}([i]) &= \{j \in [n] \mid a_j \leq i\} \\ &= \{\text{drivers who want to park in the first } i \text{ places}\} \end{aligned}$$

has more than i elements for every $i \in [n-1]$.

An m-parking function $\mathbf{a} \in \mathbb{N}^n$ of length n is prime iff the unique weakly increasing rearrangement $\mathbf{b} = (b_1, \ldots, b_n)$ of $\mathbf{a}$ satisfies

$$b_{i+1} \leq m(i-1)+1, \text{ for every } i \in [n-1],$$

i.e.,

$$\mathbf{b} \preceq (1, 1, 1+m, \ldots, 1+m(n-2)).$$

$$m\text{-}\mathrm{PF}_n = \{m\text{-parking functions of length } n\}$$
$$m\text{-}\mathrm{PF}'_n = \{\text{prime } m\text{-parking functions of length } n\}$$

$$|\operatorname{PF}_n| = (n+1)^{n-1} \qquad \text{and} \qquad |\operatorname{PF}'_n| = (n-1)^{n-1}$$

$$|m\text{-}\operatorname{PF}_n| = (mn+1)^{n-1} \qquad \text{and} \qquad |m\text{-}\operatorname{PF}'_n| = (mn-1)^{n-1}$$

Hyperplane arrangements

n-dimensional Coxeter arrangement or braid arrangement:

$$\mathrm{Cox}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0\}$$

n-dimensional Shi arrangement (Shi, 1986):

$$\mathrm{Shi}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0, x_i - x_j = 1\}$$

n-dimensional Ish arrangement (Armstrong, 2013):

$$\mathrm{Ish}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0, x_1 - x_j = i\}$$

n-dimensional Catalan arrangement:

$$\mathrm{Cat}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = 0, x_i - x_j = 1, x_i - x_j = -1\}$$

Shi_3 and Ish_3

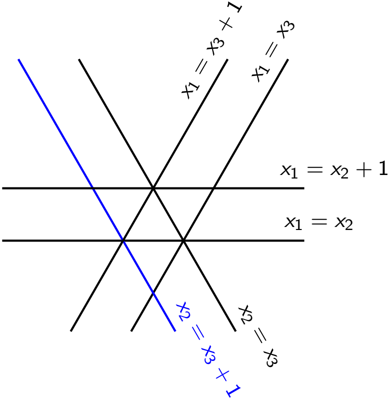

Figure: Shi_3 (left) and Ish_3 (right)

n-dimensional m-Shi arrangement:

$$m\text{-}\mathrm{Shi}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = k \mid k \in \{-(m-1), \ldots, 0, 1, \ldots, m\}\}$$

n-dimensional m-Catalan arrangement:

$$m\text{-}\mathrm{Cat}_n = \bigcup_{1 \le i < j \le n} \{x_i - x_j = k \mid k \in \{-m, \ldots, 0, 1, \ldots, m\}\}$$

hyperplane arrangement	# regions	# rel. bounded regions
Cox_n	$n!$	0
Shi_n	$(n+1)^{n-1}$	$(n-1)^{n-1}$
Ish_n	$(n+1)^{n-1}$	$(n-1)^{n-1}$
Cat_n	$\frac{(2n)!}{(n+1)!} = n!C_n$	$\frac{(2(n-1))!}{(n-1)!} = n!C_{n-1}$
m- Shi_n	$(mn+1)^{n-1}$	$(mn-1)^{n-1}$
m- Cat_n	$\frac{(mn+n)!}{(mn+1)!} = n!F(n,m)$	$\frac{(mn+n-1)!}{(mn)!}$

$C_n = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number
$F(n,m) = \frac{1}{mn+1}\binom{mn+n}{mn}$ is a Fuss-Catalan number
(Remark: $F(n,1) = C_n$)

Pak-Stanley labeling

Let $R_0 = \{\mathbf{x} \in \mathbb{R}^n \mid x_n + 1 > x_1 > \cdots > x_n\}$ be the fundamental alcove, i.e, the region bounded by the hyperplanes of equation $x_i - x_{i+1} = 0$, for $i \in [n-1]$, and $x_1 - x_n = 1$.

$\lambda(R_0) := \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

Let R_1 and R_2 be two regions separated by a unique hyperplane H, of equation $x_i - x_j = k$, such that R_0 and R_1 are on the same side of H. Then

$$\lambda(R_2) = \begin{cases} \lambda(R_1) + e_j & \text{if } k \leq 0, \\ \lambda(R_1) + e_i & \text{if } k > 0, \end{cases}$$

where $e_i = (0, \ldots, 0, \underbrace{1}_{i\text{th pos.}}, 0, \ldots, 0)$

Problem

The original Pak-Stanley labeling is not injective when applied to the Ish arrangement.

Pak-Stanley labeling (adapted)

Let $R_0 = \{\mathbf{x} \in \mathbb{R}^n \mid x_n + 1 > x_1 > \cdots > x_n\}$ be the region bounded by the hyperplanes of equation $x_i - x_{i+1} = 0$, for $i \in [n-1]$, and $x_1 - x_n = 1$.

$\ell(R_0) := \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

Let R_1 and R_2 be two regions separated by a unique hyperplane H, of equation $x_i - x_j = k$, such that R_0 and R_1 are on the same side of H. Then

$$\ell(R_2) = \begin{cases} \ell(R_1) + e_i & \text{if } k \leq 0, \\ \ell(R_1) + e_j & \text{if } k > 0, \end{cases}$$

where $e_i = (0, \ldots, 0, \underbrace{1}_{i\text{th pos.}}, 0, \ldots, 0)$

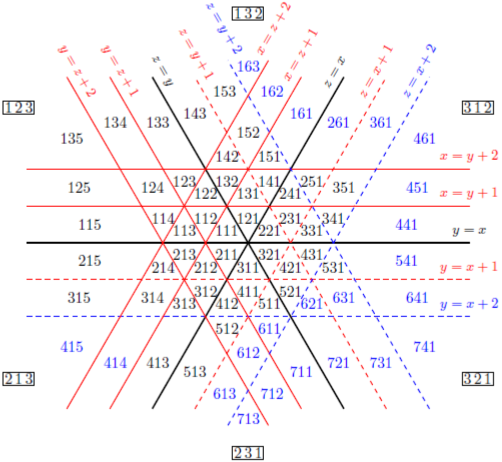

Figure: Pak-Stanley labeling of the 3-dimensional 2-Catalan arrangement.

The Pak-Stanley labels of the regions of the m-Shi arrangement are the m-parking functions and the labels of the relatively bounded regions of the m-Shi arrangement are the prime m-parking functions.

Definition (D. and Guedes de Oliveira, 2021)

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $p \in \mathbb{Z}$. The *p-center* of $\mathbf{a}$, $Z_p(\mathbf{a})$, is the largest subset $X = \{x_1, \ldots, x_q\}$ of $[n]$ such that if $1 \leq x_q < x_{q-1} < \cdots < x_1 \leq n$ then

$$a_{x_j} \leq p + j, \text{ for every } j \in [q].$$

In other words,

$$(a_{x_q}, a_{x_{q-1}}, \ldots, a_{x_2}, a_{x_1}) \preceq (p + q, p + q - 1, \ldots, p + 2, p + 1).$$

Example

Let $\mathbf{a} = 1352$.

$\mathbf{a} = 1352$ ($a_1 = 1 \preceq 1$ and there is no $a_{i_1} a_{i_2} \preceq 21$)

$Z_0(\mathbf{a}) = \{1\}$

$\mathbf{a} = 1352$ ($a_1 a_2 a_4 = 132 \preceq 432$ and $\mathbf{a} \not\preceq 5432$)

$Z_1(\mathbf{a}) = \{1, 2, 4\}$

$\mathbf{a} = 1352$ ($a_1 a_2 a_4 = 132 \preceq 543$ and $\mathbf{a} \not\preceq 6543$)

$Z_2(\mathbf{a}) = \{1, 2, 4\}$

$\mathbf{a} = 1352$ ($\mathbf{a} \preceq 7654$)

$Z_3(\mathbf{a}) = \{1, 2, 3, 4\}$

($Z_p(\mathbf{a}) = \varnothing$, if $p < 0$, and $Z_p(\mathbf{a}) = [4]$, if $p > 3$.)

The m-Catalan arrangement

Theorem (D. & Guedes de Oliveira, 2021)

If $\mathbf{b}$ *and* $\mathbf{c}$ *are the Pak-Stanley labels of two regions* R_1 *and* R_2 *of the m-Catalan arrangement such that* R_2 *is the image of* R_1 *by refletion on the hyperplane of equation* $x_i - x_{i+1} = 0$ *and* R_0 *and* R_1 *are on the same side of* $x_i - x_{i+1} = 0$*, then*

$$c_j = \begin{cases} b_j & \text{if } j \neq i \text{ and } j \neq i+1, \\ b_{i+1} + 1 & \text{if } j = i, \\ b_i & \text{if } j = i+1. \end{cases}$$

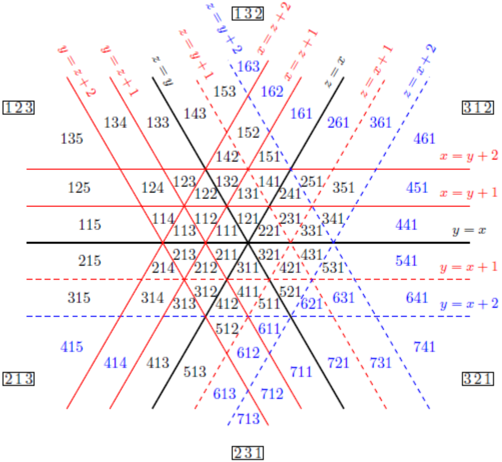

Figure: Pak-Stanley labeling of the 3-dimensional 2-Catalan arrangement.

Let $\pi \in \mathcal{S}_n$. The *inversion table* of π, $I(\pi)$, is defined by

$$I(\pi)_i = |\{j > i \mid \pi^{-1}(j) < \pi^{-1}(i)\}|, \quad \text{for every } i \in [n].$$

For every $\mathbf{a} \in \mathbb{N}^n$, let $\mathbf{p}(\mathbf{a}) \in (\mathbb{N} \cup \{0\})^n$ be defined by

$$\mathbf{p}(\mathbf{a})_i := \min\{j \in \mathbb{N} \cup \{0\} \mid i \in Z_j(\mathbf{a})\}, \quad \text{for every } i \in [n].$$

Theorem (D. & Guedes de Oliveira, 2021)

Let $\pi \in \mathcal{S}_n$, R be a region of m- Cat_n contained in $\{\mathbf{x} \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$, and $\mathbf{b}$ be the Pak-Stanley label of R. Then

$$\mathbf{b} - \mathbf{p}(\mathbf{b}) = \mathbf{1} + I(\pi).$$

$\mathbf{1} + I(\pi)$ is the minimum label among the labels of the regions of m- Cat_n contained in $\{\mathbf{x} \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$.

Theorem (D & Guedes de Oliveira, 2021)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ *is the Pak-Stanley label of a region of* $m\text{-}\mathrm{Cat}_n$ *if and only if for every* $i \in [n]$,

$$|Z_{(i-1)m}(\mathbf{a})| \geq i.$$

This labeling is bijective.

Theorem (D & Guedes de Oliveira, 2021)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ *is the Pak-Stanley label of a relatively bounded region of* $m\text{-}\mathrm{Cat}_n$ *arrangement if and only if for every* $i \in [2, n]$,

$$|Z_{(i-1)m-1}(\mathbf{a})| \geq i.$$

The Ish arrangement

Theorem (D. and Guedes de Oliveira, 2018)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ *is the Pak-Stanley label of a region of* Ish_n *if and only if* $\mathbf{a} \in [n]^n$ *and* $1 \in Z_0(\mathbf{a})$. *This labeling is bijective.*

$$\begin{aligned} \mathrm{IPF}_n =& \{\text{Pak-Stanley labels of the regions of } \mathrm{Ish}_n\} \\ \mathrm{IPF}_3 =& \{111, 112, 113, 121, 122, 123, 131, 132, 133, \\ & 211, 212, 213, 221, 231, \\ & 311, 321\} \end{aligned}$$

$133 \in \mathrm{IPF}_3 \setminus \mathrm{PF}_3$ and $312 \in \mathrm{PF}_3 \setminus \mathrm{IPF}_3$

Theorem (D.)

Let R be a region of Ish_n and $i, j \in [n]$ such that $i < j$. Then R_0 and R are separated by the hyperplane of equation $x_1 = x_j + i$ iff $j \notin Z_{i-1}(\ell(R))$.

Theorem (D.)

Given $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a}$ is the Pak-Stanley label of a relatively bounded region of Ish_n if and only if $\mathbf{a} \in [1] \times [n-1]^{n-1}$.

$$\mathrm{IPF}'_n = \{\text{Pak-Stanley labels of the rel. bounded regions of } \mathrm{Ish}_n\}$$
$$\mathrm{IPF}'_3 = \{111, 112, 121, 122\}$$

Given $\mathbf{a} \in \mathbb{N}^n$, let $\mathbf{p}(\mathbf{a}), \mathbf{q}(\mathbf{a}) \in (\mathbb{N} \cup \{0\})^n$ be defined by

$$\mathbf{p}(\mathbf{a})_i := \min\{j \in \mathbb{N} \cup \{0\} \mid i \in Z_j(\mathbf{a})\}$$

and

$$\mathbf{q}(\mathbf{a})_i := \min\{\mathbf{p}(\mathbf{a})_i, i - 1\},$$

i.e.,

$$\mathbf{q}(\mathbf{a}) := \mathbf{p}(\mathbf{a}) \wedge (\mathrm{Id} - \mathbf{1})$$

Theorem (D.)

Let R be a region of Ish_n, $\mathbf{a} := \ell(R)$ *and* $\pi \in \mathcal{S}_n$. *Then*

$$R \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$$

if and only if

$$\mathbf{a} - \mathbf{q}(\mathbf{a}) = \mathbf{1} + I(\pi).$$

$\mathbf{1} + I(\pi)$ is the minimum label among the labels of the regions of Ish_n contained in $\{\mathbf{x} \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$.

Example

Let R be the region of Ish_n such that $\mathbf{a} = \ell(R) = 1424$. Then $Z_0 = \{1\}$, $Z_1 = \{1,3\}$, $Z_2 = \{1,2,3\}$, $Z_3 = \{1,2,3,4\}$. It follows that $\mathbf{p}(\mathbf{a}) = 0213$ and $\mathbf{q}(\mathbf{a}) = 0213 \wedge 0123 = 0113$. Hence $1 + I(\pi) = \mathbf{a} - \mathbf{q}(\mathbf{a}) = 1311$ and $\pi = 1342$. Finally,

$$R = \{\mathbf{x} \in \mathbb{R}^4 \mid \overbrace{x_1 > x_3 > x_4 > x_2}^{\pi=1342}, \underbrace{x_1 > x_2 + 1}_{q_2=1}, \underbrace{x_3 + 1 < x_1 < x_3 + 2}_{q_3=1}, \underbrace{x_1 > x_4 + 3}_{q_4=3}\}.$$

Thank you for your attention!

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