Plethystic lifts of q-binomial identities

Álvaro Gutiérrez (University of Bristol)

Every combinatorial formula has a q-analogue

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$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

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$$\prod_{i=1}^{n}(x+q^{n-2i}y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k y^{n-k}$$

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$$\frac{1}{n+1}\binom{2n}{n}=\#\mathbf{Dyck}(n)$$

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$$\#\,\mathrm{SSYT}_n(\lambda)=\prod_{\square\in\lambda}\frac{n+\mathbf{ct}(\square)}{\mathbf{hl}_\lambda(\square)}$$

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and this is horrible news

$$\prod_{i=1}^{n}(x+q^{n-2i}y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k y^{n-k}$$

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Quantum integers

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{1-n}$$

Quantum binomials

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]}$$

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Problem ($\mathfrak{sl}_2$ plethysm)

Compute the coefficients in

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \sum_k a^k_{n[m]} \cdot [k]$$

Let $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \ : \ \mathrm{tr}(M) = 0\}$.

Facts

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2.
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Facts

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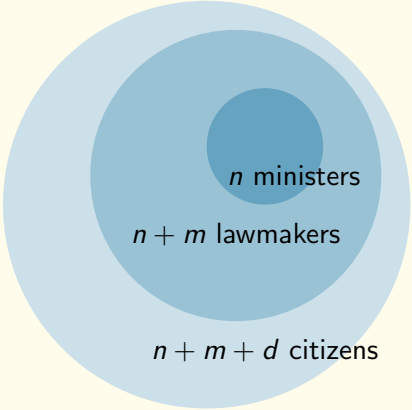
Problem ($\mathfrak{sl}_2$ plethysm)

Decompose $\mathrm{Sym}^n \mathrm{Sym}^m \mathbb{C}^2$.

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \sum_k a^k_{n[m]} \cdot [k] \qquad \Longleftrightarrow \qquad \mathrm{Sym}^n \mathrm{Sym}^m \mathbb{C}^2 = \bigoplus_k (\mathrm{Sym}^{k-1} \mathbb{C}^2)^{\oplus a^k_{n[m]}}$$

Which q-identities reflect deeper properties of plethysm?



Team-&-leader identity

$$\binom{n+m}{n}\binom{n+m+d}{n+m} = \binom{n+d}{n}\binom{n+m+d}{m}$$

$$\begin{bmatrix} n+m \\ n \end{bmatrix} \begin{bmatrix} n+m+d \\ n+m \end{bmatrix} = \begin{bmatrix} n+d \\ n \end{bmatrix} \begin{bmatrix} n+m+d \\ m \end{bmatrix}$$

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$$\mathrm{Sym}^m \mathrm{Sym}^n \mathbb{C}^2 \otimes \mathrm{Sym}^{m+n} \mathrm{Sym}^d \mathbb{C}^2 \quad \cong \quad \mathrm{Sym}^n \mathrm{Sym}^d \mathbb{C}^2 \otimes \mathrm{Sym}^m \mathrm{Sym}^{d+n} \mathbb{C}^2$$

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Theorem [G.-Martínez-Szwej-Wildon'25+]

$$\mathrm{Sym}_m \mathrm{Sym}^n \mathbb{F}^2 \otimes \mathrm{Sym}_{m+n} \mathrm{Sym}^d \mathbb{F}^2 \quad \cong \quad \mathrm{Sym}_n \mathrm{Sym}^d \mathbb{F}^2 \otimes \mathrm{Sym}_m \mathrm{Sym}^{d+n} \mathbb{F}^2$$

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$$\binom{n+m}{n} \binom{n+m+d}{n+m} = \binom{n+d}{n} \binom{n+m+d}{m}$$

$$\mathrm{Sym}_n V = (V^{\otimes n})^{\mathfrak{S}_n} = \langle P \in V^{\otimes n} \mid \sigma.P = P \ \forall \sigma \in \mathfrak{S}_n \rangle \hookrightarrow V^{\otimes n}$$

$$V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n V = V^{\otimes n} / \langle \cdots v \otimes w \otimes \cdots - \cdots w \otimes v \otimes \cdots \rangle$$

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Theorem [McDowell-Wildon'22]

$$\mathrm{Sym}_m \mathrm{Sym}^n \mathbb{F}^2 \quad \cong \quad \mathrm{Sym}^n \mathrm{Sym}_m \mathbb{F}^2$$

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m \\ n \end{bmatrix}$$

$$\begin{aligned}\mathsf{Sym}^d \mathbb{F}^2 &= \mathsf{Sym}^d \langle x, y \rangle_{\mathbb{F}} \\ &= \langle x^d, \quad x^{d-1}y, \quad x^{d-2}y^2, \quad \ldots, \quad xy^{d-1}, \quad y^d \rangle_{\mathbb{F}}\end{aligned}$$

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\mathsf{Sym}^d \mathbb{F}^2 &= \mathsf{Sym}^d \langle x, y \rangle_{\mathbb{F}} \\
&= \langle x^d, \quad x^{d-1}y, \quad x^{d-2}y^2, \quad \ldots, \quad xy^{d-1}, \quad y^d \rangle_{\mathbb{F}} \\
&\cong \langle x^d, \quad x^{d-1}, \quad x^{d-2}, \qquad \ldots, \quad x, \qquad\quad 1 \ \rangle_{\mathbb{F}} \\
&= \mathbb{F}_{\leq d}[x]
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$$
\mathsf{Sym}_n \mathsf{Sym}^d \mathbb{F}^2 = ((\mathsf{Sym}^d \mathbb{F}^2)^{\otimes n})^{\mathfrak{S}_n}
$$

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\mathsf{Sym}_n \mathsf{Sym}^d \mathbb{F}^2 &= ((\mathsf{Sym}^d \mathbb{F}^2)^{\otimes n})^{\mathfrak{S}_n} \\
&\cong ((\mathbb{F}_{\leq d}[x])^{\otimes n})^{\mathfrak{S}_n} \\
&= (\mathbb{F}_{\leq d}[x_1, \ldots, x_n])^{\mathfrak{S}_n} \\
&= \Lambda^{\mathbb{F}}_{\leq d}[x_1, \ldots, x_n]
\end{aligned}
$$

$$\mathsf{Sym}_n \mathsf{Sym}^d \mathbb{F}^2 \cong \Lambda^{\mathbb{F}}_{\leq d}[x_1, \ldots, x_n]$$

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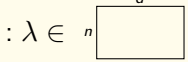
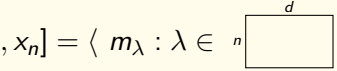
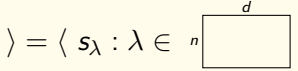
Remark. $\Lambda^{\mathbb{F}}_{\leq d}[x_1, \ldots, x_n] = \langle\ m_\lambda : \lambda \in$  $\rangle = \langle\ s_\lambda : \lambda \in$ $\rangle$

$$\mathrm{Sym}_n \mathrm{Sym}^d \mathbb{F}^2 \cong \Lambda^{\mathbb{F}}_{\leq d}[x_1, \ldots, x_n]$$

Remark. $\Lambda^{\mathbb{F}}_{\leq d}[x_1, \ldots, x_n] = \langle\ m_\lambda : \lambda \in$  $\rangle = \langle\ s_\lambda : \lambda \in$  $\rangle$

Remark. This brings symmetric polynomials into representation theory without characters!

$$\mathrm{Sym}^m \mathrm{Sym}^n \mathbb{C}^2 \otimes \mathrm{Sym}^{m+n} \mathrm{Sym}^d \mathbb{C}^2 \quad \cong \quad \mathrm{Sym}^n \mathrm{Sym}^d \mathbb{C}^2 \otimes \mathrm{Sym}^m \mathrm{Sym}^{d+n} \mathbb{C}^2$$

$$cat \uparrow \downarrow ch$$

$$\begin{bmatrix} n+m \\ n \end{bmatrix} \begin{bmatrix} n+m+d \\ n+m \end{bmatrix} = \begin{bmatrix} n+d \\ n \end{bmatrix} \begin{bmatrix} n+m+d \\ m \end{bmatrix}$$

$$\downarrow q = 1$$

$$\binom{n+m}{n} \binom{n+m+d}{n+m} = \binom{n+d}{n} \binom{n+m+d}{m}$$

Team-&-leader

$$\Lambda^{\mathbb{F}}_{\leq n}[z_1,\dots,z_m]\otimes\Lambda^{\mathbb{F}}_{\leq d}[y_1,\dots,y_m,x_1,\dots,x_n]\cong\Lambda^{\mathbb{F}}_{\leq d}[x_1,\dots,x_n]\otimes\Lambda^{\mathbb{F}}_{\leq d+n}[y_1,\dots,y_m]$$

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Team-&-leader

$$\Lambda^{\mathbb{F}}_{\leq n}[z_1,\dots,z_m]\otimes\Lambda^{\mathbb{F}}_{\leq d}[y_1,\dots,y_m,x_1,\dots,x_n]\cong\Lambda^{\mathbb{F}}_{\leq d}[x_1,\dots,x_n]\otimes\Lambda^{\mathbb{F}}_{\leq d+n}[y_1,\dots,y_m]$$

$$P(\mathbf{x},\mathbf{y},\mathbf{z})\mapsto P(\mathbf{x},\mathbf{y},\mathbf{y})$$

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Team-&-leader

Remark. Not all formulas will lift.

q-Pascal identity

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = q^n \begin{bmatrix} n+m-1 \\ n \end{bmatrix} + q^{-m} \begin{bmatrix} n+m-1 \\ n-1 \end{bmatrix}$$

List of formulas that lift

- $\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m \\ n \end{bmatrix}$ [Aprodu-Farkas-Papadima-Raicu-Weyman'19, McDowell-Wildon'22]

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- $s_{\begin{array}{|c|c|}\hline n & m \\ \hline \end{array}}(q^{-d}, q^{2-d}, ..., q^{d}) = \begin{bmatrix} m+n-1 \\ m \end{bmatrix}\begin{bmatrix} m+d+1 \\ m+n \end{bmatrix}$ [G.-Martínez-Szwej-Wildon'25+]

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- $a_{n[m]}^{\lambda_1-\lambda_2+1} = g_{(n^k),(n^k),(\lambda_1,\lambda_2)}$ [Ikenmeyer-Omar-Tsintsilidas'25+]

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Goal: q-Pfaff–Saalschütz identity + Jacobi–Trudi formula

$$s_{\begin{array}{|c|c|}\hline n & m \\ \hline n \\ \cline{1-1}\end{array}}(q^{-d}, q^{2-d}, \ldots, q^{d}) = \sum_{k=0}^{n} \begin{bmatrix} d+k \\ 2n+m \end{bmatrix} s_{\begin{array}{|c|c|}\hline k & m \\ \hline k \\ \cline{1-1}\end{array}}(q^{k-n}, q^{2-k+n}, \ldots, q^{n-k})$$

Lifts at least for $n \leq 3$ [G.-Martínez-Szwej-Wildon'?+]

List of formulas that lift

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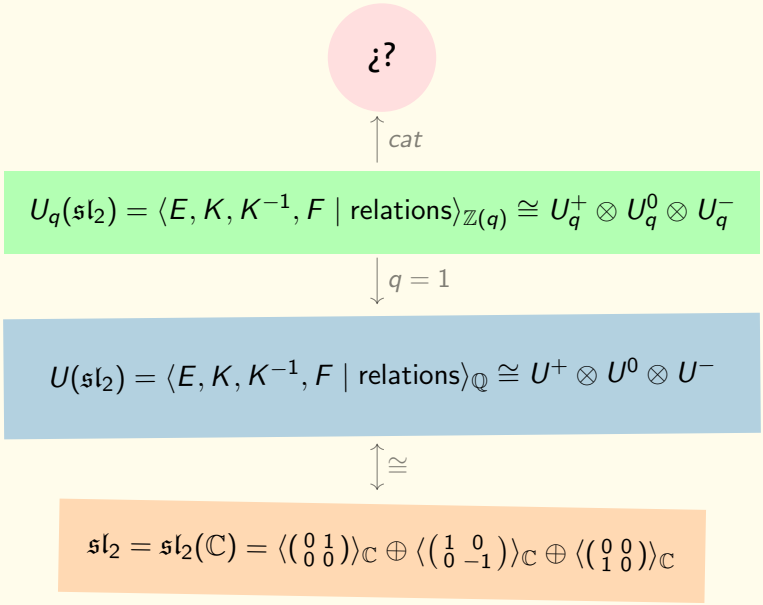
- $a^{\lambda_1-\lambda_2+1}_{n[m]} = g_{(n^k),(n^k),(\lambda_1,\lambda_2)}$ [Ikenmeyer-Omar-Tsintsilidas'25+]

Thank you!

Goal: q-Pfaff–Saalschütz identity + Jacobi–Trudi formula

$$s_{\begin{array}{|c|c|}\hline n & m \\ \hline n \\ \cline{1-1}\end{array}}(q^{-d}, q^{2-d}, ..., q^{d}) = \sum_{k=0}^{n} \begin{bmatrix} d+k \\ 2n+m \end{bmatrix} s_{\begin{array}{|c|c|}\hline k & m \\ \hline k \\ \cline{1-1}\end{array}}(q^{k-n}, q^{2-k+n}, ..., q^{n-k})$$

Lifts at least for $n \leq 3$ [G.-Martínez-Szwej-Wildon'?+]


¿?
cat
$U_q(\mathfrak{sl}_2) = \langle E, K, K^{-1}, F \mid \text{relations} \rangle_{\mathbb{Z}(q)} \cong U_q^+ \otimes U_q^0 \otimes U_q^-$
$q = 1$
$U(\mathfrak{sl}_2) = \langle E, K, K^{-1}, F \mid \text{relations} \rangle_{\mathbb{Q}} \cong U^+ \otimes U^0 \otimes U^-$
$\cong$
$\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) = \langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \rangle_{\mathbb{C}} \oplus \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle_{\mathbb{C}} \oplus \langle \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \rangle_{\mathbb{C}}$