Groups of Diffeomorphisms

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Vorwort

Die vorliegende Arbeit basiert größtenteils auf Arktikeln die von D. B. A. Epstein, J. Mather, R. P. Filipkiewicz und A. Banyaga zwischen 1970 und 1988 publiziert wurden.

Das erste Kapitel geht im wesentlichen auf D. B. A Epstein zurück. Hier wird gezeigt, daß die Kommutatorgruppen gewisser Homöomorphismengruppen einfach sind (vgl. Theorem 1.3). Dieses eher allgemein gehaltene Resultat läßt sich auf $\mathrm{Diff}^r_c(M)_\circ$ anwenden, und so hat $\mathrm{Diff}^r_c(M)_\circ$ eine einfache Kommutatorgruppe (vgl. Corollary 1.17).

In den Kapiteln 2&3 wird ein Resultat von J. Mather präsentiert, das besagt: $\mathrm{Diff}^r_c(M)_\circ$ ist einfach, falls $r \neq \dim(M) + 1$. Eigentlich wird nur gezeigt, daß $\mathrm{Diff}^r_c(M)_\circ$ perfekt ist, und dann das Ergebnis des ersten Kapitels verwendet. Für $\dim(M) + 2 \leq r \leq \infty$ ist dies im zweiten Kapitel (vgl. Corollary 2.44), und für $1 \leq r \leq \dim(M)$ im dritten Kapitel (vgl. Corollary 3.5) ausgeführt. Ob $\mathrm{Diff}^{\dim(M)+1}_c(M)_\circ$ auch perfekt ist, ist meines Wissens noch nicht geklärt, der hier präsentierte Beweis versagt jedenfalls völlig bei $r = \dim(M) + 1$.

Im vierten Kapitel wird gezeigt, daß jeder Gruppenisomorphismus zwischen gewissen Diffeomorphismengruppen von einem Diffeomorphismus der zugrundeliegenden Mannigfaltigkeiten induziert wird. Insbesondere stammen isomorphe Diffeomorphismengruppen von diffeomorphen Mannigfaltigkeiten. Auch hier wird zuerst ein allgemeineres Resultat bewiesen (vgl. Theorem 4.14), das dann auf $\mathrm{Diff}^r_c(M)_\circ$ angewandt wird (vgl. Theorem 4.17). Die Beweise dieser Sätze gehen auf R. P. Filipkiewicz und A. Banyaga zurück. Die Methoden und Ergebnisse aus Kapitel 2 & 3 werden dabei nicht benötigt, und daher kann das vierte Kapitel auch unmittelbar nach dem ersten Kapitel gelesen werden.

Der Anhang behandelt das "diskrete Analogon" zu Theorem 4.17. Die Mannigfaltigkeiten werden durch endliche Mengen, und die Diffeomorphismengruppen durch Permutationsgruppen ersetzt. Sind die Mengen nicht sechselementig, wird wieder gezeigt, daß ein Gruppenisomorphismus zwischen Permutationsgruppen von einer Bijektion der zugrundeliegenden Mengen induziert wird (vgl. Corollary A.8). Für sechselementige Mengen stimmt das nicht (vgl. Theorem A.10). Im Anhang werden nur elementare gruppentheoretische Methoden verwendet, er kann also unabhängig von den anderen Kapiteln gelesen werden.

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Contents

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Abstract

Mainly this work is based on articles that were published by D. B. A. Epstein, J. Mather, R. P. Filipkiewicz and A. Banyaga between 1970 and 1988.

The first chapter is essentially due to D. B. A. Epstein. Here it is shown that the commutator groups of certain groups of homeomorphisms are simple (cf. Theorem 1.3). This result can be applied to $\mathrm{Diff}^r_c(M)_\circ$ and thus $\mathrm{Diff}^r_c(M)_\circ$ has a simple commutator group (cf. Corollary 1.17).

In chapter 2 & 3 a result of J. Mather is presented, that says: $\mathrm{Diff}^r_c(M)_\circ$ is simple, provided $r \neq \dim(M) + 1$. Actually it is shown that $\mathrm{Diff}^r_c(M)_\circ$ is perfect, then the result of the first chapter is used. For $\dim(M) + 2 \leq r \leq \infty$ this is carried out in the second chapter (cf. Corollary 2.44), and for $1 \leq r \leq \dim(M)$ in the third one (cf. Corollary 3.5). As far as I know it is not yet known whether $\mathrm{Diff}^{\dim(M)+1}_c(M)_\circ$ is simple as well. However, the proof presented here fails completely in the case $r = \dim(M) + 1$.

In the fourth chapter it is shown, that every isomorphism of groups between certain groups of diffeomorphisms is induced by a diffeomorphism of the underlying manifolds. Especially isomorphic groups of diffeomorphisms come from diffeomorphic manifolds. First the more general result Theorem 4.14 is proved, then it is applied to $\mathrm{Diff}^r_c(\mathbf{R}^n)_\circ$ (cf. Theorem 4.17). The proofs of these theorems are due to R. P. Filipkiewicz and A. Banyaga. The methods and results of chapter 2 & 3 are not required for chapter 4, and thus it can be read immediately after the first chapter.

The appendix deals with the discrete analogon of Theorem 4.17. The manifolds are replaced by finite sets and the groups of diffeomorphisms by groups of permutations. If the sets do not consist of six elements, it is again shown that an isomorphism of groups between two groups of permutations is induced by a bijection of the underlying sets (cf. Corollary A.8). If the sets do consist of six elements this is not true (cf. Theorem A.10). In the appendix only elementary group theoretic methods are used, hence it can be read independently of the other chapters.

1. The Simplicity of $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$

1.1. Definition. Let G be a group and $g, h \in G$. The *commutator* of g and h is $[g,h] := g^{-1}h^{-1}gh$. If E, F are two arbitrary subsets of G then $\langle E \rangle_G$ denotes the subgroup of G generated by E, and we let $[E,F] := \langle \{[e,f] : e \in E, f \in F\} \rangle_G$. $[G,G]$ is called the *commutator subgroup* of G, it's a normal subgroup of G. The *normalizer* of a subgroup $U \subseteq G$ is $\mathrm{N}_G(U) := \{g \in G : g^{-1}Ug = U\}$. It's the largest subgroup of G that contains U as normal subgroup. If N is a normal subgroup of G we write $N \lhd G$. A group G is said to be *simple* iff it has no normal subgroups except $\{e\}$ and G. G is called *perfect* iff $[G,G] = G$.

If X is a topological space, and $g : X \to X$ a homeomorphism then the support of g is

$$\mathrm{supp}(g) := \overline{\{x \in X : g(x) \neq x\}}.$$

1.2. Definition (Epstein's Axioms). Let X be a paracompact Hausdorff space, $\mathcal{U}$ a basis of open subsets of X and G a group of homeomorphisms of X. The triple $(G, X, \mathcal{U})$ is said to satisfy *Epstein's Axioms* iff the following holds:

Axiom 1: $U \in \mathcal{U} \implies g(U) \in \mathcal{U}$.

Axiom 2: G acts transitively on $\mathcal{U}$.

Axiom 3: Let $g \in G$, $U \in \mathcal{U}$ and let $\mathcal{V} \subseteq \mathcal{U}$ be a covering of X. Then there exists an integer n, elements $g_1, \dots, g_n \in G$ and $V_1, \dots, V_n \in \mathcal{V}$ such that $g = g_n \cdots g_1$, $\mathrm{supp}(g_i) \subseteq V_i$ and $\mathrm{supp}(g_i) \cup g_{i-1} \cdots g_1(\overline{U}) \neq X$ for $1 \leq i \leq n$.

1.3. Theorem (Epstein). *[3] Let $(X, G, \mathcal{U})$ satisfy Epstein's Axioms and let $\{1\} \neq N \subseteq G$ be a subgroup such that $[G,G] \subseteq \mathrm{N}_G(N)$. Then $[G,G] \subseteq N$.*

Proof. Obviously we may assume $G \neq \{Id\}$.

Claim 1. *X is connected, and no open subset of X is finite.*

Proof of Claim 1. X is connected, for suppose conversely $X = U \cup V$ where U and V are disjoint open subsets of X. Then $\mathcal{W} := \{W \in \mathcal{U} : W \subseteq U \,\mathrm{or}\, W \subseteq V\}$ is a covering of X, and hence by Axiom 3, G is generated by elements $g \in G$ supported on $W \in \mathcal{W}$. Therefore G is generated by elements supported either in U or in V. It follows, that each element of G maps U to U and V to V, but this contradicts Axiom 2 and hence X is connected.

Suppose F is a finite open subset of X. Choose $x \in F$. Since X is Hausdorff $\{x\}$ is an open and closed subset of X, and hence $X = \{x\}$, but this is a contradiction to the assumption $G \neq \{Id\}$. $\square$

Claim 2. *Let* $U, V \subseteq X$ *be open subsets and let* $g, h \in G$ *be homeomorphisms with* $\mathrm{supp}(h) \subseteq V$*,* $U \cap \mathrm{supp}(h) = \emptyset$ *and* $g(U) = V$*. Then*

$$[g,h] = \begin{cases} g^{-1}h^{-1}g & \text{on } g^{-1}(\mathrm{supp}(h)) \\ h & \text{elsewhere} \end{cases} \tag{1.1}$$

Especially we have $\mathrm{supp}([g,h]) \subseteq \mathrm{supp}(h) \cup g^{-1}(\mathrm{supp}(h)) \subseteq V \cup U$*.*

Proof of Claim 2. On $g^{-1}(\mathrm{supp}(h))$ (1.1) follows from $g^{-1}(\mathrm{supp}(h)) \cap \mathrm{supp}(h) = \emptyset$, on $\mathrm{supp}(h)$ it follows from $g(\mathrm{supp}(h)) \cap \mathrm{supp}(h^{-1}) = \emptyset$ and elsewhere it's obvious. □

Claim 3. $[G, G]$ *acts transitively on* $\mathcal{U}$*.*

Proof of Claim 3. Let $U_1, U_2 \in \mathcal{U}$ and choose $g \in G$ with $g(U_1) = U_2$ (Axiom 2). We have to find $g' \in [G, G]$ with $g'(U_1) = U_2$. By Axiom 3 there exist $n \in \mathbf{N}$, $V_1, \ldots, V_n \in \mathcal{U}$ and $h_1, \ldots, h_n \in G$ with $g = h_n \cdots h_1$, $\mathrm{supp}(h_i) \subseteq V_i$ and

$$K_i := \mathrm{supp}(h_i) \cup h_{i-1} \cdots h_1(\overline{U_1}) \neq X \qquad \forall 1 \leq i \leq n.$$

Choose $W_i \in \mathcal{U}$ with $W_i \subseteq X \setminus K_i$ and $g_i \in G$ with $g_i(W_i) = V_i$ (Axiom 2). Since $W_i \cap \mathrm{supp}(h_i) = \emptyset$ we can apply Claim 2 and obtain

$$[g_i, h_i] = \begin{cases} g_i^{-1}h_i^{-1}g_i & \text{on } g_i^{-1}(\mathrm{supp}(h_i)) \\ h_i & \text{elsewhere} \end{cases} \tag{1.2}$$

From $g_i^{-1}(\mathrm{supp}(h_i)) \cap h_{i-1} \cdots h_1(\overline{U_1}) \subseteq W_i \cap K_i = \emptyset$ and (1.2) we get

$$[g_i, h_i]|_{h_{i-1}\cdots h_1(\overline{U_1})} = h_i|_{h_{i-1}\cdots h_1(\overline{U_1})}$$

and thus inductively

$$\begin{aligned} \underbrace{[g_n, h_n] \cdots [g_1, h_1]}_{\in [G,G]}(U_1) &= [g_n, h_n] \cdots [g_2, h_2] h_1(U_1) \\ &\vdots \\ &= h_n \cdots h_1(U_1) = g(U_1) = U_2. \end{aligned}$$

□

Claim 4. *Let* $U \in \mathcal{U}$*,* $h \in G$ *with* $\mathrm{supp}(h) \subseteq U$*. Then there exists* $g \in N$ *with* $g|_U = h|_U$*.*

Proof of Claim 4. Since $N \neq \{Id\}$ we can choose $\alpha \in N, \alpha \neq Id$ and $x \in X$, $\alpha(x) \neq x$. Further we choose $\widetilde{U} \in \mathcal{U}$ with $x \in \widetilde{U}$ and $\widetilde{U} \cap \alpha^{-1}(\widetilde{U}) = \emptyset$. Since a paracompact Hausdorff space is normal we find $V, W \in \mathcal{U}$ such that $x \in V$, $V \cap W = \emptyset$ and $\overline{V} \cup \overline{W} \subseteq \widetilde{U}$. By Axiom 2 there exists $\widetilde{g} \in G$ with $\widetilde{g}(W) = V$, and by Claim 3 we find $k \in [G, G]$ with $k(V) = U$. Let $\widetilde{h} := k^{-1}hk$ and $\rho := [\alpha, [\widetilde{g}, \widetilde{h}]]$. We have $\mathrm{supp}(\widetilde{h}) \subseteq V$ and $\rho \in [N, [G, G]] \subseteq N$, for $[G, G] \subseteq \mathrm{N}_G(N)$. From Claim 2 we obtain $[\tilde g, \tilde h]|_V = h|_V$ and $[\alpha, [\tilde g, \tilde h]]|_{\tilde U} = [\tilde g, \tilde h]|_{\tilde U}$, since $\mathrm{supp}([\tilde g, \tilde h]) \subseteq \tilde U$. Hence $\rho|_V = \tilde h|_V$. Since $k \in [G,G] \subseteq \mathrm{N}_G(N)$ we have $g := k\rho k^{-1} \in N$ and

$$g|_U = k\rho k^{-1}|_U = k\tilde h k^{-1}|_U = kk^{-1}hkk^{-1}|_U = h|_U.$$

□

Claim 5. *The orbits of N are the same as the orbits of G. In particular they are dense by Axiom 2.*

Proof of Claim 5. Let $x \in X$ and $y \in G(x)$. Choose n minimal with respect to the condition that there exist $g_1, \dots, g_n \in G$, $U_1, \dots, U_n \in \mathcal{U}$ with $\mathrm{supp}(g_i) \subseteq U_i$ and $y = g_n \cdots g_1(x)$. By Claim 4 we obtain $h_i \in N$ with $h_i|_{U_i} = g_i|_{U_i}$. Since n is minimal $g_{i-1} \cdots g_1(x) \in U_i$, $1 \le i \le n$, for otherwise we could omit g_i and this would contradict the minimality of n. Thus we obtain inductively

$$\begin{aligned} y = g_n \cdots g_1(x) &= h_n g_{n-1} \cdots g_1(x) \\ &\vdots \\ &= \ldots = h_n \cdots h_1(x) \in N(x). \end{aligned}$$

□

Claim 6. *Let $h_1, h_2 \in G$, $V \in \mathcal{U}$, $\mathrm{supp}(h_i) \subseteq V$. Then $[h_1, h_2] \in N$.*

Proof of Claim 6. Let $x \in X$. By Claim 5 we find $\alpha_1, \alpha_2 \in N$ with $x, \alpha_1^{-1}(x), \alpha_2^{-1}(x)$ distinct. Now choose $U \in \mathcal{U}$ with $x \in U$ and $U, \alpha_1^{-1}(U), \alpha_2^{-1}(U)$ disjoint and choose $g_1, g_2 \in G$ such that $x, g_1^{-1}(x), g_2^{-1}(x)$ are distinct elements of U. Further we select $\tilde V \in \mathcal{U}$ with $x \in \tilde V$ and $\overline{\tilde V}, g_1^{-1}(\overline{\tilde V}), g_2^{-1}(\overline{\tilde V})$ disjoint subsets of U. By Claim 3 we find $k \in [G,G]$ with $k(\tilde V) = V$. As we did in Claim 4, we let $\tilde h_i := k^{-1}h_i k$. Whereas h_i are supported in an arbitrary $V \in \mathcal{U}$, $\tilde h_i$ are supported in the suitable $\tilde V \in \mathcal{U}$. Suppose we have $[\tilde h_1, \tilde h_2] \in N$ then $[h_1, h_2] = kk^{-1}[h_1, h_2]kk^{-1} = k[k^{-1}h_1k, k^{-1}h_2k]k^{-1} = k[\tilde h_1, \tilde h_2]k^{-1} \in N$, since $[G,G] \subseteq \mathrm{N}_G(N)$.

So it remains to show $[\tilde h_1, \tilde h_2] \in N$. Let $\rho_i := [\alpha_i, [g_i, \tilde h_i]] \in [N, [G,G]] \subseteq N$. As in Claim 4 it follows $\rho_i|_{\tilde V} = \tilde h_i|_{\tilde V}$ and hence

$$[\rho_1, \rho_2]|_{\tilde V} = [h_1, h_2]|_{\tilde V}. \tag{1.3}$$

$\rho_i|_{X \setminus \tilde V}$ is supported on $g_i^{-1}(\tilde V) \cup \alpha_i^{-1}(U)$. Because of the choice of the sets U and $\tilde V$, $\mathrm{supp}(\rho_1|_{X \setminus \tilde V}) \cap \mathrm{supp}(\rho_2|_{X \setminus \tilde V}) = \emptyset$ and hence

$$[\rho_1, \rho_2]|_{X \setminus \tilde V} = Id|_{X \setminus \tilde V} = [\tilde h_1, \tilde h_2]|_{X \setminus \tilde V}. \tag{1.4}$$

From equations (1.3) and (1.4) we obtain $[h_1, h_2] = [\rho_1, \rho_2] \in N$. □

Claim 7. *There exists a covering $\mathcal{V} \subseteq \mathcal{U}$ of X, such that the following holds:*

$$V_1, V_2 \in \mathcal{V}, V_1 \cap V_2 \neq \emptyset \implies \exists U \in \mathcal{U} : V_1 \cup V_2 \subseteq U.$$

Proof of Claim 7. Since X is paracompact, we can choose $\{U_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{U}$, $\{\widetilde{U}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{U}$ with $\overline{U_\lambda} \subseteq \widetilde{U}_\lambda$ and such that $\{U_\lambda\}_{\lambda\in\Lambda}$ is a locally finite cover of X. For every $x \in X$ we choose $\widetilde{V}_x \in \mathcal{U}$, $x \in \widetilde{V}_x$, such that $\widetilde{V}_x$ intersects only finitely many $U_{\lambda_1}, \dots U_{\lambda_{n_x}}$. We can assume $x \in \overline{U_{\lambda_i}} \subseteq \widetilde{U}_{\lambda_i}$, $1 \le i \le n_x$, by shrinking $\widetilde{V}_x$ if necessary. Let $V_x \in \mathcal{U}$, $x \in V_x$ and

$$V_x \subseteq \widetilde{V}_x \cap \bigcap_{i=1}^{n_x} \widetilde{U}_{\lambda_i}$$

Let $\mathcal{V} := \{V_x\}_{x\in X}$. Obviously $\mathcal{V}$ is a covering of X and $\mathcal{V} \subseteq \mathcal{U}$. Furthermore we have

$$V \in \mathcal{V}, V \cap U_\lambda \neq \emptyset \Longrightarrow V \subseteq \widetilde{U}_\lambda \tag{1.5}$$

Now take $V_1, V_2 \in \mathcal{V}$ with $V_1 \cap V_2 \neq \emptyset$. Since $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of X we find U_λ intersecting V_1 and V_2. By (1.5) we obtain $V_1 \subseteq \widetilde{U}_\lambda$ and $V_2 \subseteq \widetilde{U}_\lambda$. Since $\widetilde{U}_\lambda \in \mathcal{U}$, we are done. □

Let $\mathcal{V}$ be the covering constructed in Claim 7, and consider

$$E := \{h \in G : \exists V \in \mathcal{V} \text{ with } \mathrm{supp}(h) \subseteq V\}.$$

E generates G by Axiom 3. Let $h_1, h_2 \in E$ and choose $V_1, V_2 \in \mathcal{V}$ with $\mathrm{supp}(h_i) \subseteq V_i$. If $V_1 \cap V_2 = \emptyset$ then $[h_1, h_2] = Id \in N$. If $V_1 \cap V_2 \neq \emptyset$ then by Claim 7 there exists $U \in \mathcal{U}$ with $V_1 \cup V_2 \subseteq U$ and by Claim 6 we obtain $[h_1, h_2] \in N$ too. Summing up we have

$$[h_1, h_2] \in N \qquad \forall h_1, h_2 \in E. \tag{1.6}$$

In general $[E, E]$ need not generate $[G, G]$, so we are not finished yet.

Claim 8. *$g \in G$, $h_1, h_2 \in E \implies g[h_1, h_2]g^{-1} \in N$.*

Proof of Claim 8. Choose $W_i \in \mathcal{V}$ with $\mathrm{supp}(h_i) \in W_i$ for $i = 1, 2$. We can assume $W_1 \cap W_2 \neq \emptyset$, for otherwise $[h_1, h_2] = Id$. Let $U \in \mathcal{U}$ with $W_1 \cup W_2 \subseteq U$. By Axiom 3 we obtain $g_1, \dots, g_n \in G$, $V_1, \dots, V_n \in \mathcal{U}$ with $\mathrm{supp}(g_i) \subseteq V_i$, $g = g_n \cdots g_1$ and $K_i := \mathrm{supp}(g_i) \cup g_{i-1} \cdots g_1(\overline{U}) \neq X$. Now choose $U_i \in \mathcal{U}$ with $U_i \subseteq X \setminus K_i$ and choose $\beta_i \in G$ with $\beta_i(U_i) = V_i$. From Claim 2 we get

$$\gamma_i := [\beta_i, g_i] = \begin{cases} \beta_i^{-1} g_i^{-1} \beta_i & \text{on } U_i \\ g_i & \text{on } X \setminus U_i \end{cases} \tag{1.7}$$

We will show

$$g[h_1, h_2]g^{-1} = \gamma_n \cdots \gamma_1 [h_1, h_2] \gamma_1^{-1} \cdots \gamma_n^{-1} \tag{1.8}$$

for we are done then since $\gamma_i \in [G, G] \subseteq \mathrm{N}_G(N)$ and $[h_1, h_2] \in N$ by (1.6). To see (1.8) notice that $\mathrm{supp}(g_{i-1} \cdots g_1 [h_1, h_2] g_1^{-1} \cdots g_{i-1}^{-1}) \subseteq g_{i-1} \cdots g_1(U)$ since $\mathrm{supp}([h_1, h_2]) \subseteq U$ and hence

$$g_{i-1} \cdots g_1 [h_1, h_2] g_1^{-1} \cdots g_{i-1}^{-1}|_{U_i} = Id|_{U_i} \tag{1.9}$$

for we have $U_i \cap g_{i-1} \cdots g_1(U) = \emptyset$ by the choice of U_i. Since $g_i(U_i) = U_i$ and $\gamma_i(U_i) = U_i$ by (1.7) we obtain from (1.9)

$$g_i \cdots g_1 [h_1, h_2] g_1^{-1} \cdots g_i^{-1}|_{U_i} = Id|_{U_i} = \gamma_i g_{i-1} \cdots g_1 [h_1, h_2] g_1^{-1} \cdots g_{i-1}^{-1} \gamma_i^{-1}|_{U_i}. \tag{1.10}$$

From (1.10) and since $g_i|_{X\setminus U_i} = \gamma_i|_{X\setminus Ui}$ by (1.7), we have

$$g_i \cdots g_1[h_1,h_2]g_1^{-1}\cdots g_i^{-1} = \gamma_i g_{i-1}\cdots g_1[h_1,h_2]g_1^{-1}\cdots g_{i-1}^{-1}\gamma_i^{-1},$$

and thus (1.8) follows inductively. □

Consider $S := \langle\{g[h_1,h_2]g^{-1} : g \in G, h_1, h_2 \in E\}\rangle_G$. Obviously this is a normal subgroup of G and by Claim 8 we have $S \subseteq N$. Since G is generated by E, G/S is generated by $\{hS : h \in E\}$. Since $[h_1,h_2] \in S$ for $h_i \in E$ the generators of G/S commute and hence G/S is abelian. Consequently

$$[g,h]S = [gS,hS] = S \qquad \forall g,h \in G$$

and hence $[G,G] \subseteq S \subseteq N$. This completes the proof of Theorem 1.3. □

1.4. Corollary. *Let* $(G,X,\mathcal{U})$ *satisfy Epstein's Axioms, then* $[G,G]$ *is simple.*

Proof. Suppose $\{Id\} \neq N \lhd [G,G]$ then $[G,G] \subseteq \mathrm{N}_G(N)$. By Theorem 1.3 $[G,G] \subseteq N$ and hence $[G,G]$ is simple. □

1.5. Corollary. *Let* $(G,X,\mathcal{U})$ *satisfy Epstein's Axioms, then every nontrivial normal subgroup of* G *contains* $[G,G]$. *Thus* $[G,G]$ *is the unique minimal normal subgroup of* G.

Proof. Let $\{Id\} \neq N \lhd G$. This implies $[G,G] \subseteq \mathrm{N}_G(N) = G$, and hence by Theorem 1.3 $[G,G] \subseteq N$. □

1.6. Definition. Throughout the text a *smooth manifold* is a connected, paracompact, separable, boundaryless C^∞-manifold, $1 \leq \dim(M) < \infty$. Suppose M, N are smooth manifolds and $0 \leq r \leq \infty$, then $\mathrm{C}^r(M,N)$ denotes the space of C^r-mappings $M \to N$ equipped with the *Whitney topology* (cf. Proposition 1.7). Further $\mathrm{Diff}^r(M)$ resp. $\mathrm{Diff}^r_c(M)$ denotes the group of C^r- resp. compactly supported C^r-diffeomorphisms of M. If A is a fixed subset of M we let $\mathrm{Diff}^r_A(M)$ be the group of C^r-diffeomorphisms supported on A. We equip these groups with the Whitney- or C^r-topology i.e. the initial topology with respect to the inclusions $\mathrm{Diff}^r(M) \subseteq \mathrm{C}^r(M,M)$, $\mathrm{Diff}^r_c(M) \subseteq \mathrm{C}^r(M,M)$ and $\mathrm{Diff}^r_A(M) \subseteq \mathrm{C}^r(M,M)$.

If G is a topological group we denote by G_o the component containing Id. Further we let $\mathbf{B}^n_r(x) := \{y \in \mathbf{R}^n : \|x-y\| < r\}$, the open ball in $\mathbf{R}^n$ with radius r and center x, and $\mathbf{B}^n := \mathbf{B}^n_1(0)$, the open unit ball.

1.7. Proposition. *We list some facts concerning the Whitney- or* C^r*-topology. For details see [6] and [9].*

(1) $\mathrm{Diff}^r(M)$ *is open in* $\mathrm{C}^r(M,M)$ *for* $1 \leq r \leq \infty$.

(2) The set $\mathrm{Emb}^r(M,N) \subseteq \mathrm{C}^r(M,N)$ *of* C^r*-embeddings of* M *in* N *is open for* $1 \leq r \leq \infty$.

(3) Let $\mathrm{C}^r_{prop}(M,N) \subseteq \mathrm{C}^r(M,N)$ *denote the subspace of proper mappings (*$f : M \to N$ *is proper iff* $K \subseteq N$ *compact* $\Rightarrow f^{-1}(K)$ *is compact). Then*

$$\begin{array}{rcl} \mathrm{C}^r(N,P) \times \mathrm{C}^r_{prop}(M,N) & \to & \mathrm{C}^r(M,P) \\ (g,h) & \mapsto & g \circ h \end{array}$$

is continuous for $0 \leq r \leq \infty$.

(4) The following mapping is continuous for $0 \leq r \leq \infty$*:*

$$\begin{array}{rcl} \mathrm{C}^r(M,N_1) \times \mathrm{C}^r(M,N_2) & \to & \mathrm{C}^r(M,N_1 \times N_2) \\ (f_1,f_2) & \mapsto & (f_1,f_2) \end{array}$$

(5) If $0 \leq r \leq \infty$ *and* $E \xrightarrow{\pi} M$ *is a finite dimensional, smooth vector bundle over* M*, then* $\Gamma^r_c(E \xrightarrow{\pi} M)$*, the set of compactly supported* C^r*-sections of* $E \xrightarrow{\pi} M$ *equipped with the Whitney-topology, i.e. the initial topology with respect to the inclusion* $\Gamma^r_c(E \xrightarrow{\pi} M) \subseteq \mathrm{C}^r(M,E)$*, is a locally convex vector space. (compact support is essential here, for otherwise scalar-multiplication won't be continuous if* M *is not compact!)*

(6) If $0 \leq r \leq s \leq \infty$ *then* $\mathrm{C}^s(M,N) \subseteq \mathrm{C}^r(M,N)$ *is a dense subset. Further if* $f \in \mathrm{C}^r(M,N)$*,* $\mathcal{N}$ *a neighborhood of* $f \in \mathrm{C}^r(M,N)$*,* $A \subseteq M$ *closed, and* f *is already* C^s *on a neighborhood of* A*, then there exists* $g \in \mathcal{N} \cap \mathrm{C}^s(M,N)$ *with* $g|_A = f|_A$*.*

(7) $\mathrm{Diff}^r(M)$ *is a topological group for* $0 \leq r \leq \infty$*. Since* $\mathrm{Diff}^r_c(M)$ *and* $\mathrm{Diff}^r_c(M)_\circ$ *are subgroups of* $\mathrm{Diff}^r(M)$ *equipped with the initial topology, they are topological groups too.*

(8) We have

$$\mathrm{C}^\infty(M,N) = \varprojlim \mathrm{C}^r(M,N),$$

where the projective limit is over all $0 \leq r < \infty$ *and the inclusions* $\mathrm{C}^r(M,N) \hookrightarrow \mathrm{C}^k(M,N)$ *for* $r \geq k$*.*

(9) Given a locally finite family of charts $\Phi = (u_i, U_i)_{i \in I}$ *on* M*, a family of compact subsets* $K = (K_i)_{i \in I}$ *with* $K_i \subset U_i$*, a family of charts* $\Psi = (v_i, V_i)_{i \in I}$ *on* N *with* $f(K_i) \subseteq V_i$ *and a family of positive numbers* $\epsilon = (\epsilon_i)_{i \in I}$*, then we define* $\mathcal{N}^r(f; \Phi, \Psi, K, \epsilon)$ *to be the space of* $g \in \mathrm{C}^r(M,N)$ *with* $g(K_i) \subseteq V_i$ $\forall i \in I$ *and*

$$\|D^k(v_i f u_i^{-1})(x) - D^k(v_i g u_i^{-1})(x)\| < \epsilon_i$$

$\forall i \in I$*,* $\forall x \in u_i(K_i)$ *and* $\forall 0 \leq k \leq r$*. The sets* $\mathcal{N}^r(f; \Phi, \Psi, K, \epsilon)$ *as* Φ*,* Ψ*,* K*,* ϵ *vary form a neighborhood basis of* $f \in \mathrm{C}^r(M,N)$ *for* $0 \leq r < \infty$*.*

(10) *For* $0 \le r < \infty$ *consider the vector bundle of* r*-jets* $J^r(M,N) \xrightarrow{\pi} M$ *and the canonical injective mapping*

$$j^r : \mathrm{C}^r(M,N) \hookrightarrow \mathrm{C}^0(M, J^r(M,N)).$$

If we put the graph topology on $\mathrm{C}^0(M, J^r(M,N))$ *then the Whitney-topology on* $\mathrm{C}^r(M,N)$ *is the topology induced by the inclusion* j^r*. In other words the Whitney-topology is the coarsest topology on* $\mathrm{C}^r(M,N)$ *such that* j^r *becomes continuous.*

(11) *If* $0 \le r < \infty$ *and* d_r *is a metric on* $J^r(M,N)$ *generating the topology, then a neighborhood basis for* $f \in \mathrm{C}^r(M,N)$ *is given by:*

$$\mathcal{N}(f;r,\epsilon) := \{g \in \mathrm{C}^r(M,N) : d_r(j^r f(x), j^r g(x)) < \epsilon(x)\}$$

where ϵ *varies over* $C(M,(0,\infty))$.

1.8. Remark. We have:

$$\begin{array}{lll} \mathrm{Diff}^r_c(M)_\circ \lhd \mathrm{Diff}^r(M), & \mathrm{Diff}^r_c(M)_\circ \lhd \mathrm{Diff}^r(M)_\circ, & \mathrm{Diff}^r_c(M)_\circ \lhd \mathrm{Diff}^r_c(M) \\ \mathrm{Diff}^r_c(M) \lhd \mathrm{Diff}^r(M), & \mathrm{Diff}^r(M)_\circ \lhd \mathrm{Diff}^r(M) & \end{array}$$

Proof. If $f, g \in \mathrm{Diff}^r(M)$, then $\mathrm{supp}(gfg^{-1}) = g(\mathrm{supp}(f))$. Hence if f is compactly supported, then so is gfg^{-1}. Now everything follows from the continuity of $conj_g : \mathrm{Diff}^r(M) \to \mathrm{Diff}^r(M)$, $conj_g(f) := gfg^{-1}$ and the fact $conj_g(Id) = Id \in \mathrm{Diff}^r_c(M)_\circ$. □

1.9. Definition. Let M be a smooth manifold and $0 \le r \le \infty$. A C^r*-diffeotopy* of M is a C^r-mapping $H : M \times I \to M$ such that $H_t := H(\,\cdot\,,t) \in \mathrm{Diff}^r(M)$. H_1 is said to be C^r*-diffeotopic to the identity through* H. The *support of a diffeotopy* H is

$$\mathrm{supp}(H) := \overline{\{x \in M : \exists t \in I \text{ with } H(x,t) \neq x\}}.$$

1.10. Remark. Given a compactly supported, time dependent C^r-vector field X on M, one obtains a compactly supported C^r-diffeotopy $H : M \times I \to M$ by integrating X, i.e. the unique mapping $H : M \times I \to M$ with $H_0 = Id$ and $\frac{d}{ds}|_t H_s(x) = X(H_t(x),t)$ is a C^r-diffeotopy.

Conversely, given a compactly supported C^r-diffeotopy $H : M \times I \to M$, one obtains a compactly supported, time dependent C^{r-1}-vector field $\dot{H}$ on M given by $\dot{H}(x,t) := \frac{d}{ds}|_t H_s(H_t^{-1}(x))$. Clearly these two constructions are inverse to each other.

1.11. Lemma. *Let* M *be a connected, smooth manifold,* $\sigma : I \to M$ *a smooth path in* M *and let* U *be an open neighborhood of* $\mathrm{Im}(\sigma) \subseteq M$*. Then there exists* $f \in \mathrm{Diff}^\infty_U(M)_\circ$ *with* $f(\sigma(0)) = \sigma(1)$*. In particular* $\mathrm{Diff}^r_c(M)_\circ$ *acts transitively on* M *for* $1 \le r \le \infty$.

Proof. First we choose a smooth path $\widetilde{\sigma} : I \to M$ joining $x = \sigma(0)$ and $y = \sigma(1)$, such that $\widetilde{\sigma}$ is an embedding and $\mathrm{Im}(\widetilde{\sigma}) \subseteq U$. Hence there exists a C^∞-vector field X on M with $\mathrm{supp}(X) \subseteq U$ and $X(\widetilde{\sigma}(t)) = \frac{d}{ds}|_t \widetilde{\sigma}(s)$. By Remark 1.10 $\mathrm{Fl}^X : M \times I \to M$ is a compactly supported C^∞-diffeotopy and thus $\mathrm{Fl}^X_1 \in \mathrm{Diff}^\infty_U(M)_\circ$. Finally we have $\mathrm{Fl}^X_1(x) = y$ since $\widetilde{\sigma} : I \to M$ is an integral curve of X. □

1.12. Lemma. *If* $1 \leq r \leq \infty$ *and* $H : M \times I \to M$ *is a compactly supported* C^r*-diffeotopy then* $\check{H} : I \to \mathrm{Diff}^r_c(M)$ *is continuous. Especially we have* $\check{H} : I \to \mathrm{Diff}^r_c(M)_\circ$ *since* $\check{H}(0) = Id$.

Proof. For finite r this follows easily from the description of the Whitney-topology given in Proposition 1.7(11) and for $r = \infty$ it follows from Proposition 1.7(8) and the universal property of the projective limit. □

1.13. Proposition. *[9] Let M be a smooth manifold, $exp : TM \to M$ the exponential mapping obtained from a Riemannian metric on M and let $1 \leq r \leq \infty$. Then there exists a convex, open neighborhood U' of the zero section in $\Gamma^1_c(TM \xrightarrow{\pi} M)$ and an open neighborhood U of $Id \in \mathrm{Diff}^1_c(M)_\circ$ such that the map*

$$\begin{aligned} \varphi : U' \cap \Gamma^r(TM \xrightarrow{\pi} M) &\longrightarrow U \cap \mathrm{Diff}^r(M) \\ \sigma &\longmapsto \exp \circ \sigma \end{aligned}$$

is well defined and a homeomorphism.

Proof. It is well known that there exists an open neighborhood V' of the image of the zero section in TM, and an open neighborhood V of the diagonal $\mathrm{diag}(M) \subseteq M \times M$ such that

$$TM \supseteq V' \xrightarrow{(\pi,\exp)} V \subseteq M \times M$$

is an C^∞-diffeomorphism. Now let

$$\begin{aligned} U' &:= \{\sigma \in \Gamma^r_c(TM \xrightarrow{\pi} M) : \sigma(M) \subseteq V'\}, \\ U &:= \{f \in \mathrm{C}^r_c(M, M) : (Id, f)(M) \subseteq V\}, \end{aligned}$$

define $\varphi : U' \to U$ like above and let

$$\begin{aligned} U &\xrightarrow{\psi} U' \\ f &\mapsto (\pi, \exp)^{-1} \circ (Id, f) \end{aligned}$$

Clearly everything is well defined, U and U' are open, $0 \in U'$ and $Id \in U$ To see that $\psi(f)$ is a section of $TM \xrightarrow{\pi} M$ compute as follows:

$$\pi \circ \psi(f) = \pi \circ (\pi, exp)^{-1} \circ (Id, f) = pr_1 \circ (\pi, exp) \circ (\pi, exp)^{-1} \circ (Id, f) = Id.$$

Since

$$\varphi(\psi(f)) = \exp \circ\ (\pi, \exp)^{-1} \circ (Id, f) = pr_2 \circ (\pi, \exp) \circ\ (\pi, \exp)^{-1} \circ (Id, f) = f$$

and

$$\psi(\varphi(\sigma)) = (\pi, \exp)^{-1} \circ (Id, \exp \circ \sigma) = (\pi, \exp)^{-1} \circ (\pi, \exp) \circ \sigma = \sigma,$$

ψ is the inverse to φ. By Proposition 1.7(3)&(4) both, φ and ψ, are continuous. Using Proposition 1.7(1)&(5) we can shrink U and U' such that the proposition holds. □

1.14. Corollary. *Let M be a smooth manifold and $1 \leq r \leq \infty$. Then $\mathrm{Diff}^r_c(M)_\circ$ is locally contractible and consists of exactly those C^r-diffeomorphisms that are diffeotopic to the identity through a compactly supported C^r-diffeotopy.*

Proof. Let $f \in \mathrm{Diff}^r_c(M)_\circ$. Then $f_* : \mathrm{Diff}^r_c(M)_\circ \to \mathrm{Diff}^r_c(M)_\circ$, $f_*(g) := fg$ is a homeomorphism by Proposition 1.7(3). Hence with the notation from Proposition 1.13 $f_* \circ \varphi : U' \to f_*(U)$ is a homeomorphism onto an open neighborhood of f. Since U' is a convex subset of a topological vector space it is contractible, and thus the neighborhood $f_*(U)$ of f is contractible.

By Lemma 1.12 $\mathrm{Diff}^r_c(M)_\circ$ contains all C^r-diffeomorphisms that are diffeotopic to the identity through a compactly supported C^r-diffeotopy. Conversely if $f \in U \subseteq \mathrm{Diff}^r_c(M)_\circ$ we have a compactly supported C^r-diffeotopy with $H_0 = Id$ and $H_1 = f$ given by $H(x,t) := \varphi(t\varphi^{-1}(f))(x)$. Since $\mathrm{Diff}^r_c(M)_\circ$ is generated by U it remains to show that 'being C^r-diffeotopic to Id' is compatible with composition. Therefore consider two compactly supported C^r-diffeotopies F resp. G with $F_0 = G_0 = Id$. Then

$$H(x,t) := F(G_1^{-1}(G(x, 1-t)), t)$$

is a compactly supported C^r-diffeotopy from $F_1 G_1^{-1}$ to Id. □

1.15. Example. Consider $M = \mathbf{R}^n$ and the exponential mapping obtained from the usual Riemannian metric on $\mathbf{R}^n$. Via the homeomorphism (cf. Proposition 1.7(3))

$$\begin{array}{rcl} \Gamma^r_c(T\mathbf{R}^n \xrightarrow{\pi} \mathbf{R}^n) \cong \Gamma^r_c(\mathbf{R}^n \times \mathbf{R}^n \xrightarrow{pr_1} \mathbf{R}^n) & \xrightarrow{\cong} & C^r_c(\mathbf{R}^n, \mathbf{R}^n) \\ \sigma & \mapsto & pr_2 \circ \sigma \end{array}$$

(here $\sigma \in C^r_c(\mathbf{R}^n, \mathbf{R}^n)$ means: $\sigma \in C^r(\mathbf{R}^n, \mathbf{R}^n)$ and σ is zero outside a compact set) the mapping φ defined in Proposition 1.13 is given by $\varphi : C^r_c(\mathbf{R}^n, \mathbf{R}^n) \supseteq U' \to U \subseteq \mathrm{Diff}^r_c(\mathbf{R}^n)_\circ$, $\sigma \mapsto \sigma + Id$.

We have $\mathrm{Diff}^r_c(\mathbf{R}^n) = \mathrm{Diff}^r_c(\mathbf{R}^n)_\circ$, since for arbitrary $f \in \mathrm{Diff}^r_c(\mathbf{R}^n)$ we have a compactly supported C^r-diffeotopy $H : \mathbf{R}^n \times [\epsilon, 1] \to \mathbf{R}^n$, given by $H(x,t) := tf(\frac{1}{t}x)$. By choosing ϵ sufficiently small we can achieve, that $H_\epsilon \in \mathrm{Diff}^0_c(\mathbf{R}^n)_\circ$ and hence there is a prolongation $H : \mathbf{R}^n \times I \to \mathbf{R}^n$ that is a C^0-diffeotopy from f to Id with $H_t = Id$ $\forall t \in [0, \frac{\epsilon}{2}]$. Since C^0-mappings are dense in C^r-mappings and since H is already C^r on $\mathbf{R}^n \times ([0, \frac{1}{2}\epsilon] \cup [\epsilon, 1]$ we find a compactly supported C^r-diffeotopy $\widetilde{H}$ from f to Id, and thus $f \in \mathrm{Diff}^r_c(\mathbf{R}^n)_\circ$. Moreover, if $\mathrm{supp}(f) \subseteq \mathbf{B}^n$, then $\widetilde{H}$ can be chosen such that $\mathrm{supp}(\widetilde{H}) \subseteq \mathbf{B}^n$. To see the last assertion let $\widetilde{H}$ be a diffeotopy from f to Id like above, choose $1 \leq D \in \mathbf{R}$ with $\mathrm{supp}(\widetilde{H}) \subseteq \mathbf{B}^n_D(0)$ and assume that $\widetilde{H}_t = f$

$\forall t \in [\frac{1}{2}, 1]$. If we choose $\lambda \in \mathrm{C}^\infty([0,1],[1,D])$ with $\lambda|_{[0,\frac{1}{2}]} = D$ and $\lambda(1) = 1$, then $G(x,t) := \frac{1}{\lambda(t)}\widetilde{H}(\lambda(t)x,t)$ is a C^r-diffeotopy from f to Id and $\mathrm{supp}(G) \subseteq \mathbf{B}^n$.

1.16. Lemma. *[3] Let M be a smooth manifold, $1 \leq r \leq \infty$, $n = \dim(M)$ and*

$$\mathcal{U}^r(M) := \{\varphi(\mathbf{B}^n) : \mathbf{R}^n \overset{\varphi}{\hookrightarrow} M \quad \mathrm{C}^r\text{−embedding}\}.$$

Then $(\mathrm{Diff}_c^r(M)_\circ, M, \mathcal{U}^r(M))$ satisfies Epstein's Axioms.

Proof. Obviously $(\mathrm{Diff}_c^r(M)_\circ, M, \mathcal{U}^r(M))$ satisfies Axiom 1.

To see, that Axiom 2 is satisfied, consider C^r-embeddings $\varphi_1, \varphi_2 : \mathbf{R}^n \hookrightarrow M$. We have to find $f \in \mathrm{Diff}_c^r(M)_\circ$ with and $f(\varphi_1(\mathbf{B}^n)) = \varphi_2(\mathbf{B}^n)$. Since $\mathrm{Diff}_c^r(M)_\circ$ acts transitively on M we can assume $\varphi_1(0) = \varphi_2(0)$, and since we can shrink $\varphi_i(\mathbf{B}^n)$ by elements of $\mathrm{Diff}_c^r(M)_\circ$ we can additionally assume that $\varphi_i(\mathbf{B}^n) \subseteq U$, where (u, U) is a chart of M centered at $\varphi_1(0) = \varphi_2(0)$. So it suffices to find $f \in \mathrm{Diff}_c^r(\mathbf{R}^n)_\circ$ with $f(\varphi(\mathbf{B}^n)) = \mathbf{B}^n$ for all embeddings $\varphi : \mathbf{R}^n \hookrightarrow \mathbf{R}^n$ with $\varphi(0) = 0$. Besides we may assume that $\det(D\varphi(0)) > 0$, for if we define $\sigma : \mathbf{R}^n \to \mathbf{R}^n$ by $\sigma(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$, we have $\varphi \circ \sigma(\mathbf{B}^n) = \varphi(\mathbf{B}^n)$ and $\det(D(\varphi \circ \sigma)(0)) = -\det(D\varphi(0))$.

Claim 1. *There exists a C^r-isotopy from φ to Id, i.e. a C^r-mapping $H : \mathbf{R}^n \times I \to \mathbf{R}^n$ such that $H_t := H(\cdot, t) : \mathbf{R}^n \to \mathbf{R}^n$ is an embedding $\forall t \in I$, $H_0 = Id$ and $H_1 = \varphi$.*

Proof of Claim 1. Consider the following C^{r-1}-isotopy from $D\varphi(0)$ to φ:

$$\begin{array}{rcl} \mathbf{R}^n \times I & \to & \mathbf{R}^n \\ (x,t) & \mapsto & \begin{cases} \frac{1}{t}\varphi(tx) & \text{if } t \neq 0 \\ D\varphi(0) \cdot x & \text{if } t = 0 \end{cases} \end{array}$$

Since $D\varphi(0) \in GL^+(n, \mathbf{R})$ and $GL^+(n, \mathbf{R})$ is connected, we find a C^{r-1}-isotopy from Id to $D\varphi(0)$ and by composing and reparametrizing we obtain a C^{r-1}-isotopy $G : \mathbf{R}^n \times I \to \mathbf{R}^n$ with $G_t = Id$ $\forall t \in [0, \frac{1}{3}]$ and $G_t = \varphi$ $\forall t \in [\frac{2}{3}, 1]$. Finally, we approximate G by a C^r-mapping H that equals G on $\mathbf{R}^n \times [0, \frac{1}{4}] \cup \mathbf{R}^n \times [\frac{3}{4}, 1]$. This is possible, since G is already C^r on $\mathbf{R}^n \times [0, \frac{1}{3}) \cup \mathbf{R}^n \times (\frac{2}{3}, 1]$. Choosing H sufficiently near G we can achieve that H_t remain embeddings (cf. Proposition 1.7(2)) and hence H is a C^r-isotopy. □

Now consider the locally defined, time dependent C^{r-1}-vector field on $\mathbf{R}^n$ given by $\dot{H}(x,t) := \frac{d}{ds}|_t H_s(H_t^{-1}(x))$. We then have $\frac{d}{ds}|_t H_s(x) = \dot{H}(H_t(x), t)$. The domain of $\dot{H}$ is $(H, pr_I)(\mathbf{R}^n \times I)$. Hence we can find a compactly supported, globally defined, time dependent C^{r-1}-vector field $\dot{G}$ on $\mathbf{R}^n$ such that $\dot{G}$ equals $\dot{H}$ on $(H, pr_I)(\mathbf{B}_2^n(0) \times I)$. Thus, integrating $\dot{G}$ we obtain a compactly supported C^{r-1}-diffeotopy G with $G_0 = Id$ and $G|_{\mathbf{B}_2^n(0) \times I} = H|_{\mathbf{B}_2^n(0) \times I}$, especially G is C^r on $\mathbf{B}_2^n(0) \times I$. By approximation we find a compactly supported C^r-diffeotopy $\widetilde{G}$ with $\widetilde{G}|_{\mathbf{B}^n \times I} = G|_{\mathbf{B}^n \times I}$ and thus $\widetilde{G}_1(\mathbf{B}^n) = G_1(\mathbf{B}^n) = H_1(\mathbf{B}^n)$. Since $\widetilde{G}_1 \in \mathrm{Diff}_c^r(\mathbf{R}^n)_\circ$, we have verified Axiom 2.

Claim 2. (cf. (3.1) Lemma in [11]) *Let $f \in \mathrm{Diff}_c^r(M)_\circ$, $\mathcal{V} \subseteq \mathcal{U}$ be a cover of M. Then there exists an integer n, $V_1, \ldots, V_n \in \mathcal{V}$ and $f_i \in \mathrm{Diff}_c^r(M)_\circ$, $1 \leq i \leq n$, with* $\mathrm{supp}(f_i) \subseteq V_i$ *and $f = f_n \cdots f_1$.*

Proof of Claim 2. Choose a partition of unity $(\lambda_i)_{i\in\mathbf{N}}$ subordinated to the cover $\mathcal{V}$ and choose $V_i \in \mathcal{V}$ with $\mathrm{supp}(\lambda_i) \subseteq V_i$. Let $\varphi : U' \to U$ as in Proposition 1.13 and define $W' \subseteq \Gamma^r_c(TM \xrightarrow{\pi} M)$ by

$$W' := \{\sigma \in U' : (\sum_{i=1}^{n} \lambda_i)\sigma \in U', \varphi((\sum_{i=1}^{n} \lambda_i)\sigma)(\mathrm{supp}(\lambda_{n+1})) \subseteq V_{n+1} \quad \forall n \in \mathbf{N}\}.$$

W' is an open neighborhood of the zero section and hence $W := \varphi(W')$ is an open neighborhood of $Id \in \mathrm{Diff}^r_c(M)_\circ$. We can assume that $f \in W$, for W generates $\mathrm{Diff}^r_c(M)_\circ$. Since f is compactly supported there exists $n \in \mathbf{N}$ such that

$$\mathrm{supp}(\lambda_i) \cap \mathrm{supp}(\varphi^{-1}(f)) \quad = \quad \emptyset \qquad \forall i > n. \tag{1.11}$$

Now define $g_i := \varphi((\sum_{j=1}^{i} \lambda_j)\varphi^{-1}(f)) \in \mathrm{Diff}^r_c(M)_\circ$ and let $f_i := g_i g_{i-1}^{-1} \in \mathrm{Diff}^r_c(M)_\circ$, $i = 1, \ldots, n$. Using (1.11) we then have

$$f_n \cdots f_1 = g_n \underbrace{g_0}_{=Id} = \varphi(\underbrace{(\sum_{i=1}^{n} \lambda_i)\varphi^{-1}(f)}_{=\varphi^{-1}(f)}) = f.$$

Further, since $g_i|_{M\setminus \mathrm{supp}(\lambda_i)} = g_{i-1}|_{M\setminus \mathrm{supp}(\lambda_i)}$ we obtain

$$\mathrm{supp}(f_i) = \mathrm{supp}(g_i g_{i-1}^{-1}) \subseteq g_{i-1}(\mathrm{supp}(\lambda_i)) \subseteq V_i.$$

This finishes the proof of Claim 2. $\square$

Claim 3. *Let $U, V \in \mathcal{U}$, $f \in \mathrm{Diff}^r_c(M)_\circ$ and $\mathrm{supp}(f) \subseteq V$. Then there exist $f_1, f_2 \in \mathrm{Diff}^r_c(M)_\circ$ with $\mathrm{supp}(f_i) \subseteq V$, $f = f_2 f_1$, $\mathrm{supp}(f_1) \cup \overline{U} \neq M$ and $\mathrm{supp}(f_2) \cup f_1(\overline{U}) \neq M$.*

Proof of Claim 3. Suppose we have neighborhoods N_i of $x_i \in N_i \subseteq M \setminus \overline{U}$, $i = 1, 2$ and $f_1 \in \mathrm{Diff}^r_c(M)_\circ$ with $\mathrm{supp}(f_1) \subseteq V$, $f_1|_{N_1} = f|_{N_1}$ and $f_1|_{N_2} = Id$. We then let $f_2 := f f_1^{-1}$ and obtain $f_2 \in \mathrm{Diff}^r_c(M)_\circ$, $\mathrm{supp}(f_2) \subseteq V$, $f = f_2 f_1$. Since $x_2 \notin \mathrm{supp}(f_1) \cup \overline{U}$, we have $\mathrm{supp}(f_1) \cup \overline{U} \neq M$. Further we have $f_2 f_1|_{N1} = f|_{N_1} = f_1|_{N_1}$ and thus $f_2|_{f_1(N_1)} = Id$. Hence $f_1(x_1) \notin \mathrm{supp}(f_2)$, and since $f_1(x_1) \notin f_1(\overline{U})$ we obtain $\mathrm{supp}(f_2) \cup f_1(\overline{U}) \neq M$. So it remains to find x_i, N_i and f_1 presumed above.

By Example 1.15 we can choose a C^r-diffeotopy $H : M \times I \to M$ with $H_0 = Id$, $H_1 = f$ and $\mathrm{supp}(H) \subseteq V$. In the case $n = 1$ we think of V as $V \subseteq \mathbf{R}$ via a chart, and let $H(x, t) := tf(x) + (1-t)x \; \forall x \in V, t \in I$, extended trivially on $M \setminus V$. We have $\overline{U} \neq M$ and thus we find x_1 and N_1 with $x_1 \in N_1 \subseteq M \setminus \overline{U}$, such that $\overline{H(N_1 \times I)} \not\supseteq M \setminus \overline{U}$. If $n > 1$ this is possible since $H(\{x_1\} \times I)$ is compact and $M \setminus \overline{U}$ is open, in the case $n = 1$ the existence of x_1 and N_1 follows from our special choice of H. Now choose x_2 and N_2 such that $x_2 \in N_2 \subseteq M \setminus (\overline{U} \cup \overline{H(N_1 \times I)})$, $\lambda \in \mathrm{C}^\infty(M, \mathbf{R})$ with $\lambda|_{N_2} = 0$, $\lambda|_{H(N_1\times I)} = 1$, let $\dot{H}(x, t) := \frac{d}{ds}|_t H_s(H_t^{-1}(x))$ and define $\dot{G}(x, t) := \lambda(x)\dot{H}(x, t)$. Integration of $\dot{G}$ gives rise to a C^{r-1}-diffeotopy $G : M \times I \to M$ supported on V with $G_t|_{N_2} = Id \; \forall t \in I$ and $G|_{N_1\times I} = H|_{N_1\times I}$. Hence we can approximate G by an C^r-diffeotopy $\widetilde{G}$, supported on V, which equals G on $(\widetilde{N_1} \cup \widetilde{N_2}) \times I$, where $\widetilde{N_i}$ are smaller neighborhoods of x_i. Thus we have $\widetilde{G}_1 \in \mathrm{Diff}^r_c(M)_\circ$, $\mathrm{supp}(\widetilde{G}_1) \subseteq V$, $\widetilde{G}_1|_{\widetilde{N}_2} = Id$ and $\widetilde{G}_1|_{\widetilde{N}_1} = G_1|_{\widetilde{N}_1} = H_1|_{\widetilde{N}_1} = f|_{\widetilde{N}_1}$. □

Now let $f \in \mathrm{Diff}^r_c(M)_\circ$, $\mathcal{V} \subseteq \mathcal{U}$ a cover of M and let $U \in \mathcal{U}$. By Claim 2 we obtain $V_1, \dots, V_n \in \mathcal{V}$ and $f_i \in \mathrm{Diff}^r_c(M)_\circ$, $1 \le i \le n$ with $\mathrm{supp}(f_i) \subseteq V_i$ and $f = f_n \cdots f_1$. Since $f_{i-1} \cdots f_1(U) \in \mathcal{U}$ we obtain from Claim 3 $f_i^1, f_i^2 \in \mathrm{Diff}^r_c(M)_\circ$ with $\mathrm{supp}(f_i^j) \subseteq V_i$, $f_i = f_i^2 f_i^1$, $\mathrm{supp}(f_i^1) \cup f_{i-1} \cdots f_1(\overline{U}) \neq M$ and $\mathrm{supp}(f_i^2) \cup f_i^1 f_{i-1} \cdots f_1(\overline{U}) \neq M$. Since $f_i \cdots f_1 = f_i^2 f_i^1 \cdots f_1^2 f_1^1$ we have verified Axiom 3. □

1.17. Corollary. *Let $1 \le r \le \infty$ and let M be a connected smooth manifold. Then* $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$ *is simple,* $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ] \lhd \mathrm{Diff}^r(M)$ *and every non-trivial, normal subgroup of* $\mathrm{Diff}^r(M)$ *contains* $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$. *Consequently* $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$ *is the unique minimal, normal subgroup of* $\mathrm{Diff}^r(M)$. *The same is true when* $\mathrm{Diff}^r(M)$ *is replaced by* $\mathrm{Diff}^r_c(M)$, $\mathrm{Diff}^r(M)_\circ$ *or* $\mathrm{Diff}^r_c(M)_\circ$.

Proof. The simplicity of $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$ follows immediately from Corollary 1.4 and Lemma 1.16. To see the second assertion notice, that if G is an arbitrary group and $H \lhd G$ then $[H, H] \lhd G$, since $g[h_1, h_2]g^{-1} = [gh_1g^{-1}, gh_2g^{-1}]$. By Remark 1.8 we thus obtain $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ] \lhd \mathrm{Diff}^r(M)$. Let $\{Id\} \neq H \lhd \mathrm{Diff}^r(M)$. We'll show

$$H \cap [\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ] \neq \{Id\}, \tag{1.12}$$

for we are done then, since we have

$$H \cap [\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ] \lhd [\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$$

and $[\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$ is simple. To see (1.12) choose $h \in H$, $U \in \mathcal{U}$ and $f_1, f_2 \in \mathrm{Diff}^r_c(M)_\circ$, such that $U \cap h(U) = \emptyset$, $\mathrm{supp}(f_i) \subseteq U$ and $[f_1, f_2] \neq Id$. We have $g_i := [h, f_i] \in H \cap \mathrm{Diff}^r_c(M)_\circ$ since $H \lhd \mathrm{Diff}^r(M)$ and $\mathrm{Diff}^r_c(M)_\circ \lhd \mathrm{Diff}^r(M)$. Hence $[g_1, g_2] \in H \cap [\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ]$, but we have $g_i|_U = f_i|_U : U \to U$ (cf. Claim 2 on page 2) and hence $[g_1, g_2]|_U = [f_1, f_2]|_U \neq Id$ by the choice of f_i. This shows (1.12).

Nothing essential in the proof changes, when $\mathrm{Diff}^r(M)$ is replaced by $\mathrm{Diff}^r_c(M)$, $\mathrm{Diff}^r(M)_\circ$ or $\mathrm{Diff}^r_c(M)_\circ$. To see the last case one could also apply Corollary 1.5. □

2. The Simplicity of $\mathrm{Diff}_c^r(M)_\circ$, $\dim(M)+2 \leq r \leq \infty$

2.1. Definition. A *modulus of continuity* is a continuous, strictly increasing function $\alpha : [0,\infty) \to \mathbf{R}$, such that $\alpha(0) = 0$ and $\alpha(tx) \leq t\alpha(x)$, $\forall x \in [0,\infty)$ $\forall t \geq 1$.

Consider two metric spaces (X, d_X) and (Y, d_Y), a modulus of continuity α, and a mapping $f : X \to Y$. f is said to be *α-continuous*, if there exist $C > 0$ and $\epsilon > 0$, such that

$$x_1, x_2 \in X, d_X(x_1, x_2) \leq \epsilon \Longrightarrow d_Y(f(x_1), f(x_2)) \leq C\alpha(d_X(x_1, x_2)).$$

f is called *locally α-continuous* if each point has a neighborhood U, such that $f|_U$ is α-continuous.

2.2. Remark. Id is a modulus of continuity and a mapping is Id-continuous, iff it is Lipschitz. If $0 < t < s$ then $\alpha(s) = \alpha(t\frac{s}{t}) \leq \frac{s}{t}\alpha(t)$, hence $t \mapsto \frac{t}{\alpha(t)}$ is an increasing function, and thus $Id : (X, d) \to (X, d)$ is α-continuous for all moduli of continuity α. If α, α' are moduli of continuity that equal on a neighborhood of 0 then a mapping is α-continuous iff it is α'-continuous.

2.3. Lemma. *Let X and Y be two metric spaces, each of them equipped with two metrics which are equivalent in the following sense: there exist positive constants m_X, M_X, m_Y, M_Y, with*

$$\begin{array}{ccccc} m_X d_X(x_1, x_2) & \leq & \widetilde{d}_X(x_1, x_2) & \leq & M_X d_X(x_1, x_2) \\ m_Y d_Y(y_1, y_2) & \leq & \widetilde{d}_Y(y_1, y_2) & \leq & M_Y d_Y(y_1, y_2) \end{array}$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then a mapping $f : (X, d_X) \to (Y, d_Y)$ is α-continuous, iff $f : (X, \widetilde{d}_X) \to (Y, \widetilde{d}_Y)$ is α-continuous.

Proof. Suppose $f : (X, d_X) \to (Y, d_Y)$ is α-continuous and choose $C > 0$ and $\epsilon > 0$ with

$$d_Y(f(x_1), f(x_2)) \leq C\alpha(d_X(x_1, x_2)) \qquad \forall d_X(x_1, x_2) \leq \epsilon.$$

We assume $m_X \leq 1$ and let $\widetilde{\epsilon} := \epsilon m_X$. If $\widetilde{d}_X(x_1, x_2) \leq \widetilde{\epsilon}$ we have $d_X(x_1, x_2) \leq \epsilon$, hence

$$\widetilde{d}_Y(f(x_1), f(x_2)) \leq M_Y C\alpha(d_X(x_1, x_2)) \leq \underbrace{M_Y C \frac{1}{m_X}}_{\widetilde{C}:=} \alpha(\widetilde{d}_X(x_1, x_2)),$$

and thus $f : (X, \widetilde{d}_X) \to (Y, \widetilde{d}_Y)$ is α-continuous. The second implication is proven similarly, but now the constants m_Y and M_X have to be used. □

2.4. Lemma. *Let (X, d) be a metric space and $(V, \|\cdot\|)$ a normed vector space. If $f, g : X \to V$ are α-continuous and $\lambda, \mu \in \mathbf{R}$ then $\lambda f + \mu g$ is α-continuous too.*

Proof. Choose $\epsilon_f, \epsilon_g > 0$ and $C_f, C_g > 0$, such that

$$\begin{array}{rl} \|f(x_1) - f(x_2)\| \le C_f \alpha(d(x_1,x_2)) & \forall d(x_1,x_2) \le \epsilon_f \\ \|g(x_1) - g(x_2)\| \le C_g \alpha(d(x_1,x_2)) & \forall d(x_1,x_2) \le \epsilon_g, \end{array}$$

and let $\epsilon := \min\{\epsilon_f, \epsilon_g\}$, $C := |\lambda| C_f + |\mu| C_g$. If $d(x_1,x_2) \le \epsilon$ then we have

$$\begin{aligned} \|(\lambda f + \mu g)(x_1) - (f+g)(x_2)\| &\le |\lambda| \|f(x_1) - f(x_2)\| + |\mu| \|g(x_1) - g(x_2)\| \\ &\le (|\lambda| C_f + |\mu| C_g)\alpha(d(x_1,x_2)). \end{aligned}$$

and this shows Lemma 2.4. □

2.5. Lemma. *Let* $(X, d_X) \xrightarrow{f} (Y, d_Y) \xrightarrow{g} (Z, d_Z)$ *be mappings of metric spaces, and let* α, β *be moduli of continuity. If* f *is* α*-continuous and* g *is* β*-continuous then* $g \circ f$ *is* $\beta \circ \alpha$*-continuous.*

Proof. We choose $\epsilon_f, \epsilon_g > 0$ and $C_f, C_g > 1$ analogously to the proof of Lemma 2.4, let $C := C_g C_f$ and choose $0 < \epsilon \le \epsilon_f$, such that $C_f \alpha(\epsilon) \le \epsilon_g$. If $d_X(x_1,x_2) \le \epsilon$ then

$$d_Y(f(x_1), f(x_2)) \le C_f \alpha(d_X(x_1,x_2)) \le C_f \alpha(\epsilon) \le \epsilon_g$$

hence

$$\begin{aligned} d_Z((g\circ f)(x_1), (g \circ f)(x_2)) &\le C_g \beta(d_Y(f(x_1), f(x_2))) \\ &\le C_g \beta(C_f \alpha(d_X(x_1,x_2))) \\ &\le C_g C_f (\beta \circ \alpha)(d_X(x_1,x_2)). \end{aligned}$$

and thus Lemma 2.5 is proven. □

2.6. Remark. From Remark 2.2 and Lemma 2.5 we obtain, that every $f : (X, d_X) \to (Y, d_Y)$ that is Lipschitz, is α-continuous for all moduli of continuity α. Especially a C^1-mapping $\mathbf{R}^n \supset U \xrightarrow{f} \mathbf{R}^m$ is locally α-continuous for all moduli of continuity α.

2.7. Lemma. *Let* $f_i : (X, d_X) \to (Y_i, d_{Y_i})$ *be* α*-continuous mappings,* $1 \le i \le n$*, and let* $d((y_1,\dots,y_n),(z_1,\dots,z_n)) := \sum_{i=1}^n d_{Y_i}(y_i, z_i)$*. Then* $(f_1,\dots,f_n) : (X, d_X) \to (Y_1 \times \dots \times Y_n, d)$ *is* α*-continuous. By Remark 2.3* d *can be replaced by many common metrics on* $Y_1 \times \dots \times Y_n$ *(e.g.* $\max\{d_{Y_1}, \dots, d_{Y_n}\}$ *or* $\sqrt{d_{Y_1}^2 + \dots + d_{Y_n}^2}$*).*

Proof. Let $\epsilon := \min\{\epsilon_{f_1}, \dots, \epsilon_{f_n}\}$, $C := \sum_{i=1}^n C_{f_i}$, and $d_X(x_1,x_2) \le \epsilon$. Then we can estimate

$$\begin{aligned} d((f_1,\dots,f_n)(x_1),(f_1,\dots,f_n)(x_2)) &= \sum_{i=1}^n d_{Y_i}(f_i(x_1), f_i(x_2)) \\ &\le (\sum_{i=1}^n C_{f_i}) \alpha(d_X(x_1,x_2)), \end{aligned}$$

and this is Lemma 2.7. □

2.8. Lemma. *Consider a multi linear mapping* $B : V_1 \times \cdots \times V_n \to W$ *of finite dimensional, normed vector spaces, and* α*-continuous mappings* $f_i : (X,d) \to V_i$*. If the images of* f_i *are bounded, then* $B \circ (f_1, \dots, f_n)$ *is* α*-continuous.*

Proof. Since B is C^1 it is locally Lipschitz, and since $K := \overline{\mathrm{Im}(f_1) \times \cdots \times \mathrm{Im}(f_n)}$ is compact we $B|_K$ is Lipschitz. Thus by Lemma 2.5 and Lemma 2.7 $B \circ (f_1, \dots, f_n) = B|_K \circ (f_1, \dots, f_n)$ is α-continuous. □

2.9. Lemma. *Let* $f : (X, \|\cdot\|) \to (Y, d)$ *be a continuous map from a compact, convex subset of a normed vector space to a metric space. Then there exists a modulus of continuity* α*, such that* f *is* α*-continuous.*

Proof. Let $\beta(t) := \sup_{\|x_1 - x_2\| \le t} d(f(x_1), f(x_2))$. Since X is compact, $\beta : \mathbf{R} \to \mathbf{R}$ is well defined and $\beta < \infty$. Obviously $\beta(0) = 0$, $t_1 \le t_2 \Longrightarrow \beta(t_1) \le \beta(t_2)$ and

$$d(f(x_1), f(x_2)) \le \beta(\|x_1 - x_2\|) \qquad \forall x_1, x_2 \in X. \tag{2.1}$$

It remains to find a modulus of continuity α with $\beta \le \alpha$. First, in order to show that β is continuous, let $\epsilon > 0$. For X is compact, f is uniformly continuous, and thus there exists $\delta > 0$ with

$$d(f(x), f(x')) \le \frac{\epsilon}{2} \qquad \forall \|x - x'\| \le \delta.$$

If $|t_2 - t_1| \le \delta$ and $\|x_1 - x_2\| \le t_2$ we find $x_1', x_2' \in X$ with $\|x_1 - x_1'\| \le \delta$, $\|x_2 - x_2'\| \le \delta$ and $\|x_1' - x_2'\| \le t_1$ (here we used the convexity of X). We then have

$$\begin{aligned} d(f(x_1), f(x_2)) &\le \underbrace{d(f(x_1), f(x_1'))}_{\le \frac{\epsilon}{2}} + d(f(x_1'), f(x_2')) + \underbrace{d(f(x_2'), f(x_2))}_{\le \frac{\epsilon}{2}} \\ &\le \epsilon + d(f(x_1'), f(x_2')) \end{aligned}$$

This shows

$$\Big| \sup_{\|x_1 - x_2\| \le t_2} d(f(x_1), f(x_2)) - \sup_{\|x_1 - x_2\| \le t_1} d(f(x_1), f(x_2)) \Big| \le \epsilon \qquad \forall |t_2 - t_1| \le \delta,$$

and hence β is continuous.

Since X is compact there exist C and N, such that $\beta|_{[N,\infty]} \equiv C$. In the sequel we will construct a sequence $\{\beta_n\}_{n \in \mathbf{N}}$ with the following properties:

(1) $\beta_n : [0, \infty] \to [0, \infty]$ is continuous, increasing and $\beta_n|_{[N,\infty]} \equiv C$

(2) $\lim_{n \to \infty} \beta_n(0) = 0$

(3) $\beta_n(tx) \le t\beta_n(x) \qquad \forall x \in [0, \infty] \forall 1 \le t$

(4) $\beta \le \beta_n$

(5) $\|\beta_n - \beta_{n-1}\|_0 \le \frac{C}{2^n}$, where $\|\cdot\|_0$ denotes the sup-norm.

Suppose we have such a sequence, then $\alpha := \lim_{n\to\infty}\beta_n + Id_{\mathbf{R}}$ exists and is continuous, since the sequence converges uniformly by (5). From the other properties we obtain, that α is a modulus of continuity, and $\beta\le\alpha$.

So it remains to show the existence of the sequence $\{\beta_n\}_{n\in\mathbf{N}}$. Let $\beta_0 :\equiv C$. There exists x_1 with $\beta(x_1)=\frac{C}{2}$. We let

$$\beta_1|_{[x_1,\infty]} = \beta_0|_{[x_1,\infty]}, \qquad \beta_1(x) := \frac{C}{2} + x\frac{C}{2x_1} \quad \forall x\in[0,x_1].$$

Inductively we choose x_n with $\beta(x_n)=\frac{C}{2^n}$ and let

$$\beta_n|_{[x_n,\infty]} = \beta_{n-1}|_{[x_n,\infty]}, \qquad \beta_n(x) := \frac{C}{2^n} + x\Big(\sum_{i=1}^{n}\frac{C}{2^i x_1}\Big) \quad \forall x\in[0,x_n].$$

It is easily seen, that this sequence has all the desired properties. □

2.10. Definition. Let $f\in \mathrm{C}^r(U,\mathbf{R}^m)$, where U is an open subset of $\mathbf{R}^n$. Then $D^r f : U\to \mathrm{L}^r(\mathbf{R}^n,\mathbf{R}^m)$ denotes the r-th derivative of f, where $\mathrm{L}^r(E,F)$ denotes the vector space of r-linear mappings $E\times\cdots\times E\to F$. We say f is of class $\mathrm{C}^{r,\alpha}$, iff it is C^r and $D^r f$ is locally α-continuous. By Remark 2.3 this doesn't depend on the choice of a norm on $L^r(\mathbf{R}^n,\mathbf{R}^m)$. $\mathrm{C}^{\infty,\alpha}$ will mean C^∞. A mapping of smooth manifolds $f: M\to N$ is of class $\mathrm{C}^{r,\alpha}$, iff for every $x\in M$ and for every chart (V,v) on N with $f(x)\in V$, there exists a chart (U,u) on M with $x\in U$, $f(U)\subseteq V$ and $v\circ f\circ u^{-1}$ is $\mathrm{C}^{r,\alpha}$. We define

$$\mathrm{C}^{r,\alpha}(M,N) := \{f: M\to N : f \text{ is } \mathrm{C}^{r,\alpha}\}\subseteq \mathrm{C}^r(M,N)$$

equipped with the initial topology with respect to the inclusion. By Remark 2.6 we have $\mathrm{C}^{r+1}(M,N)\subset \mathrm{C}^{r,\alpha}(M,N)$ for all moduli of continuity α and $\forall 0\le r<\infty$. If $E\xrightarrow{\pi} M$ is a smooth vector bundle over M we denote by

$$\Gamma^{r,\alpha}(E\xrightarrow{\pi} M) := \Gamma^r(E\xrightarrow{\pi} M)\cap \mathrm{C}^{r,\alpha}(M,E)$$

the set of all $\mathrm{C}^{r,\alpha}$-sections of $E\xrightarrow{\pi} M$.

2.11. Lemma. *Let M be a smooth manifold, $E\xrightarrow{\pi} M$ be a smooth vector bundle over M and V a finite dimensional vector space. Then $\mathrm{C}^{r,\alpha}(M,V)$ and $\Gamma^{r,\alpha}(E\xrightarrow{\pi} M)$ are vector spaces.*

Proof. Everything follows immediately from Lemma 2.4. □

2.12. Lemma. *Let $1\le r\le\infty$, M,N,P be smooth manifolds, $g\in \mathrm{C}^{r,\alpha}(M,N)$ and $f\in \mathrm{C}^{r,\alpha}(N,P)$. Then $f\circ g\in \mathrm{C}^{r,\alpha}(M,P)$.*

Proof. Obviously we may assume $g\in \mathrm{C}^{r,\alpha}(U,V)$ and $f\in \mathrm{C}^{r,\alpha}(V,\mathbf{R}^p)$, where U resp. V are open subsets in $\mathbf{R}^m$ resp. $\mathbf{R}^n$. Recall the formulas

$$D(f\circ g) = (Df\circ g)\cdot Dg \tag{2.2}$$

$$\begin{aligned} D^r(f\circ g) &= (D^r f\circ g)\cdot(Dg\times\cdots\times Dg) + (Df\circ g)\cdot D^r g \\ &\quad + \sum_{\substack{1<i<r,\ 1\le j_s \\ j_1+\cdots+j_i=r}} C_{i,j_1,\dots,j_i}(D^i f\circ g)\cdot(D^{j_1}g\times\cdots\times D^{j_i}g) \end{aligned} \tag{2.3}$$

where $C_{i,j_1,\dots,j_i}$ are positive integers independent of f and g. In the case $r=1$, we have Dg and $Df\circ g$ are locally α-continuous (cf. Lemma 2.5). Since the dot in (2.2) is bilinear, we obtain from Lemma 2.8, that $D(f\circ g)$ is locally α-continuous, and hence $f\circ g$ is $\mathrm{C}^{r,\alpha}$.

In the case $r>1$ we use (2.3). By similar arguments one sees, that $D^r f\circ g$, Dg, $Df\circ g$ and $D^r g$ are locally α-continuous. Hence by Lemma 2.8 we have $(D^r\circ g)\cdot(Dg\times\cdots\times Dg)$ and $(Df\circ g)\cdot D^r g$ are locally α-continuous, for the dots in (2.3) are multi linear. Each of the remaining summands of (2.3) is C^1 and hence locally α-continuous by Remark 2.6. Thus, by Lemma 2.4, $D^r(f\circ g)$ is locally α-continuous, and hence $f\circ g$ is $\mathrm{C}^{r,\alpha}$. □

2.13. Lemma. *If $1\le r$, $f\in\mathrm{C}^{r,\alpha}(M,N)$, and if f has a C^1 inverse, then $f^{-1}\in\mathrm{C}^{r,\alpha}(N,M)$.*

Proof. We assume once again, that $f\in\mathrm{C}^{r,\alpha}(U,V)$, where U and V are open subsets of $\mathbf{R}^n$. Recall that we have

$$D(f^{-1}) = \mathrm{Inv}\circ Df\circ f^{-1}, \tag{2.4}$$

where Inv is the inversion of linear mappings, which is known to be C^∞. Since f^{-1} is C^r, f^{-1} is $\mathrm{C}^{r-1,\alpha}$, and Df is $\mathrm{C}^{r-1,\alpha}$. So we obtain from (2.4), Lemma 2.12 and Lemma 2.5 ($r=1$), that $D(f^{-1})$ is $\mathrm{C}^{r-1,\alpha}$, and thus f^{-1} is $\mathrm{C}^{r,\alpha}$. □

2.14. Definition. Let M be a smooth manifold, $E\xrightarrow{\pi}M$ a smooth vector bundle, α a modulus of continuity, $1\le r\le\infty$ and $K\subseteq M$ an arbitrary subset. We then we define

$$\begin{aligned} \mathrm{Diff}^{r,\alpha}_{[c][K]}(M) &:= \mathrm{Diff}^r_{[c][K]}(M)\cap\mathrm{C}^{r,\alpha}(M,M) \\ \Gamma^{r,\alpha}_{[c][K]}(E\xrightarrow{\pi}M) &:= \Gamma^r_{[c][K]}(E\xrightarrow{\pi}M)\cap\mathrm{C}^{r,\alpha}(M,E) \end{aligned}$$

and equip each of these spaces with the initial topology. (Here $[c][K]$ means: optionally c, K or nothing)

2.15. Lemma. *In the situation of Definition 2.14 $\mathrm{Diff}^{r,\alpha}_c(M)$ is a subgroup of $\mathrm{Diff}^r_c(M)$ and $\Gamma^{r,\alpha}_c(TM\xrightarrow{\pi}M)$ is a linear subspaces of $\Gamma^r_c(TM\xrightarrow{\pi}M)$. The homeomorphism $\varphi: U'\to U$ of Proposition 1.13 restricts to a homeomorphism*

$$\varphi: U'\cap\Gamma^{r,\alpha}(TM\xrightarrow{\pi}M)\to U\cap\mathrm{Diff}^{r,\alpha}(M).$$

Further $\mathrm{Diff}^{r,\alpha}_c(M)$ is locally contractible, and $\mathrm{Diff}^{r,\alpha}_c(M)_\circ$ consists exactly of those $\mathrm{C}^{r,\alpha}$-diffeomorphisms that are compactly supported $\mathrm{C}^{r,\alpha}$-diffeotopic to Id.

Proof. Using Lemma 2.11, Lemma 2.12 and Lemma 2.13 we obtain the first two assertions. To see that $\varphi: U'\to U$ restricts to $\mathrm{C}^{r,\alpha}$ mappings as stated above, notice that $\varphi=\exp_*$, exp is C^∞ and use Lemma 2.12. The proof of Corollary 1.14 is exactly the same in the $C^{r,\alpha}$-case and hence we obtain the remaining assertions of Lemma 2.15 □

2.16. Example. Since $\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)$ is locally contractible and since $\mathrm{Diff}^{r+1}_c(\mathbf{R}^n)_\circ = \mathrm{Diff}^{r+1}_c(\mathbf{R}^n)$ (cf. Example 1.15) is dense in $\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)$ (cf. Proposition 1.7(6)), we obtain $\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ = \mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)$. Similarly we have $\mathrm{Diff}^{r,\alpha}_{\mathbf{B}^n}(\mathbf{R}^n)_\circ = \mathrm{Diff}^{r,\alpha}_{\mathbf{B}^n}(\mathbf{R}^n)$.

2.17. Lemma. *Let M be a connected, smooth manifold and let $1\le r\le\infty$. Then*

$$\mathrm{Diff}^r_c(M)_\circ = \bigcup_\alpha \mathrm{Diff}^{r,\alpha}_c(M)_\circ, \tag{2.5}$$

where the union is taken over all moduli of continuity α.

Proof. This is trivial for $r=\infty$, and hence we may assume $r<\infty$. One inclusion ($\supseteq$) is obvious, so it remains to show the other one.

Let $f\in\mathrm{Diff}^r_c(M)_\circ$ and let $\mathcal{U}^r(M)$ be the covering from Lemma 1.16. By Claim 2 on page 10 we obtain $m\in\mathbf{N}$, $U_1,\dots,U_m\in\mathcal{U}^r(M)$ and $f_1,\dots,f_m\in\mathrm{Diff}^r_c(M)_\circ$ with $\mathrm{supp}(f_i)\in U_i$ and $f=f_m\cdots f_1$. Using the charts φ_i belonging to U_i, we have $\widetilde{f}_i := \varphi_i\circ f_i\circ\varphi_i^{-1}\in\mathrm{Diff}^r_{\mathbf{B}^n}(\mathbf{R}^n)_\circ$ (cf. Example 1.15). Hence there exist C^r-diffeotopies H_i, $1\le i\le m$, supported on $\mathbf{B}^n$ with $H_i(\cdot,1)=\widetilde{f}_i$.

$$D^r(H):\mathbf{R}^n\times I\to \mathrm{L}^r(\mathbf{R}^n\times I,\mathbf{R}^n)$$

is continuous and supported on the convex, compact set $\mathbf{B}^n\times I$. Thus, using Lemma 2.9, we obtain moduli of continuity α_i, such that D^rH_i is α_i-continuous, $1\le i\le m$. Now let $\alpha:=\sum_{i=1}^m\alpha_i$. This is again a modulus of continuity and D^rH_i is α-continuous $\forall 1\le i\le m$. Consequently H is a $C^{r,\alpha}$-diffeotopy and hence $\widetilde{f}_i\in\mathrm{Diff}^{r,\alpha}_{\mathbf{B}^n}(\mathbf{R}^n)_\circ$. So $f_i\in\mathrm{Diff}^{r,\alpha}_c(M)_\circ$ and thus $f\in\mathrm{Diff}^{r,\alpha}_c(M)_\circ$. □

2.18. Remark. In the next Chapter we will need the following improvement: Lemma 2.17 remains valid, if the union is only taken over all moduli of continuity satisfying

$$\alpha(tx)\le\sqrt{t}\alpha(x)\qquad\forall t\ge 1\forall x\in[0,\infty).$$

To see this let $\widetilde{\alpha}:=\sqrt{\alpha}$, where α is the modulus of continuity constructed in Lemma 2.17. Then $\widetilde{\alpha}$ is a modulus of continuity and

$$\widetilde{\alpha}(tx)=\sqrt{\alpha(tx)}\le\sqrt{t\alpha(x)}=\sqrt{t}\widetilde{\alpha}(x)\qquad\forall t\ge 1\forall x\in[0,\infty).$$

Since $\alpha\le\widetilde{\alpha}$ in a neighborhood of 0 D^rH_i are locally $\widetilde{\alpha}$-continuous, hence $\widetilde{f}_i\in\mathrm{Diff}^{r,\widetilde{\alpha}}_{\mathbf{B}^n}(\mathbf{R}^n)_\circ$, and thus $f\in\mathrm{Diff}^{r,\widetilde{\alpha}}_c(M)_\circ$.

2.19. Theorem (Mather, Epstein). *[7] [4] Let M be a smooth manifold, $n=\dim(M)$, $n+2\le r\le\infty$ and let α be a modulus of continuity. Then $\mathrm{Diff}^{r,\alpha}_c(M)_\circ$ is perfect, i.e. equals its own commutator group.*

It will take us the rest of this chapter to prove this theorem. We start with:

2.20. Definition. Let $\varphi : G \to G'$ be a homomorphism of groups. We define $H_1(G) := G/[G,G]$, and let $H_1(\varphi) : H_1(G) \to H_1(G')$ be the induced homomorphism. H_1 is a covariant functor from the category of groups to the category of abelian groups.

2.21. Lemma. *Let M be a smooth manifold, $\varphi : \mathbf{R}^n \hookrightarrow M$ a C^∞-embedding, $1 \le r \le \infty$, and α a modulus of continuity. We define homomorphisms of groups $\tau^{r,[\alpha]}_\varphi : \mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ \to \mathrm{Diff}^{r,[\alpha]}_c(M)_\circ$ by*

$$f \mapsto \begin{cases} \varphi \circ f \circ \varphi^{-1} & \text{on } \varphi(\mathbf{R}^n) \\ Id & \text{elsewhere} \end{cases}$$

Then $H_1(\tau^{r,[\alpha]}_\varphi) : H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ) \to H_1(\mathrm{Diff}^{r,[\alpha]}_c(M)_\circ)$ is surjective.

Proof. Clearly everything is well defined, and $\tau^{r,[\alpha]}_\varphi$ is a homomorphism of groups. Let $f \in \mathrm{Diff}^{r,[\alpha]}_c(M)_\circ$. By Claim 2 on page 10, which is proven similarly in the $\mathrm{C}^{r,\alpha}$-case, we obtain $m \in \mathbf{N}$, $V_1, \dots, V_m \in \mathcal{U}^r(M)$ and $f_1, \dots, f_m \in \mathrm{Diff}^{r,[\alpha]}_c(M)_\circ$, with $\mathrm{supp}(f_i) \subseteq V_i$ and $f = f_m \cdots f_1$. Choose $g_1, \dots, g_m \in \mathrm{Diff}^\infty_c(M)_\circ$ with $g_i(V_i) \subseteq \varphi(\mathbf{R}^n)$. Thus we have $g_i f_i g_i^{-1} \in \mathrm{Im}(\tau^{r,[\alpha]}_\varphi)$ (cf. Example 2.16), and hence

$$[f] = [f_m] \cdots [f_1] = [g_m f_m g_m^{-1}] \dots [g_1 f_1 g_1^{-1}] \in \mathrm{Im}(H_1(\tau^{r,[\alpha]}_\varphi)).$$

□

2.22. Remark. By Lemma 2.21 it suffice to show Theorem 2.19 in the case $M = \mathbf{R}^n$. For then we have $H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ) = \{e\}$, and hence using the surjectivity of $\tau^{r,[\alpha]}_\varphi$ $H_1(\mathrm{Diff}^{r,[\alpha]}_c(M)_\circ) = \{e\}$, but this is the perfectness of $\mathrm{Diff}^{r,[\alpha]}_c(M)_\circ$.

Moreover it suffices to find a compact set $K \subseteq \mathbf{R}^n$ with $K^\circ \ne \emptyset$ and

$$\mathrm{Diff}^{r,[\alpha]}_K(\mathbf{R}^n)_\circ \subseteq [\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ].$$

For then given $f \in \mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ$ and a compactly supported $\mathrm{C}^{r,[\alpha]}$-diffeotopy F from f to Id, we choose $g \in \mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ$ with $g(\mathrm{supp}(F)) \subseteq K^\circ$ and obtain $gfg^{-1} \in \mathrm{Diff}^{r,[\alpha]}_K(\mathbf{R}^n)_\circ$ since $(x,t) \mapsto g(F(g^{-1}(x),t))$ is a $\mathrm{C}^{r,[\alpha]}$-diffeotopy from gfg^{-1} to Id, supported on K. Hence

$$gfg^{-1} \in \mathrm{Diff}^{r,[\alpha]}_K(\mathbf{R}^n)_\circ \subseteq [\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ],$$

and thus $\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ$ is perfect since

$$[f] = [g][f][g]^{-1} = [gfg^{-1}] = [Id] \in H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ).$$

2.23. Definition. Let $A \ge 1$. Choose $\widetilde{\rho}_A \in \mathrm{C}^\infty(\mathbf{R}, [0,1])$ with $\mathrm{supp}(\widetilde{\rho}_A) = [-2A - 1, 2A+1]$ and $\widetilde{\rho}_A|_{[-2A,2A]} \equiv 1$. We define $\rho_A : \mathrm{C}^\infty(\mathbf{R}^n, [0,1])$ by $\rho_A(x_1, \dots, x_n) :=$

$\widetilde{\rho}_A(x_1)\cdots\widetilde{\rho}_A(x_n)$. Then $\operatorname{supp}(\rho_A)=[-2A-1,2A+1]^n$ and $\rho_A|_{[-2A,2A]^n}\equiv 1$. Let $\tau_{i,A}:=\mathrm{Fl}_1^{\rho_A\partial_i}\in\mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ$, and

$$\begin{aligned}\{x\in\mathbf{R}^n:|x_j|<2A+1, j\ne i\}&\xrightarrow{\varphi_{i,A}}\mathbf{R}^n\\ \varphi_{i,A}(x_1,\dots,x_n)&:=\mathrm{Fl}_{x_i}^{\rho_A\partial_i}(x_1,\dots,0,\dots,x_n).\end{aligned}$$

Here ∂_i denotes the unit vector field on $\mathbf{R}^n$ in the direction of the ith coordinate, and Fl^X_t denotes the flow of the vector field X. Further we denote by T_i the unit translation in the direction of the ith coordinate, i.e. $T_i:=\mathrm{Fl}_1^{\partial_i}$.

2.24. Lemma. *In this situation we have*

(1) $\varphi_{i,A}(\{x_1\}\times\dots\times\mathbf{R}\times\dots\times\{x_n\})\subseteq\{x_1\}\times\dots\times\mathbf{R}\times\dots\times\{x_n\}$.

(2) $\varphi_{i,A}|_{[-2A,2A]^n}=Id$

(3) $\varphi_{i,A}\in\mathrm{Diff}^\infty(\operatorname{dom}(\varphi_{i,A}),(-2A-1,2A+1)^n)$

(4) $\varphi_{i,A}^*(\rho_A\partial_i)=\partial_i$

(5) $\tau_{i,A}\circ\varphi_{i,A}=\varphi_{i,A}\circ\mathrm{T}_i$.

(6) $|pr_i\circ\varphi_{i,A}^{-1}(x)|\ge|pr_i(x)|\qquad\forall x\in(-2A-1,2A+1)^n$

(7) $pr_i\circ\varphi_{i,A}\circ\widetilde{pr}_i=pr_i\circ\varphi_{i,A}\qquad\forall x\in[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}$

Here $pr_i:\mathbf{R}^n\to\mathbf{R}$ *and* $\widetilde{pr}_i:\mathbf{R}^n\to\mathbf{R}^n$ *are given by* $pr_i(x_1,\dots,x_n):=x_i$ *and* $\widetilde{pr}_i(x_1,\dots,x_n):=(0,\dots,x_i,\dots,0)$.

Proof. (1) and (2) are obvious. To see (3), recall that $\varphi_{i,A}$ is C^∞, $\operatorname{supp}(\rho_A)=[-2A-1,2A+1]^n$, and thus by the uniqueness of integral curves $\varphi_{i,A}:\operatorname{dom}(\varphi_{i,A})\to(-2A-1,2A+1)^n$ is bijective.

$$\begin{aligned}\frac{\partial\varphi_{i,A}}{\partial x_i}\Big|_x&=\frac{\partial}{\partial x_i}\Big|_x\mathrm{Fl}_{x_i}^{\rho_A\partial_i}(x_1,\dots,0,\dots,x_n)\\&=(\rho_A\partial_i)(\mathrm{Fl}_{x_i}^{\rho_A\partial_i}(x_1,\dots,0,\dots,x_n))=\rho_A(\varphi_{i,A}(x))\partial_i\end{aligned}\tag{2.6}$$

From (2.6) we obtain $\frac{\partial}{\partial x_i}\big|_x(pr_i\circ\varphi_{i,A})=pr_i(\rho_A(\varphi_{i,A}(x))\partial_i)=\rho_A(\varphi_{i,A}(x))$ and thus, since $pr_j\circ\varphi_{i,A}=pr_j$, $j\ne i$, we have $\det(D\varphi_{i,A})(x))=\rho_A(\varphi_{i,A}(x))\ne 0$. Hence $\varphi_{i,A}\in\mathrm{Diff}^\infty(\operatorname{dom}(\varphi_{i,A}),(-2A-1,2A+1)^n)$. (4) follows from (2.6) and (5) is an immediate consequence of (4). From $0\le\rho_A\le 1$ we obtain (6). Because of the special choice of ρ_A we have $\rho_A\circ\widetilde{pr}_i=\rho_A$ on $[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}$ and this implies (7). □

2.25. Definition. Let U be an open subset of $\mathbf{R}^n$, $0\le r<\infty$, α a modulus of continuity and let $f\in\mathrm{C}^r(U,\mathbf{R}^m)$. Then we define

$$\begin{aligned}\|f\|_r&:=\sup_{x\in U}\|D^rf(x)\|\quad\le\infty\\ \|f\|_{r,\alpha}&:=\sup_{x\ne y\in U}\frac{\|D^rf(x)-D^rf(y)\|}{\alpha(\|x-y\|)}\quad\le\infty\end{aligned}$$

where $\|D^r f(x)\|$ denotes the norm of the multi linear mapping $D^r f(x) \in L^r(\mathbf{R}^n, \mathbf{R}^m)$, i.e.

$$\|D^r f(x)\| = \sup_{v_j \in \mathbf{R}^n} \frac{\|D^r f(x) \cdot (v_1, \dots, v_r)\|}{\|v_1\| \cdots \|v_r\|}$$

if $1 \le r$ and $\|D^0 f(x)\| = \|f(x)\|$. Further we let $\mu_r(f) := \|f - Id\|_r$, $\mu_{r,\alpha}(f) := \|f - Id\|_{r,\alpha}$ and $M_r(f) := \max\{\mu_1(f), \dots, \mu_r(f)\}$. Finally we define the distance of two nonempty sets $A, B \subseteq \mathbf{R}^n$ to be

$$\mathrm{dist}(A, B) := \inf_{a \in A, b \in B} \|x - y\|.$$

2.26. Lemma. *We list some useful properties of* $\mu_{r,[\alpha]}$ *and* $\|\cdot\|_{r,[\alpha]}$*, we'll need in the sequel.*

(1) $\mu_r(f) = \|f\|_r \qquad \forall r \ge 2, \forall U \subseteq \mathbf{R}^n$ *open and* $\forall f \in \mathrm{C}^r(U, \mathbf{R}^m)$

(2) $\mu_1(f) \le \|f\|_1 + 1$, $\|f\|_1 \le \mu_1(f) + 1 \qquad \forall U \subseteq \mathbf{R}^n$ *open and* $\forall f \in \mathrm{C}^1(U, \mathbf{R}^m)$

(3) $\mu_{r,\alpha}(f) = \|f\|_{r,\alpha} \qquad \forall \alpha, \forall r \ge 1, \forall U \subseteq \mathbf{R}^n$ *open and* $\forall f \in \mathrm{C}^r(U, \mathbf{R}^m)$

(4) Let $0 \le r < \infty$ *and* $f \in \mathrm{C}^{r+1}(\mathbf{R}^n, \mathbf{R}^m)$*. If there exists* $y_0 \in \mathbf{R}^n$ *with* $D^r f(y_0) = 0$ *then*

$$\|f\|_r \le \Big(\sup_{x \in \mathbf{R}^n} \mathrm{dist}\Big(x, \{y \in \mathbf{R}^n : D^r f(y) = 0\}\Big)\Big) \|f\|_{r+1}.$$

If there exists $y_0 \in \mathbf{R}^n$ *with* $D^r(f - Id)(y_0) = 0$ *then*

$$\mu_r(f) \le \Big(\sup_{x \in \mathbf{R}^n} \mathrm{dist}\Big(x, \{y \in \mathbf{R}^n : D^r(f - Id)(y) = 0\}\Big)\Big) \mu_{r+1}(f).$$

(5) Let $0 \le r < \infty$, $f \in \mathrm{C}^r(\mathbf{R}^n, \mathbf{R}^m)$*, suppose there exists* $y_0 \in \mathbf{R}^n$ *with* $D^r f(y_0) = 0$, $C := \sup_{x \in \mathbf{R}^n} \mathrm{dist}\Big(x, \{y \in \mathbf{R}^n : D^r f(y) = 0\}\Big) < \infty$*, and choose* $N \in \mathbf{N}$ *with* $\alpha(\frac{C}{N}) \le 1$*. Then*

$$\|f\|_r \le N \|f\|_{r,\alpha}.$$

If there exists $y_0 \in \mathbf{R}^n$ *with* $D^r f(y_0) = 0$, $C' := \sup_{x \in \mathbf{R}^n} \mathrm{dist}\Big(x, \{y \in \mathbf{R}^n : D^r(f - Id)(y) = 0\}\Big) < \infty$ *and* $\alpha(\frac{C'}{N'}) \le 1$ *then*

$$\mu_r(f) \le N' \mu_{r,\alpha}(f).$$

(6) Let $0 \le r < \infty$, $f \in \mathrm{C}^{r+1}(\mathbf{R}^n, \mathbf{R}^m)$ *and suppose there exists a closed subset* $A \subseteq \mathbf{R}^n$ *such that* $D^r f|_A$ *is constant. Then*

$$\|f\|_{r,\alpha} \le \frac{1}{\alpha(1)} \Big(1 + 2 \sup_{x \in \mathbf{R}^n} \mathrm{dist}(x, A)\Big) \|f\|_{r+1}.$$

If $D^r(f - Id)|_A$ *is constant then*

$$\mu_{r,\alpha}(f) \le \frac{1}{\alpha(1)} \Big(1 + 2 \sup_{x \in \mathbf{R}^n} \mathrm{dist}(x, A)\Big) \mu_{r+1}(f).$$

Proof. (1), (2) and (3) follow immediately from $\|Id_{\mathbf{R}^n}\|_1 = 1$ and $\|Id_{\mathbf{R}^n}\|_r = 0$ if $r \geq 2$. To see (4) let $x \in \mathbf{R}^n$, and choose $z \in \mathbf{R}^n$ with $D^r f(z) = 0$ and $\|x - z\| = \mathrm{dist}\big(x, \{y \in \mathbf{R}^n : D^r f(y) = 0\}\big)$. Then we have

$$\begin{aligned} \|D^r f(x)\| &= \|D^r f(x) - D^r f(z)\| \\ &= \|\int_0^1 D^{r+1} f(tx + (1-t)z) \cdot (z - x) dt\| \\ &\leq \|f\|_{r+1} \|x - z\|, \end{aligned}$$

and thus we get

$$\|f\|_r = \sup_{x \in \mathbf{R}^n} \|D^r f(x)\| \leq \Big(\sup_{x \in \mathbf{R}^n} \mathrm{dist}\big(x, \{y \in \mathbf{R}^n : D^r f(y) = 0\}\big) \Big) \|f\|_{r+1}.$$

The second assertion in (4) is an immediate consequence of the first one.

Too see (5) let $x \in \mathbf{R}^n$, choose $z \in \mathbf{R}^n$ with $D^r f(z) = 0$ and $\|x - z\| = \mathrm{dist}\big(x, \{y \in \mathbf{R}^n : D^r f(y) = 0\}\big)$. and let $x_i := z + i\frac{x-z}{N}$, $i = 0, \ldots, N$. Then

$$\begin{aligned} \|D^r f(x)\| &= \|D^r f(x_N) - D^r f(x_0)\| \\ &\leq \sum_{i=1}^N \frac{\|D^r f(x_i) - D^r f(x_{i-1})\|}{\alpha(\|x_i - x_{i-1}\|)} \\ &\leq N \|f\|_{r,\alpha}, \end{aligned}$$

where we used $\alpha(\|x_i - x_{i-1}\|) = \alpha(\frac{\|x-z\|}{N}) \leq \alpha(\frac{C}{N}) \leq 1$ for the first inequality. Since N doesn't depend on x we obtain $\|f\|_r \leq N\|f\|_{r,\alpha}$. The second assertion of (5) is now obvious.

Let f and A as assumed in (6). If $x, y \in \mathbf{R}^n$ with $\|x - y\| \leq 1$ then we can estimate as follows

$$\frac{\|D^r f(x) - D^r f(y)\|}{\alpha(\|x - y\|)} \leq \frac{\|f\|_{r+1}\|x - y\|}{\alpha(\|x - y\|)} \leq \frac{\|f\|_{r+1}}{\alpha(1)}, \tag{2.7}$$

since $t \mapsto \frac{t}{\alpha(t)}$ is increasing (cf. Remark 2.2). If $\|x - y\| > 1$ we choose $x_0, y_0 \in A$ with $\|x - x_0\| = \mathrm{dist}(x, A)$ and $\|y - y_0\| = \mathrm{dist}(x, A)$. Since $D^r f(x_0) = D^r f(y_0)$ we obtain

$$\begin{aligned} \frac{\|D^r f(x) - D^r f(y)\|}{\alpha(\|x - y\|)} &\leq \frac{\|D^r f(x) - D^r f(x_0)\| + \|D^r f(y_0) - D^r f(y)\|}{\alpha(1)} \\ &\leq \frac{\|f\|_{r+1}}{\alpha(1)} (\|x - x_0\| + \|y - y_0\|) \\ &\leq \frac{2\|f\|_{r+1}}{\alpha(1)} \sup_{z \in \mathbf{R}^n} \mathrm{dist}(z, A) \end{aligned} \tag{2.8}$$

Estimations (2.7) and (2.8) prove the first assertion of (6), the second follows immediately. □

2.27. Remark. Let $K \subseteq \mathbf{R}^n$ be a closed subset satisfying

$$C_K := \sup_{x\in K} \operatorname{dist}\Big(x, \overline{\mathbf{R}^n \setminus K}\Big) < \infty,$$

let α be a modulus of continuity, $0 \le r < \infty$ and let f be a diffeomorphism supported on K. Then, by Lemma 2.26, there exists a constant $0 < C < \infty$, depending only on C_K and α with

$$\mu_r(f) \le C\mu_{r+1}(f) \qquad \mu_r(f) \le C\mu_{r,\alpha}(f) \qquad \mu_{r,\alpha}(f) \le C\mu_{r+1}(f),$$

provided f is of sufficiently high differentiability class.

2.28. Definition. An admissible polynomial is a polynomial whose coefficients are nonnegative and which has no constant or linear terms.

2.29. Lemma.

(1) *If* $f_1, \dots, f_k$, $1 \le k$, *are* C^1*-mappings between open subsets of Euclidean spaces, such that the composition* $f_1 \cdots f_k$ *is defined, then*

$$\mu_1(f_1 \cdots f_k) \le k\Big(\sup_{1\le i\le k} \mu_1(f_i)\Big)\Big(1 + \sup_{1\le i\le k} \mu_1(f_i)\Big)^{k-1}.$$

(2) *If* $r \ge 2$ *then there exists an admissible polynomial* F *of one variable with*

$$\mu_r(f_1 \cdots f_k) \le k\Big(\sup_{1\le i\le k} \mu_r(f_i)\Big)\Big(1 + \sup_{1\le i\le k} \mu_1(f_i)\Big)^{r(k-1)} + F\big(\sup_{1\le i\le k} M_{r-1}(f_i)\big)$$

for all C^r*-mappings* $f_1, \dots, f_k$ *between open subsets of Euclidean spaces, such that the composition* $f_1 \cdots f_k$ *is defined.*

(3) *If* $f \in \mathrm{Diff}^1(\mathbf{R}^n)$ *and* $\mu_1(f) \le \frac{1}{2}$ *then*

$$\mu_1(f^{-1}) \le 2\mu_1(f)$$

(4) *If* $r \ge 2$, *then there exists an admissible polynomial* F' *of one variable, such that*

$$\mu_r(f^{-1}) \le \mu_r(f)\Big(1 + 2\mu_1(f)\Big)^{r+1} + F'(M_{r-1}(f))$$

$\forall f \in \mathrm{Diff}^r(\mathbf{R}^n)$ *with* $\mu_1(f) \le \frac{1}{2}$.

Proof. We will prove (1) by induction on k. For $k = 1$ this is obvious, so suppose (1) is proven for $k-1$. Then we have

$$\begin{aligned} &\|D(f_1 \cdots f_{k-1} f_k)(x) - Id\| = \\ &\quad = \|D(f_1 \cdots f_{k-1})(f_k(x)) Df_k(x) - Id\| \\ &\quad \le \|\Big(D(f_1 \cdots f_{k-1})(f_k(x)) - Id\Big) Df_k(x)\| + \|Df_k(x) - Id\| \end{aligned}$$

$$
\begin{aligned}
&\le \mu_1(f_1\cdots f_{k-1})\underbrace{\|f_k\|_1}_{\le 1+\mu_1(f_k)}+\mu_1(f_k)\\
&\le (k-1)\Big(\sup_{1\le i\le k-1}\mu_1(f_i)\Big)\Big(1+\sup_{1\le i\le k-1}\mu_1(f_i)\Big)^{k-2}\Big(1+\mu_1(f_k)\Big)+\mu_1(f_k)\\
&\le (k-1)\Big(\sup_{1\le i\le k}\mu_1(f_i)\Big)\Big(1+\sup_{1\le i\le k}\mu_1(f_i)\Big)^{k-1}+\Big(\sup_{1\le i\le k}\mu_1(f_i)\Big)\\
&\le k\Big(\sup_{1\le i\le k}\mu_1(f_i)\Big)\Big(1+\sup_{1\le i\le k}\mu_1(f_i)\Big)^{k-1}
\end{aligned}
$$

and thus (1) is proven. We will prove (2) by induction on k too. For $k=1$ it is again obvious and hence we assume it is yet shown for all $i\le k-1$. Then, using (2.3) and (1), we obtain

$$
\begin{aligned}
&\|D^r(f_1\cdots f_k)(x)\| \le\\
&\le \|f_1\cdots f_{k-1}\|_r\|f_k\|_1^r+\|f_1\cdots f_{k-1}\|_1\|f_k\|_r\\
&\quad+\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\cdots+j_i=r}} C_{i,j_1,\ldots,j_i}\|f_1\cdots f_{k-1}\|_i\|f_k\|_{j_1}\ldots\|f_k\|_{j_i}\\
&\le (k-1)\Big(\sup_{1\le l\le k-1}\mu_r(f_l)\Big)\Big(1+\sup_{1\le l\le k-1}\mu_1(f_l)\Big)^{r(k-2)}\Big(1+\mu_1(f_k)\Big)^r\\
&\quad+F_1(\sup_{1\le l\le k-1}M_{r-1}(f_l))\Big(1+\mu_1(f_k)\Big)^r\\
&\quad+(k-1)\Big(\sup_{1\le l\le k-1}\mu_1(f_l)\Big)\Big(1+\sup_{1\le l\le k-1}\mu_1(f_l)\Big)^{k-2}\mu_r(f_k)\\
&\quad+\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\cdots+j_i=r}} C_{i,j_1,\ldots,j_i}\|f_1\cdots f_{k-1}\|_i M_{r-1}(f_k)\Big(1+M_{r-1}(f_k)\Big)^{i-1}\\
&\le (k-1)\Big(\sup_{1\le l\le k}\mu_r(f_l)\Big)\Big(1+\sup_{1\le l\le k}\mu_1(f_l)\Big)^{r(k-1)}+F_2(\sup_{1\le l\le k}M_{r-1}(f_l))\\
&\quad+\Big(1+\sup_{1\le l\le k}\mu_1(f_l)\Big)^{k-1}\Big(1+\sup_{1\le l\le k}\mu_1(f_l)\Big)^{k-1}\mu_r(f_k)\\
&\quad+\Big(\sup_{1\le l\le k}M_{r-1}(f_l)\Big)\Big(1+\sup_{1\le l\le k}M_{r-1}(f_l)\Big)^{r-1}\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\cdots+j_i=r}} C_{i,j_1,\ldots,j_i}\|f_1\cdots f_{k-1}\|_i\\
&\le k\Big(\sup_{1\le l\le k}\mu_r(f_l)\Big)\Big(1+\sup_{1\le l\le k}\mu_1(f_l)\Big)^{r(k-1)}+F(\sup_{1\le l\le k}M_{r-1}(f_l)),
\end{aligned}
$$

where F is an admissible polynomial of one variable. Here we used

$$
\begin{aligned}
&\|f_1\cdots f_{k-1}\|_i =\\
&= \mu_i(f_1\cdots f_{k-1})\\
&\le (i-1)\Big(\sup_{1\le l\le k-1}\mu_i(f_l)\Big)\Big(1+\sup_{1\le l\le k-1}\mu_1(f_l)\Big)^{i(k-1)}+F_3(\sup_{1\le l\le k-1}M_{i-1}(f_l))\\
&\le (i-1)\Big(\sup_{1\le l\le k}M_{r-1}(f_l)\Big)\Big(1+\sup_{1\le l\le k}M_{r-1}(f_l)\Big)^{i(k-1)}+F_3(\sup_{1\le l\le k}M_{r-1}(f_l)),
\end{aligned}
$$

and thus F really doesn't have any constant or linear terms. If $f\in\mathrm{Diff}^1(\mathbf{R}^n)$ and $\mu_1(f)<1$ we have

$$(Df(x))^{-1}=\sum_{m=0}^{\infty}\Big(-(Df(x)-Id)\Big)^m$$

and thus

$$\begin{aligned}\mu_1(f^{-1}) &= \sup_{x\in\mathbf{R}^n}\|D(f^{-1}-Id)(x)\| \;=\; \sup_{x\in\mathbf{R}^n}\|(Df)^{-1}(f^{-1}(x))-Id\|\\ &= \sup_{x\in\mathbf{R}^n}\|\sum_{m=1}^{\infty}\Big(-(Df(f^{-1}(x))-Id)\Big)^m\|\\ &\le \sum_{m=1}^{\infty}\sup_{x\in\mathbf{R}^n}\|(Df(x)-Id)\|^m\\ &\le \sum_{m=1}^{\infty}\mu_1(f)^m=\frac{1}{1-\mu_1(f)}-1=\frac{\mu_1(f)}{1-\mu_1(f)}\le 2\mu_1(f),\end{aligned}$$

provided $\mu_1(f)\le\frac{1}{2}$. We will prove (4) by induction on r. From (2.3) we get

$$\begin{aligned}0=D^r(ff^{-1}) &= (D^rf\circ f^{-1})D(f^{-1})^r+(Df\circ f^{-1})D^r(f^{-1})\\ &\quad+\underbrace{\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\dots+j_i=r}}C_{i,j_1,\dots,j_i}(D^if)\Big(D^{j_1}(f^{-1})\times\dots\times D^{j_i}(f^{-1})\Big)}_{:=R_{f,f^{-1}}}\end{aligned}$$

and thus

$$D^r(f^{-1})=-D(f^{-1})(D^rf\circ f^{-1})D(f^{-1})^r-D(f^{-1})R_{f,f^{-1}}. \tag{2.9}$$

If $r=2$ and $\mu_1(f)\le\frac{1}{2}$ we obtain from (2.9) and (3)

$$\begin{aligned}\|D^2(f^{-1})(x)\| &\le \|f^{-1}\|_1\|f\|_2\|f^{-1}\|_1^2\\ &\le \mu_2(f)\Big(1+\mu_1(f^{-1})\Big)^3\le\mu_2(f)\Big(1+2\mu_1(f)\Big)^3,\end{aligned}$$

this shows (4) in the case $r=2$. Now suppose the formula in (4) above is proven for all $j\le r-1$. Using once again (2.9) and (3) we get

$$\begin{aligned}&\|D^r(f^{-1})(x)\|\le\\ &\le \|f^{-1}\|_1\|f\|_r\|f^{-1}\|_1^r+\|f^{-1}\|_1\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\dots+j_i=r}}C_{i,j_1,\dots,j_i}\|f\|_i\|f^{-1}\|_{j_1}\cdots\|f^{-1}\|_{j_i}\\ &\le \mu_r(f)\Big(1+2\mu_1(f)\Big)^{r+1}\\ &\quad+\Big(1+2\mu_1(f)\Big)\sum_{\substack{1<i<r,\ 1\le j_s\\ j_1+\dots+j_i=r}}C_{i,j_1,\dots,j_i}M_{r-1}(f)\mu_{j_{l_0}}(f^{-1})\prod_{\substack{1\le l\le i\\ l\ne l_0}}\Big(1+\mu_{j_l}(f^{-1})\Big)\\ &\le \mu_r(f)\Big(1+2\mu_1(f)\Big)^{r+1}+F'(M_{r-1}(f)),\end{aligned}$$

where F' is an admissible polynomial of one variable (here we used that $j_i<r$, and that there exists at least one $j_{l_0}\ge 2$). □

2.30. Lemma. *Let* $1\le r<\infty$, α *a modulus of continuity and let* $K\subseteq\mathbf{R}^n$ *be a closed subset, satisfying*

$$C_K:=\sup_{x\in\mathbf{R}^n}\operatorname{dist}\Big(x,\overline{\mathbf{R}^n\setminus K}\Big)<\infty.$$

(1) There exist $\delta>0$ *and* $C>0$ *depending on* n, r, α *and* C_K, *such that*

$$\mu_{r,\alpha}(f\circ g)\ \le\ \mu_{r,\alpha}(f)+\mu_{r,\alpha}(g)+C\mu_{r,\alpha}(f)\mu_{r,\alpha}(g)$$

$\forall f,g\in\mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ *with* $\mu_{r,\alpha}(f),\mu_{r,\alpha}(g)\le\delta$ *and* $D^1(f-Id)|_{\mathbf{R}^n\setminus K}=D^1(g-Id)|_{\mathbf{R}^n\setminus K}\equiv 0$.

(2) Given $\lambda>1$ *there exists* $\delta>0$ *depending on* n, r, α *and* C_K, *such that*

$$\mu_{r,\alpha}(f^{-1})\le\lambda\mu_{r,\alpha}(f)$$

$\forall f\in\mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ *with* $\mu_{r,\alpha}(f)\le\delta$ *and* $D^1(f-Id)|_{\mathbf{R}^n\setminus K}\equiv 0$.

Proof. In view of (2.3) we take $f,g\in C^{r,\alpha}(\mathbf{R}^n,\mathbf{R}^n)$ and estimate

$$\begin{aligned}
&\frac{\|\Big((D^if\circ g)(D^{j_1}g\times\cdots\times D^{j_i}g)\Big)\Big|_y^x\|}{\alpha(\|x-y\|)}\le\\
&\le\ \frac{\|(D^if\circ g)\Big|_y^x\|}{\alpha(\|x-y\|)}\prod_{1\le l\le i}\|D^{j_l}g(x)\|+\frac{\|D^if(g(y))\|}{\alpha(\|x-y\|)}\prod_{1\le l\le i}\|D^{j_l}g\Big|_y^x\|\\
&\le\ \mu_{i,\alpha}(f)\frac{\alpha(\|g(x)-g(y)\|)}{\alpha(\|x-y\|)}\prod_{1\le l\le i}\|g\|_{j_l}+\|f\|_i\frac{\|D^{j_{l_0}}g\Big|_y^x\|}{\alpha(\|x-y\|)}\prod_{\substack{1\le l\le i\\ l\ne l_0}}2\|g\|_{j_l}\\
&\le\ \mu_{i,\alpha}(f)\frac{\alpha(\|g\|_1\|x-y\|)}{\alpha(\|x-y\|)}\prod_{1\le l\le i}\|g\|_{j_l}+2^{i-1}\|f\|_i\mu_{j_{l_0},\alpha}(g)\prod_{\substack{1\le l\le i\\ l\ne l_0}}\|g\|_{j_l}\\
&\le\ \mu_{i,\alpha}(f)\Big(\mu_1(g)+1\Big)\prod_{1\le l\le i}\|g\|_{j_l}+2^{i-1}\|f\|_i\mu_{j_{l_0},\alpha}(g)\prod_{\substack{1\le l\le i\\ l\ne l_0}}\|g\|_{j_l}
\end{aligned}\tag{2.10}$$

where $1\le l_0\le i$ can be chosen arbitrarily. Together with (2.3) and Lemma 2.26 this yields

$$\begin{aligned}
\mu_{r,\alpha}(f\circ g)\ \le\ &\mu_{r,\alpha}(f)\Big(\mu_1(g)+1\Big)\|g\|_1^r+2^{r-1}\|f\|_r\mu_{1,\alpha}(g)\|g\|_1^{r-1}\\
&+\mu_{1,\alpha}(f)\Big(\mu_1(g)+1\Big)\|g\|_r+\|f\|_1\mu_{r,\alpha}(g)\\
&+\sum_{\substack{2\le i\le r-1,1\le j_i\\ j_1+\cdots+j_i=r}}C_{i,j_1,\ldots,j_i}\Bigg(\mu_{i,\alpha}(f)(\mu_1(g)+1)\prod_{1\le l\le i}\|g\|_{j_l}\\
&\qquad+2^{i-1}\|f\|_i\mu_{j_1,\alpha}(g)\prod_{2\le l\le i}\|g\|_{j_l}\Bigg)\\
\le\ &\mu_{r,\alpha}(f)+\mu_{r,\alpha}(g)+C_1\mu_{r,\alpha}(f)\mu_{r,\alpha}(g)
\end{aligned}$$

provided $\mu_{r,\alpha}(f), \mu_{r,\alpha}(g)$ are sufficiently small. Here C_1 is a constant, depending only on C_K, n, r and α. This shows (1) of Lemma 2.30.

In order to prove (2) we first assume $r = 1$. Then if $\mu_{1,\alpha}(f)$ is sufficiently small we have $\mu_1(f) < 1$ (cf. Lemma 2.26) and thus

$$(Df)^{-1}(x) = \sum_{m=0}^{\infty} \Big(-(Df(x) - Id)\Big)^m.$$

Hence

$$\begin{aligned}
\mu_{1,\alpha}(f^{-1}) &= \sup_{x \neq y \in \mathbf{R}^n} \frac{\|D(f^{-1})(x) - D(f^{-1})(y)\|}{\alpha(\|x-y\|)} \\
&= \sup_{x \neq y \in \mathbf{R}^n} \frac{\|\Big((Df)^{-1} \circ f^{-1}\Big)\Big|_y^x\|}{\alpha(\|f^{-1}(x) - f^{-1}(y)\|)} \frac{\alpha(\|f^{-1}(x) - f^{-1}(y)\|)}{\alpha(\|x-y\|)} \\
&\le \sup_{x \neq y \in \mathbf{R}^n} \frac{\|(Df)^{-1}\Big|_y^x\|}{\alpha(\|x-y\|)} \underbrace{\sup_{x \neq y \in \mathbf{R}^n} \frac{\alpha(\|f^{-1}\|_1 \|x-y\|)}{\alpha(\|x-y\|)}}_{\le \mu_1(f^{-1})+1 \le \sqrt{\lambda}} \\
&\le \sqrt{\lambda} \sup_{x \neq y \in \mathbf{R}^n} \sum_{m=1}^{\infty} \frac{\|(Df(x) - Id)^m - (Df(y) - Id)^m\|}{\alpha(\|x-y\|)} \\
&\le \sqrt{\lambda} \sum_{m=1}^{\infty} \frac{\sum_{k=0}^{m-1} \|(Df(x) - Id)^{m-k-1}\| \|Df\Big|_y^x\| \|(Df(y) - Id)^k\|}{\alpha(\|x-y\|)} \\
&\le \sqrt{\lambda} \underbrace{\sum_{m=1}^{\infty} m \cdot \mu_1(f)^{m-1}}_{\le \sqrt{\lambda}} \mu_{1,\alpha}(f) \; \le \; \lambda \mu_{1,\alpha}(f)
\end{aligned}$$

provided $\mu_{1,\alpha}(f)$ is small (cf. Lemma 2.26 and Lemma 2.29(3)). Now suppose $r \ge 2$. Recall formula (2.9) and look at

$$\begin{aligned}
&\frac{\|D(f^{-1})(D^r f \circ f^{-1})(D(f^{-1}))^r\Big|_y^x\|}{\alpha(\|x-y\|)} = \\
&\quad \le \frac{\|D(f^{-1})(x) - D(f^{-1})(x)\|}{\alpha(\|x-y\|)} \|f\|_r \|f^{-1}\|_1^r \\
&\qquad + \|f^{-1}\|_1 \frac{\|D^r f(f^{-1}(x)) - D^r f(f^{-1}(x))\|}{\alpha(\|x-y\|)} \|f^{-1}\|_1^r \\
&\qquad + \|f^{-1}\|_1 \|f\|_r \frac{\|\Big(D(f^{-1})(x) - D(f^{-1})(y)\Big)^r\|}{\alpha(\|x-y\|)} \\
&\quad \le \mu_{1,\alpha}(f^{-1}) \|f\|_r \|f^{-1}\|_1^r + \|f^{-1}\|_1^{r+1} \mu_{r,\alpha}(f) \frac{\|\alpha(f^{-1}(x) - f^{-1}(y))\|}{\alpha(\|x-y\|)} \\
&\qquad + \|f^{-1}\|_1 \|f\|_r \mu_{1,\alpha}(f^{-1}) \Big(2\|f^{-1}\|_1\Big)^{r-1}
\end{aligned}$$

$$\begin{aligned}
&\le \mu_{1,\alpha}(f^{-1})\|f\|_r\|f^{-1}\|_1^r + \|f^{-1}\|_1^{r+1}\mu_{r,\alpha}(f)(\mu_1\big(f^{-1}) + 1\big) \\
&\quad +2^{r-1}\|f^{-1}\|_1^r\|f\|_r\mu_{1,\alpha}(f^{-1}) \\
&\le \underbrace{\left(1 + \frac{\lambda-1}{2}\right)}_{>1}\mu_{r,\alpha}(f) \qquad (2.11)
\end{aligned}$$

provided $\mu_{r,\alpha}(f)$ is sufficiently small (cf. Lemma 2.29(3) and Lemma 2.26). Further we have

$$\begin{aligned}
&\frac{\|D(f^{-1})R_{f,f^{-1}}\big|_y^x\|}{\alpha(\|x-y\|)} \le \\
&\le \frac{\|D(f^{-1})\big|_y^x\|\|R_{f,f^{-1}}(x)\|}{\alpha(\|x-y\|)} + \frac{\|D(f^{-1})(y)\|\|R_{f,f^{-1}}\big|_y^x\|}{\alpha(\|x-y\|)} \\
&\le \mu_{1,\alpha}(f^{-1}) \sum_{\substack{1<i<r,\ 1\le j_s \\ j_1+\cdots+j_i=r}} C_{i,j_1,\dots,j_i}\|f\|_i \prod_{1\le l\le i}\|f^{-1}\|_{j_l} \\
&\quad +\|f^{-1}\|_1 \sum_{\substack{1<i<r,\ 1\le j_s \\ j_1+\cdots+j_i=r}} C_{i,j_1,\dots,j_i}\mu_{i,\alpha}(f)(\mu_1(f^{-1})+1)\prod_{1\le l\le i}\|f^{-1}\|_{j_i} \\
&\quad +\|f^{-1}\|_1 \sum_{\substack{1<i<r,\ 1\le j_s \\ j_1+\cdots+j_i=r}} C_{i,j_1,\dots,j_i}\|f\|_i\mu_{j_1,\alpha}(f^{-1})\prod_{2\le l\le i}\Big(2\|f^{-1}\|_{j_l}\Big) \\
&\le \underbrace{\frac{\lambda-1}{2}}_{>0}\mu_{r,\alpha}(f) \qquad (2.12)
\end{aligned}$$

provided $\mu_{r,\alpha}(f)$ is sufficiently small (cf. Lemma 2.29(3)&(4) and Lemma 2.26). Using (2.9) (2.11) and (2.12) we obtain $\delta > 0$ depending on n, r, α and C_K with

$$\begin{aligned}
\mu_{r,\alpha}(f^{-1}) &= \sup_{x\ne y\in\mathbf{R}^n}\frac{\|D^r(f^{-1})(x) - D^r(f^{-1})(y)\|}{\alpha(\|x-y\|)} \\
&\le \left(1+\frac{\lambda-1}{2}\right)\mu_{r,\alpha}(f) + \left(\frac{\lambda-1}{2}\right)\mu_{r,\alpha}(f) = \lambda\mu_{r,\alpha}(f)
\end{aligned}$$

$\forall f \in \mathrm{Diff}^r(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f) \le \delta$ and $D^1(f-Id)|_{\mathbf{R}^n\setminus K} \equiv 0$. □

2.31. Corollary. *Let $K \subseteq \mathbf{R}^n$ be a compact, convex subset, $1 \le r < \infty$ and let α be a modulus of continuity. Then*

$$(f,g) \mapsto \|f-g\|_{r,\alpha}$$

is a metric on $\mathrm{Diff}_K^{r,\alpha}(\mathbf{R}^n)$. The induced topology is called $\mathrm{C}^{r,\alpha}$-topology. It is finer than the C^r-topology and coarser than the C^{r+1}-topology on $\mathrm{Diff}_K^{r+1}(\mathbf{R}^n)$. $\mathrm{Diff}_K^{r,\alpha}(\mathbf{R}^n)$ equipped with the $\mathrm{C}^{r,\alpha}$-topology is a connected, topological group.

Proof. Let $f,g\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$. Since $D^r(f-g)$ is locally α-continuous, $f-g|_{\mathbf{R}^n\setminus K}\equiv 0$ and since K is compact, the set

$$\left\{\frac{\|D^rf(x)-D^rf(y)\|}{\alpha\|x-y\|}:x,y\in\mathbf{R}^n\right\}$$

is bounded, and thus $\|f-g\|_{r,\alpha}<\infty$. Obviously $\|f-g\|_{r,\alpha}=\|g-f\|_{r,\alpha}$ and $\|f-h\|_{r,\alpha}\leq\|f-g\|_{r,\alpha}+\|g-h\|_{r,\alpha}$ for all $f,g,h\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$. Now suppose $f,g\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ and $\|f-g\|_{r,\alpha}=0$. Then $D^r(f-g)$ is constant, and since $f-g|_{\mathbf{R}^n\setminus K}=0$ this yields $f=g$, and thus we have shown $\|\cdot-\cdot\|_{r,\alpha}$ is a metric. By Lemma 2.26(5) the induced topology is finer than the C^r-topology and by Lemma 2.26(6) it is coarser than the C^{r+1}-topology on $\mathrm{Diff}^{r+1}_K(\mathbf{R}^n)$. If $f,f',g\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ we obtain from (2.3) and (2.10)

$$\begin{aligned}
&\|fg-f'g\|_{r,\alpha}=\|(f-f')\circ g\|_{r,\alpha}=\mu_{r,\alpha}((f-f')\circ g)\leq\\
&\leq\sum_{\substack{1\leq i\leq r,1\leq j_l\\ j_1+\cdots+j_i=r}}C_{i,j_1,\ldots,j_i}\Big(\mu_{i,\alpha}(f-f')\big(1+\mu_1(g)\big)\prod_{1\leq l\leq i}\|g\|_{j_l}\\
&\qquad\qquad+2^{i-1}\|f-f'\|_i\mu_{j_1,\alpha}(g)\prod_{2\leq l\leq i}\|g\|_{j_l}\Big)\\
&\leq C\Big(\|f-f'\|_{r,\alpha}\big(1+\mu_{r,\alpha}(g)\big)^{r+1}+\|f-f'\|_{r,\alpha}\mu_{r,\alpha}(g)\big(1+\mu_{r,\alpha}(g)\big)^{r-1}\Big)
\end{aligned}$$

where C is a constant depending on n, r, α and C_K (cf. Lemma 2.26). Thus $\mathrm{comp}(\cdot,g):\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)\to\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ is continuous with respect to $\mathrm{C}^{r,\alpha}$-topology for all $g\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$. Similarly one shows that $\mathrm{comp}(g,\cdot)$ is continuous for all $g\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$. By Lemma 2.30(1) comp is continuous at (Id,Id), and hence

$$\mathrm{comp}=\mathrm{comp}(f_0,\cdot)\circ\mathrm{comp}(\cdot,g_0)\circ\mathrm{comp}\circ\Big(\mathrm{comp}(f_0^{-1},\cdot)\times\mathrm{comp}(\cdot,g_0^{-1})\Big)$$

is continuous at (f_0,g_0). Since f_0 and g_0 were arbitrary this yields: comp is continuous with respect to $\mathrm{C}^{r,\alpha}$-topology.

From Lemma 2.30(2) we obtain immediately $\mathrm{inv}:f\mapsto f^{-1}$ is continuous at Id. If $f_0\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ we have

$$\mathrm{inv}=\mathrm{comp}(f_0^{-1},\cdot)\circ\mathrm{inv}\circ\mathrm{comp}(\cdot,f_0^{-1})$$

and inv is continuous at f_0. Consequently $\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ together with the $\mathrm{C}^{r,\alpha}$-topology is a topological group, and it remains to show its connectedness.

Suppose $f\in\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ is C^1 near Id. Then

$$\begin{array}{rcl}
\mathbf{R}^n\times I & \xrightarrow{H} & \mathbf{R}^n\\
(x,t) & \mapsto & H_t(x):=tf(x)+(1-t)x
\end{array}$$

is a $C^{r,\alpha}$-diffeotopy to Id, supported on K. Further we have

$$\begin{aligned}\|H_t-H_{t_0}\|_{r,\alpha} &= \sup_{x\neq y\in\mathbf{R}^n}\frac{\|D^r(H_t-H_{t_0})(x)-D^r(H_t-H_{t_0})(y)\|}{\alpha(\|x-y\|)}\\ &= \sup_{x\neq y\in\mathbf{R}^n}\frac{\|(t-t_0)D^rf(x)-(t-t_0)D^rf(y)\|}{\alpha(\|x-y\|)}\\ &\le |t-t_0|\|f\|_{r,\alpha}\end{aligned}$$

and thus $\check{H}: I\to\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ is continuous with respect to $C^{r,\alpha}$-topology. Since $\mathrm{Diff}^{r+1}_K(\mathbf{R}^n)$ is dense in $\mathrm{Diff}^r_K(\mathbf{R}^n)$ (cf. Proposition 1.7(6)) and since $\mathrm{Diff}^{r+1}_K(\mathbf{R}^n)$ is connected (cf. Example 1.15) this shows that $\mathrm{Diff}^{r,\alpha}_K(\mathbf{R}^n)$ is connected. □

2.32. Definition. We let $C_i:=\mathbf{R}^{i-1}\times S^1\times\mathbf{R}^{n-i}$, where we think of S^1 as $S^1\cong\mathbf{R}/\mathbf{Z}$. Let $\pi_i:\mathbf{R}^n\to C_i$ be the covering projection, and let $\tilde{p}_i:\mathbf{R}^n\to\mathbf{R}^{n-1}$ resp. $p_i: C_i\to\mathbf{R}^{n-1}$ be the projections, which omit the ith coordinate. Clearly $p_i\circ\pi_i=\tilde{p}_i$.

The mapping $\pi_i:\mathbf{R}^n\to C_i$ gives us a preferred system of coordinates in a neighborhood of any point of C_i. The transition mappings between these coordinate systems are all translations and hence if $r\ge 1$ and $f\in C^r(C_i,C_i)$ then $D^rf: C_i\to L^r(\mathbf{R}^n,\mathbf{R}^n)$ is defined independently of the choice of one of these preferred charts. Thus the (semi)norms introduced in Definition 2.25 do make sense on C_i too, provided $r\ge 1$.

Let $A\ge 1$. If $\mathrm{supp}(f)\subseteq[-2A,2A]^n$ and $\mu_0(f)\le\frac{1}{2}$, we define $\Gamma_{i,A}(f): C_i\to C_i$ by: Given $\theta\in C_i$ we choose $x\in\mathbf{R}^n$ with $\pi_i(x)=\theta$ and $pr_i(x)<-2A$. Then we choose $N\in\mathbf{N}$ such that $pr_i((T_if)^N(x))>2A$, and let

$$\Gamma_{i,A}(f)(\theta):=\pi_i((T_if)^N(x)),$$

which is independent of the choice of N and x, since $\mathrm{supp}(f)\subseteq[-2A,2A]^n$.

2.33. Lemma. *[7] [4] There exists a neighborhood U of $Id\in\mathrm{Diff}^1_c(\mathbf{R}^n)_\circ$ such that*

(1) $\Gamma_{i,A}(Id)=Id\qquad\forall A\ge 1\ \forall 1\le i\le n$

(2) Let $2A\ge a_i\ge 2$, $1\le i\le n$, and $\mathrm{supp}(f)\subseteq[-a_1,a_1]\times\cdots\times[-a_n,a_n]$. *Then* $\mathrm{supp}(f)\subseteq[-a_1,a_1]\times\cdots\times S^1\times\cdots\times[-a_n,a_n]$.

(3) Let α be a modulus of continuity, $r\ge 1$, $A\ge 1$ and $1\le i\le n$. Then

$$\Gamma_{i,A}:\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n)\cap U\to\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^{i-1}\times S^1\times[-2A,2A]^{n-i}}(C_i)_\circ$$

is continuous with respect to C^r-topologies.

(4) If $r\ge 1$, $A\ge 1$ and if α is a modulus of continuity, then there exists $\delta>0$, depending on n, r, α and A, such that

$$\mu_{r,\alpha}(\Gamma_{i,A}(f))\le 9A\mu_{r,\alpha}(f)$$

$\forall 1 \le i \le n$, $\forall f \in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$ *with* $\mu_{r,\alpha}(f) \le \delta$, *and*

$$\mu_{r,\alpha}(\Gamma_{i,A}(f)) \le 9\mu_{r,\alpha}(f)$$

$\forall 1 \le i \le n$, $\forall f \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n) \cap U$ *with* $\mu_{r,\alpha}(f) \le \delta$.

(5) If $A \ge 1$ *and* $r \ge 2$ *then there exists an admissible polynomial of one variable* F*, such that*

$$\mu_r(\Gamma_{i,A}(f)) \le 18A\mu_r(f)(1+\mu_1(f))^{18Ar} + F(M_{r-1}(f)) \tag{2.13}$$

$\forall f \in \mathrm{Diff}^r_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$*, and*

$$\mu_1(\Gamma_{i,A}(f)) \le 18A\mu_1(f)(1+\mu_1(f))^{18A} \tag{2.14}$$

$\forall f \in \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$.

Proof. (1) and (2) are obvious. If $f \in \mathrm{Diff}^r_{[-2A,2A]}(\mathbf{R}^n)_\circ$, then $(\Gamma_{i,A}(f))^{-1}$ is constructed similar to $\Gamma_{i,A}(f)$, but now choose $x \in \mathbf{R}^n$ with $pr_i(x) > 2A$, N such that $pr_i((T_i f)^{-N}(x)) < -2A$, and let $\Gamma_{i,A}(f)^{-1}(\theta) := \pi_i((T_i f)^{-N}(x))$.

Since composition is continuous $\Gamma_{i,A}$ is continuous. If we can choose U, such that $\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$ is connected $\forall 1 \le r \le \infty$, $\forall A \ge 1$ and for all moduli of continuity α, then (3) follows immediately. So we choose $0 < \epsilon \le \frac{1}{2}$, such that

$$\mu_0(f) < \epsilon, \mu_1(f) < \epsilon \Longrightarrow f \in \mathrm{Diff}^1(\mathbf{R}^n),$$

and let

$$U := \{f \in \mathrm{Diff}^1_c(\mathbf{R}^n) : \mu_0(f), \mu_1(f) < \epsilon\}.$$

If $f \in \mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$, it is easily seen that $H(x,t) := tf(x) + (1-t)x$ is a $\mathrm{C}^{r,[\alpha]}$-diffeotopy in $\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$ to Id, and thus $\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n) \cap U$ is connected. To see (4), we first show the following

Sublemma. *Given* $N \in \mathbf{N}$ *there exists* $\delta > 0$*, depending on* n*,* r*,* α*,* A *and* N*, such that*

$$\mu_{r,\alpha}((T_i f)^N) \le (N+1)\mu_{r,\alpha}(f)$$

$\forall f \in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n)$ *with* $\mu_{r,\alpha}(f) \le \delta$.

Proof of the Sublemma. By Lemma 2.30(1) there exist $\delta_1 > 0$ and $C_1 > 0$, such that

$$\mu_{r,\alpha}(fg) \le \mu_{r,\alpha}(f) + \mu_{r,\alpha}(g) + C_1\mu_{r,\alpha}(f)\mu_{r,\alpha}(g) \tag{2.15}$$

$\forall f, g \in \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f), \mu_{r,\alpha}(g) \le \delta_1$, $D^1(f - Id)|_{\mathbf{R}^n \setminus K} = D^1(g - Id)|_{\mathbf{R}^n \setminus K} = 0$, where $K := [-2A-N, 2A+N]^n$. Now choose $\epsilon > 0$ with $1 + (1+\epsilon) + \ldots + (1+\epsilon)^{N-1} \le N+1$, and let $\delta := \min\{\delta_1, \frac{\delta_1}{N+1}, \frac{\epsilon}{C_1}\}$. From $\mu_{r,\alpha}(f) = \mu_{r,\alpha}(T_i f)$ and induction on m we get

$$\mu_{r,\alpha}((T_i f)^m) \le (1 + (1+\epsilon) + \ldots + (1+\epsilon)^{m-1})\mu_{r,\alpha}(f)$$

$\forall 1\le m\le N$ and $\forall f\in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f)\le\delta$. In the case $m=N$ this is

$$\mu_{r,\alpha}((T_if)^N)\le(1+(1+\epsilon)+\ldots+(1+\epsilon)^{N-1})\mu_{r,\alpha}(f)\le(N+1)\mu_{r,\alpha}(f)$$

$\forall f\in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f)\le\delta$. □

Choose $N\in\mathbf{N}$ with $8A+3>N>8A+1$ and use the Sublemma above to obtain

$$\mu_{r,\alpha}(\Gamma_{i,A}(f))\le\mu_{r,\alpha}((T_if)^N)\le(N+1)\mu_{r,\alpha}(f)\le 9A\mu_{r,\alpha}(f)$$

$\forall f\in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n)\cap U$ with $\mu_{r,\alpha}(f)\le\delta$, and

$$\mu_{r,\alpha}(\Gamma_{i,A}(f))\le\mu_{r,\alpha}((T_if)^N)=\mu_{r,\alpha}((T_if)^8)\le 9\mu_{r,\alpha}(f)$$

$\forall f\in \mathrm{Diff}^{r,\alpha}_{[-2,2]^i\times[-2A,2A]^{n-i}}(\mathbf{R}^n)\cap U$ with $\mu_{r,\alpha}(f)\le\delta$.

To see (5) choose an integer N with $8A+1<N<8A+3$. Since $\mu_r(T_i)=0$ for $r\ge 1$, we obtain from Lemma 2.29(1)

$$\begin{aligned}\mu_1(\Gamma_{i,A}(f)) &= \mu_1((T_if)^N)\le 2N\mu_1(f)(1+\mu_1(f))^{2N-1}\\ &\le 18A\mu_1(f)(1+\mu_1(f))^{18A},\end{aligned}$$

and this is (2.14). To see (2.13) we estimate

$$\begin{aligned}\mu_r(\Gamma_{i,A}(f)) &= \mu_r((T_if)^N)\le 2N\mu_r(f)(1+\mu_1(f))^{r(2N-1)}+F(M_{r-1}(f))\\ &\le 18A\mu_r(f)(1+\mu_1(f))^{18Ar}+F(M_{r-1}(f)),\end{aligned}$$

where F is the admissible polynomial of Lemma 2.29(2). This finishes the proof of (5). □

2.34. Definition. Consider the following S^1-action $\mathrm{S}^1\times C_i\to C_i$

$$\beta\cdot(x_1,\ldots,\vartheta,\ldots,x_n):=(x_1,\ldots,\beta+\vartheta,\ldots,x_n)$$

and let $G_i^{r,[\alpha]}$ denote the group of equivariant $\mathrm{C}^{r,[\alpha]}$-diffeomorphisms of C_i,

$$G_i^{r,[\alpha]}:=\{f\in\mathrm{Diff}^{r,[\alpha]}(C_i): f(\beta\cdot\theta)=\beta\cdot f(\theta)\quad\forall\beta\in\mathrm{S}^1\forall\theta\in C_i\}.$$

2.35. Remark. If $f\in G_i^{r,[\alpha]}$ then there exists a mapping $f':\mathbf{R}^{n-1}\to\mathbf{R}^{n-1}$ with $p_i\circ f=f'\circ p_i$, for f is S^1-equivariant. f' is $\mathrm{C}^{r,[\alpha]}$, since there exists a global cross section $\mathbf{R}^{n-1}\to C_i$. The inverse of f' is given by $(f')^{-1}=(f^{-1})'$ and is $\mathrm{C}^{r,[\alpha]}$ by a similar argument. Thus $f'\in\mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^{n-1})$.

2.36. Lemma (Mather). *[7] Let α be a modulus of continuity, $1\le r\le\infty$ and $A\ge 1$. Then there exists a neighborhood U' of $Id\in\mathrm{Diff}^1_c(\mathbf{R}^n)$ depending on n, r, α and A, such that the following holds: If $f,g\in\mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^n}(\mathbf{R}^n)\cap U'$ and $\Gamma_{i,A}(f)\Gamma_{i,A}(g)^{-1}\in G_i^{r,[\alpha]}$ then $\tau_{i,A}f$ and $\tau_{i,A}g$ are conjugated in $\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ$.*

Proof. First we assume $U' \subseteq U$ (cf. Lemma 2.33), thus $\Gamma_{i,A}$ is well defined on U'. Now we define $\Lambda : \mathbf{R}^n \to \mathbf{R}^n$ by: Given $x \in \mathbf{R}^n$ choose $N \in \mathbf{N}$ such that $pr_i((T_if)^{-N}(x)) < -2A$ and let $\Lambda(x) := (T_ig)^N(T_if)^{-N}(x)$. Since $\mu_0(f) \le \frac{1}{2}$ such an N exists and since $\mathrm{supp}(f), \mathrm{supp}(g) \subseteq [-2A, 2A]^n$, Λ doesn't depend on the choice of N. Locally N can be chosen constant, so we have $\Lambda \in \mathrm{C}^{r,[\alpha]}(\mathbf{R}^n, \mathbf{R}^n)$. The inverse of Λ is constructed similarly, but now choose $N \in \mathbf{N}$ such that $pr_i((T_i \circ g)^{-N}(x)) < -2A$ and set $\Lambda^{-1}(x) := (T_if)^N(T_ig)^{-N}(x)$. Hence we obtain $\Lambda \in \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n)$. Notice that Λ depends on i, A, f and g, but we omit the indices, since Λ is used solely in this proof.

Claim 1. *We have*

(1) $\Lambda T_i f \Lambda^{-1} = T_i g$

(2) $\Lambda = Id$ *on* $\{x \in \mathbf{R}^n : \exists j \neq i : |pr_j(x)| \ge 2A \quad \textit{or} \quad pr_i(x) \le -2A\}$

(3) $\Gamma_{i,A}(g)(\Gamma_{i,A}(f))^{-1} \circ \pi_i = \pi_i \circ \Lambda$ *on* $\{x \in \mathbf{R}^n : pr_i(x) > 2A + \frac{1}{2}\}$

Proof of Claim 1. To see (1), let $x \in \mathbf{R}^n$ and $N \in \mathbf{N}$ such that $pr_i((T_ig)^{-N}(x)) < -2A$. Then we have

$$\begin{aligned}\Lambda T_i f \Lambda^{-1}(x) &= \Lambda T_i f (T_i f)^N (T_i g)^{-N}(x) \\ &= (T_i g)^{N+1}(T_i f)^{-(N+1)}(T_i f)^{N+1}(T_i g)^{-N}(x) \\ &= T_i g(x).\end{aligned}$$

(2) is an immediate consequence of $\mathrm{supp}(f), \mathrm{supp}(g) \subseteq [-2A, 2A]^n$. In order to proof (3), we have to shrink U', such that

$$pr_i((T_ig)^N(T_if)^{-N}(x)) > 2A \tag{2.16}$$

$\forall x \in \mathbf{R}^n$ with $pr_i(x) > 2A + \frac{1}{2}$, $\forall f, g \in U'$ and $\forall N \in \mathbf{N}$. To achieve this, just require $\mu_0(h) < \frac{1}{2}\frac{1}{16A}$ for all $h \in U'$. Now let $x \in \mathbf{R}^n$ with $pr_i(x) > 2A + \frac{1}{2}$ and choose $N \in \mathbf{N}$ with $pr_i((T_if)^{-N}(x)) < -2A$. We then have $\Gamma_{i,A}(f)^{-1}(\pi_i(x)) = \pi_i((T_if)^{-N}(x))$ and hence we obtain from (2.16) $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}(\pi_i(x)) = \pi_i((T_ig)^N(T_if)^{-N}(x)) = \pi_i(\Lambda(x))$. □

Let $x \in (-2A-1, 2A+1)^n$. Using Claim 1(1)&(2), $\mathrm{supp}(f), \mathrm{supp}(g) \subseteq [-2A, 2A]^n$ and Lemma 2.24 we get

$$\begin{aligned}(\varphi_{i,A}\Lambda\varphi_{i,A}^{-1})\tau_{i,A}f(\varphi_{i,A}\Lambda\varphi_{i,A}^{-1})^{-1}(x) &= \varphi_{i,A}\Lambda\varphi_{i,A}^{-1}\varphi_{i,A}T_i\varphi_{i,A}^{-1}\varphi_{i,A}f\varphi_{i,A}^{-1}\varphi_{i,A}\Lambda^{-1}\varphi_{i,A}^{-1}(x) \\ &= \varphi_{i,A}\Lambda T_i f \Lambda^{-1}\varphi_{i,A}^{-1} = \varphi_{i,A}T_i g\varphi_{i,A}^{-1}(x) \\ &= \varphi_{i,A}T_i\varphi_{i,A}^{-1}\varphi_{i,A}g\varphi_{i,A}^{-1} = \tau_{i,A}g(x).\end{aligned}$$

Suppose $\varphi_{i,A}\Lambda\varphi_{i,A}^{-1}$ could be extended to a map $\lambda \in \mathrm{Diff}_c^{r,[\alpha]}(\mathbf{R}^n)_\circ$. We are finished then, for $\tau_{i,A}|_{\mathbf{R}^n \setminus (-2A-1,2A+1)^n} = Id$, $f|_{\mathbf{R}^n \setminus [-2A,2A]^n} = g|_{\mathbf{R}^n \setminus [-2A,2A]^n} = Id$ and hence $\lambda\tau_{i,A}f\lambda^{-1} = \tau_{i,A}g$ holds on $\mathbf{R}^n$.

By Remark 2.35 and Lemma 2.33(3) $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}$ induces a mapping $\Lambda'_1 \in \mathrm{Diff}^{r,[\alpha]}_{[-2A,2A]^{n-1}}(\mathbf{R}^{n-1})_\circ$, with $\Lambda'_1\circ p_i = p_i\circ\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}$ (cf. Example 1.15 and Example 2.16). Choose a $\mathrm{C}^{r,[\alpha]}$-diffeotopy $\Lambda' : \mathbf{R}^{n-1}\times\mathbf{R}\to\mathbf{R}^{n-1}$, supported on $[-2A,2A]^{n-1}$ with $\Lambda'_t = Id\ \forall t\leq 0$ and $\Lambda'_t=\Lambda'_1\ \forall t\geq 1$. Since $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}$ is S^1-equivariant we can lift it to an $\mathrm{C}^{r,[\alpha]}$-mapping $\widetilde{\Lambda}:\mathbf{R}^n\to\mathbf{R}^n$, i.e. $\pi_i\circ\widetilde{\Lambda} = \Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}\circ\pi_i$. By Claim 1(3) we may assume

$$\widetilde{\Lambda}|_{\mathbf{R}^{i-1}\times(2A+\frac{1}{2},\infty)\times\mathbf{R}^{n-i}} = \Lambda|_{\mathbf{R}^{i-1}\times(2A+\frac{1}{2},\infty)\times\mathbf{R}^{n-i}}. \tag{2.17}$$

Claim 2. *There exists a mapping* $\widetilde{\beta}\in\mathrm{C}^{r,[\alpha]}(\mathbf{R}^{n-1},\mathbf{R})$ *with*

$$pr_i\circ\widetilde{\Lambda} = pr_i+\widetilde{\beta}\circ\widetilde{p}_i = pr_i\circ\mathrm{Fl}_1^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}$$

and $\mathrm{supp}(\widetilde{\beta})\subseteq[-2A,2A]^{n-1}$.

Proof of Claim 2. Consider the $\mathbf{R}$-action on $\mathbf{R}^n$ given by

$$(t,x)\mapsto te_i+x,$$

where e_i denotes the i-th unit vector in $\mathbf{R}^n$. Since $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1})$ is S^1-equivariant we have

$$\begin{aligned}
\pi_i(\widetilde{\Lambda}(te_i+x)) &= \Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}(\pi_i(te_i+x))\\
&= \Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}(\pi(t)\cdot\pi_i(x))\\
&= \pi(t)\cdot\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}(\pi_i(x))\\
&= \pi(t)\cdot\pi_i(\widetilde{\Lambda}(x)) = \pi_i(te_i+\widetilde{\Lambda}(x)),
\end{aligned}$$

where $\pi:\mathbf{R}\to\mathrm{S}^1\cong\mathbf{R}/\mathbf{Z}$. Hence $t\mapsto\widetilde{\Lambda}(te_i+x)$ and $t\mapsto te_i+\widetilde{\Lambda}(x)$ are both lifts of the same path. Since they coincide for $t=0$ we obtain $\widetilde{\Lambda}(te_i+x) = te_i+\widetilde{\Lambda}(x)$ $\forall t\in\mathbf{R},\forall x\in\mathbf{R}^n$ and thus $\widetilde{\Lambda}$ is equivariant with respect to the $\mathbf{R}$-action above. It follows, that $pr_i\widetilde{\Lambda}-pr_i$ is constant on the $\mathbf{R}$-orbits, and hence there exists a map $\widetilde{\beta}:\mathbf{R}^{n-1}\to\mathbf{R}$ with $pr_i\circ\widetilde{\Lambda}-pr_i = \widetilde{\beta}\circ\widetilde{p}_i$. $\widetilde{\beta}$ is $\mathrm{C}^{r,[\alpha]}$ since $\widetilde{p}_i:\mathbf{R}^n\to\mathbf{R}^{n-1}$ admits global cross sections. Since $\mathrm{Fl}_1^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}(x) = x+\widetilde{\beta}(\widetilde{p}_i(x))\partial_i$, we obtain $pr_i\circ\mathrm{Fl}_1^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i} = pr_i+\widetilde{\beta}\circ\widetilde{p}_i$.

To see, that $\mathrm{supp}(\widetilde{\beta})\subseteq[-2A,2A]^{n-1}$, let $x\in\mathbf{R}^{n-1}\setminus[-2A,2A]^{n-1}$ and choose $\widetilde{x}\in\mathbf{R}^n$ with $\widetilde{p}_i(\widetilde{x}) = x$ and $pr_i(\widetilde{x})>2A+\frac{1}{2}$. From (2.17) and Claim 1(2) we get

$$pr_i(\widetilde{x})+\widetilde{\beta}(\widetilde{p}_i(\widetilde{x})) = pr_i(\widetilde{\Lambda}(\widetilde{x})) = pr_i(\Lambda(\widetilde{x})) = pr_i(\widetilde{x})$$

and hence $\widetilde{\beta}(x)=\widetilde{\beta}(\widetilde{p}_i(\widetilde{x}))=0$. □

Using $\widetilde{\beta}$ and Λ' we now define the extension $\lambda:\mathbf{R}^n\to\mathbf{R}^n$ of $\varphi_{i,A}\Lambda\varphi_{i,A}^{-1}$, which will conjugate $\tau_{i,A}f$ and $\tau_{i,A}g$.

Claim 3. *The following mapping* $\lambda:\mathbf{R}^n\to\mathbf{R}^n$ *is well defined and* $\mathrm{C}^{r,[\alpha]}$.

$$\lambda := \begin{cases} \varphi_{i,A}\Lambda\varphi_{i,A}^{-1} & \textit{on } (-2A-1,2A+1)^n\\ ins_i(\Lambda'_{3+2A-pr_i}\circ\widetilde{p}_i, pr_i\circ\mathrm{Fl}^{\rho_{i,A}\partial_i}_{\widetilde{\beta}\circ\widetilde{p}_i}) & \textit{on } \mathbf{R}^{i-1}\times(2A+\frac{1}{2},\infty)\times\mathbf{R}^{n-i}\\ Id & \textit{otherwise}\end{cases}$$

where $ins_i : \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}^n$, $ins_i(x,t) := (x_1, \ldots, x_{i-1}, t, x_i, \ldots, x_{n-1})$. *Moreover* $\mathrm{supp}(\lambda) \subseteq [-2A, 2A]^{i-1} \times [-2A, 2A+3] \times [-2A, 2A]^{n-1}$.

Proof. Clearly λ is $\mathrm{C}^{r,[\alpha]}$, provided it is well defined. So we only have to show

$$\begin{aligned} \widetilde{p}_i \circ \varphi_{i,A} \Lambda \varphi_{i,A}^{-1} &= \Lambda'_{3+2A-pr_i} \circ \widetilde{p}_i \\ pr_i \circ \varphi_{i,A} \Lambda \varphi_{i,A}^{-1} &= pr_i \circ \mathrm{Fl}^{\rho_{i,A}\partial_i}_{\widetilde{\beta}\circ\widetilde{p}_i} \end{aligned}$$

hold on $(-2A, 2A)^{i-1} \times (2A + \frac{1}{2}, 2A+1) \times (-2A, 2A)^{n-i}$, the rest is obvious from $\mathrm{supp}(\widetilde{\beta}) \subseteq [-2A, 2A]^{n-1}$, $\mathrm{supp}(\rho_A) \subseteq [-2A-1, 2A+1]^n$, $\mathrm{supp}(\Lambda') \subseteq [-2A, 2A]^{n-1}$, Claim 1(2) and Lemma 2.24.

From Lemma 2.24(1) we get $\widetilde{p}_i \varphi_{i,A} = \widetilde{p}_i \varphi_{i,A}^{-1} = \widetilde{p}_i$, and hence we have

$$\begin{aligned} \widetilde{p}_i \circ \varphi_{i,A} \Lambda \varphi_{i,A}^{-1} &= \widetilde{p}_i \circ \Lambda \varphi_{i,A}^{-1} = \widetilde{p}_i \circ \widetilde{\Lambda} \varphi_{i,A}^{-1} \\ &= \Lambda'_1 \circ \widetilde{p}_i \circ \varphi_{i,A}^{-1} = \Lambda'_1 \circ \widetilde{p}_i = \Lambda'_{3+2A-pr_i} \circ \widetilde{p}_i, \end{aligned}$$

on $(-2A, 2A)^{i-1} \times (2A + \frac{1}{2}, 2A+1) \times (-2A, 2A)^{n-i}$, where we used Lemma 2.24(6) and (2.17) for the second equality.

Since $\widetilde{\beta} \circ \widetilde{p}_i$ is constant on the integral curves of the vector field $\rho_{i,A}\partial_i$, we have

$$\mathrm{Fl}^{\rho_{i,A}\partial_i}_{(\widetilde{\beta}\circ\widetilde{p}_i)(x)t}(x) = \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\rho_{i,A}\partial_i}_t(x) \qquad \forall x \in \mathbf{R}^n \forall t \in \mathbf{R}$$

and thus

$$\begin{aligned} \mathrm{Fl}^{\rho_{i,A}\partial_i}_{\widetilde{\beta}\circ\widetilde{p}_i(x)}(x) &= \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\rho_{i,A}\partial_i}_1(x) = \mathrm{Fl}^{(\varphi_{i,A}^{-1})^*(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}_1(x) \\ &= (\varphi_{i,A}^{-1})^* \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}_1(x) = \varphi_{i,A} \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}_1 \varphi_{i,A}^{-1}(x), \end{aligned}$$

where we used Lemma 2.24(4) to get

$$(\varphi_{i,A}^{-1})^*(\widetilde{\beta} \circ \widetilde{p}_i)\partial_i = (\widetilde{\beta} \circ \widetilde{p}_i \circ \varphi_{i,A}^{-1})(\varphi_{i,A}^{-1})^*\partial_i = (\widetilde{\beta} \circ \widetilde{p}_i)\rho_{i,A}\partial_i$$

From Lemma 2.24(6)&(7) and Claim 2 we obtain

$$\begin{aligned} pr_i \circ \varphi_{i,A} \Lambda \varphi_{i,A}^{-1} &= pr_i \circ \varphi_{i,A} \widetilde{\Lambda} \varphi_{i,A}^{-1} = pr_i \circ \varphi_{i,A} \circ \widetilde{pr}_i \circ \widetilde{\Lambda} \varphi_{i,A}^{-1} \\ &= pr_i \circ \varphi_{i,A} \circ \widetilde{pr}_i \circ \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}_1 \varphi_{i,A}^{-1} = pr_i \circ \varphi_{i,A} \mathrm{Fl}^{(\widetilde{\beta}\circ\widetilde{p}_i)\partial_i}_1 \varphi_{i,A}^{-1} \\ &= pr_i \mathrm{Fl}^{\rho_{i,A}\partial_i}_{\widetilde{\beta}\circ\widetilde{p}_i} \end{aligned}$$

holds on $(-2A, 2A)^{i-1} \times (2A + \frac{1}{2}, 2A+1) \times (-2A, 2A)^{n-i}$. □

Clearly $\lambda|_{\{x \in \mathbf{R}^n : pr_i(x) < 2A+1\}} \in \mathrm{Diff}^{r,[\alpha]}(\{x \in \mathbf{R}^n : pr_i(x) < 2A+1\})$. Further, since $\lambda|_{\{x \in \mathbf{R}^n : x_i \geq 2A+1\}} = ins_i(\Lambda'_{3+2A-pr_i} \circ \widetilde{p}_i, pr_i)$, λ is bijective. We have $\det(D\lambda(x)) = \det(D\Lambda'_{3+2A-x_i}(\widetilde{p}_i(x))) \neq 0$ if $pr_i(x) \geq 2A+1$. Thus we obtain $\lambda \in \mathrm{Diff}_c^{r,[\alpha]}(\mathbf{R}^n)_\circ$ (cf. Example 1.15 and Example 2.16), and the proof of Lemma 2.36 is finished. □

2.37. Remark. We used the condition $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1} \in G_i^{r,[\alpha]}$ only to extend $\varphi_{i,A}\Lambda\varphi_{i,A}^{-1}$. Now suppose 'conversely' $\varphi_{i,A}\Lambda\varphi_{i,A}^{-1}$ extends to a continuous mapping λ defined on $[-2A-1, 2A+1]^n$. Then we will show

$$\widetilde{p}_i\widetilde{\Lambda}(x') = \widetilde{p}_i\widetilde{\Lambda}(x) \qquad \forall x, x' \in \mathbf{R}^n : \widetilde{p}_i(x) = \widetilde{p}_i(x'), \tag{2.18}$$

and hence $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1}$ maps S^1-orbits to S^1-orbits. Recall that we defined $\widetilde{\Lambda}$ without using $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1} \in G_i^{r,[\alpha]}$. Since $\widetilde{\Lambda}$ is a lift of $\Gamma_{i,A}(g)\Gamma_{i,A}(f)^{-1} \in \mathrm{Diff}_c^{r,[\alpha]}(C_i)$ we have

$$\widetilde{\Lambda}(ne_i + x) = ne_i + \widetilde{\Lambda}(x) \qquad \forall x \in \mathbf{R}^n \quad \forall n \in \mathbf{Z}.$$

If $x, x' \in \mathrm{dom}(\varphi_{i,A})$ we thus obtain

$$\begin{aligned}
\widetilde{p}_i\widetilde{\Lambda}(x) &= \lim_{n\to\infty} \widetilde{p}_i\varphi_{i,A}(ne_i + \widetilde{\Lambda}(x)) = \lim_{n\to\infty} \widetilde{p}_i\varphi_{i,A}\widetilde{\Lambda}(ne_i + x) \\
&= \lim_{n\to\infty} \widetilde{p}_i \underbrace{\varphi_{i,A}\Lambda\varphi_{i,A}^{-1}}_{\lambda} \varphi_{i,A}(ne_i + x) = \widetilde{p}_i\lambda \lim_{n\to\infty} \varphi_{i,A}(ne_i + x) = \\
&= \widetilde{p}_i\lambda \lim_{n\to\infty} \varphi_{i,A}(ne_i + x') = \ldots = \widetilde{p}_i\widetilde{\Lambda}(x'),
\end{aligned}$$

since λ is continuous on $[-2A-1, 2A+1]^n$. Elsewhere (2.18) is obvious.

2.38. Lemma (Mather). *[7] [4] Let $A \ge 1$ and let α be a modulus of continuity. Then there exists a neighborhood U_A^n of $Id \in \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ$ and mappings*

$$\Psi_{i,A} : U_A^n \longrightarrow \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ, \qquad 1 \le i \le n$$

with the following properties:

(1) $\Psi_{i,A}(Id) = Id$

(2)

$$\Psi_{i,A} : U_A^n \cap \mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i-1}\times[-2A,2A]^{n-(i-1)}}(\mathbf{R}^n) \to \mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$$

is continuous with respect to C^r-topologies.

(3) $f \in U_A^n \cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n) \Longrightarrow [f] = [\Psi_{i,A}(f)] \in H_1(\mathrm{Diff}_c^{r,[\alpha]}(\mathbf{R}^n)_\circ)$

(4) There exists $\delta > 0$ depending on n, r, α, A and $C > 1$ depending on n, r,α but independent of A with

$$\mu_{r,\alpha}(\Psi_{i,A}(f)) \le CA\mu_{r,\alpha}(f)$$

$\forall f \in U_A^n \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ *with* $\mu_{r,\alpha}(f) \le \delta$.

(5) If $r \ge 2$ then there exists a constant $K > 1$, independent of r, A, and an admissible polynomial of one variable F, depending on n, r, α and A, such that

$$\mu_r(\Psi_{i,A}(f)) \le K^r A\mu_r(f) + F(M_{r-1}(f))$$

$\forall f \in \mathrm{Diff}^r(\mathbf{R}^n) \cap U_A^n$.

Proof. First we assume $U^n_A\subseteq U$ (cf. Lemma 2.33), and thus $\Gamma_{i,A}(f)$ is defined for all $f\in U^n_A$. Given $f\in U^n_A$ we define $g\in \mathrm{C}^1(C_i,C_i)$ by

$$g(x_1,\dots,\vartheta,\dots,x_n):=\Gamma_{i,A}(f)(x_1,\dots,0,\dots,x_n)+(0,\dots,\vartheta,\dots,0). \tag{2.19}$$

Using Proposition 1.7(3) and Lemma 2.33 one sees that g depends continuously on f, and thus we can shrink U^n_A, such that $g\in\mathrm{Diff}^1_{[-2A,2A]^{i-1}\times \mathrm{S}^1\times[-2A,2A]^{n-i}}(C_i)_\circ$. Notice, that $g\in G^1_i$ and $g|_{\{\theta\in C_i:pr_i(\theta)=0\}}=\Gamma_{i,A}(f)|_{\{\theta\in C_i:pr_i(\theta)=0\}}$ (in fact g is unique with these properties). Further let $h:=g^{-1}\Gamma_{i,A}(f)\in\mathrm{Diff}^1_{[-2A,2A]^{i-1}\times \mathrm{S}^1\times[-2A,2A]^{n-i}}(C_i)_\circ$. If $\widetilde{pr}_i(x_1,\dots,\vartheta,\dots,x_n)=(x_1,\dots,0,\dots,x_n)$ we have $D^2(\widetilde{pr}_i)=0$ and $g-Id=(\Gamma_{i,A}(f)-Id)\circ\widetilde{pr}_i$. From (2.3) we get

$$D^r(g-Id)=\Big(D^r(\Gamma_{i,A}(f)-Id)\circ\widetilde{pr}_i\Big)\Big(D\widetilde{pr}_i\Big)^r\qquad\forall r\geq1,$$

thus

$$\mu_r(g)\leq\|\Gamma_{i,A}(f)-Id\|_r\underbrace{\|\widetilde{pr}_i\|_1^r}_{=1}=\mu_r(\Gamma_{i,A}(f)),$$

and

$$\begin{aligned}\mu_{r,\alpha}(g)&=\sup_{x\neq y\in C_i}\frac{\|\Big(\Big(D^r(\Gamma_{i,A}(f)-Id)\circ\widetilde{pr}_i\Big)D\widetilde{pr}_i\Big)\Big|_y^x\|}{\alpha(\mathrm{dist}(x,y))}\\&\leq\sup_{x\neq y\in C_i}\frac{\|\Big(D^r(\Gamma_{i,A}(f)-Id)\circ\widetilde{pr}_i\Big)\Big|_y^x\|\|\widetilde{pr}_i\|_1}{\alpha(\mathrm{dist}(x,y))}\\&\leq\sup_{x\neq y\in C_i}\frac{\|\Big(D^r(\Gamma_{i,A}(f)-Id)\circ\widetilde{pr}_i\Big)\Big|_y^x\|}{\alpha(\mathrm{dist}(x,y))}\leq\mu_{r,\alpha}(\Gamma_{i,A}(f)).\end{aligned}$$

Choosing $\delta_1>0$ sufficiently small (cf. Lemma 2.33(4) and Lemma 2.30), we get

$$\begin{aligned}\mu_{r,\alpha}(h)&=\mu_{r,\alpha}(g^{-1}\Gamma_{i,A}(f))\\&\leq\mu_{r,\alpha}(g^{-1})+\mu_{r,\alpha}(\Gamma_{i,A}(f))+\underbrace{C_1\mu_{r,\alpha}(\Gamma_{i,A}(f))}_{\leq1}\mu_{r,\alpha}(g^{-1})\\&\leq2\mu_{r,\alpha}(g)+\mu_{r,\alpha}(\Gamma_{i,A}(f))+2\mu_{r,\alpha}(g)\\&\leq5\mu_{r,\alpha}(\Gamma_{i,A}(f))\end{aligned}$$

$\forall f\in U^n_A\cap\mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f)\leq\delta_1$. If we shrink U^n_A such that $\mu_1(\Gamma_{i,A}(f))\leq\frac12$ and $\mu_1(f)\leq\frac1A$ for all $f\in U^n_A$, we can use (2.14), Lemma 2.29(1)&(3) and $\mu_r(g)\leq\mu_r(\Gamma_{i,A}(f))$ to estimate

$$\begin{aligned}\mu_1(h)&=\mu_1(g^{-1}\Gamma_{i,A}(f))\\&\leq2\Big(\underbrace{\mu_1(g^{-1})}_{\leq2\mu_1(\Gamma_{i,A}(f))}+\mu_1(\Gamma_{i,A}(f))\Big)\Big(1+\underbrace{\mu_1(g^{-1})}_{\leq2\mu_1(\Gamma_{i,A}(f))}+\mu_1(\Gamma_{i,A}(f))\Big)\\&\leq6\mu_1(\Gamma_{i,A}(f))\Big(1+3\underbrace{\mu_1(\Gamma_{i,A}(f))}_{\leq\frac12}\Big)\\&\leq15\cdot18A\mu_1(f)\Big(1+\underbrace{\mu_1(f)}_{\leq\frac1A}\Big)^{18A}\leq K_1A\mu_1(f),\end{aligned}$$

where K_1 is a constant, independent of A, since $(1+\frac{1}{A})^A$ is bounded on $(0,\infty)$. From Lemma 2.29(4) we obtain an admissible polynomial of one variable F_1 with

$$\begin{aligned}\mu_r(g^{-1}) &\le \mu_r(g)\Big(1+2\mu_1(g)\Big)^{r+1} + F_1(M_{r-1}(g))\\ &\le 2^{r+1}\mu_r(\Gamma_{i,A}(f)) + F_1(M_{r-1}(\Gamma_{i,A}(f))).\end{aligned} \tag{2.20}$$

Thus there exists an admissible polynomial F_2, depending on r, such that

$$M_r(g^{-1}) \le 2^{r+1}M_r(\Gamma_{i,A}(f)) + F_2(M_{r-1}(\Gamma_{i,A}(f))) \tag{2.21}$$

(cf. Lemma 2.29(3)). Using (2.20), (2.21) and Lemma 2.29(2) we obtain

$$\begin{aligned}\mu_r(h) &= \mu_r(g^{-1}\Gamma_{i,A}(f))\\ &\le 2\Big(\mu_r(g^{-1}) + \mu_r(\Gamma_{i,A}(f))\Big)\Big(1+\underbrace{\mu_1(g^{-1})}_{\le 1}+\underbrace{\mu_1(\Gamma_{i,A}(f))}_{\le \frac{1}{2}}\Big)^r\\ &\quad +F_3(M_{r-1}(g^{-1}) + M_{r-1}(\Gamma_{i,A}(f)))\\ &\le K_4^r\mu_r(\Gamma_{i,A}(f)) + F_4(M_{r-1}(\Gamma_{i,A}(f)))\end{aligned}$$

Together with (2.13), this yields an admissible polynomial F_5 of one variable and a constant K_5, independent of r and A, such that

$$\mu_r(h) \le K_5^r A\mu_r(f) + F_5(M_{r-1}(f)).$$

Here we used again, that $(1+\mu_1(f))^A \le (1+\frac{1}{A})^A$ is bounded on $(0,\infty)$. Summing up we have

(1) $f \in U^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \Longrightarrow g \in G^{r,\alpha}_i$

(2) $h|_{\{\theta\in C_i : pr_i(\theta)=0\}} = Id$

(3) $f = Id \Longrightarrow h = Id$

(4) $f \in U^n_A \cap \mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i-1}\times[-2A,2A]^{n-(i-1)}}(\mathbf{R}^n) \Longrightarrow h \in \mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i-1}\times \mathrm{S}^1\times[-2A,2A]^{n-i}}(C_i)_\circ$ and h depends continuously on f with respect to C^r-topologies.

(5) There exists $\delta_1 > 0$ depending on n, r, α and A, such that

$$\mu_{r,\alpha}(h) \le 5\mu_{r,\alpha}(\Gamma_{i,A}(f)) \tag{2.22}$$

$\forall f \in \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)\cap U^n_A$ with $\mu_{r,\alpha}(f) \le \delta_1$

(6) There exists a constant K_5, independent of r and A, and an admissible polynomial of one variable F_5, such that

$$\mu_1(h) \le K_5 A\mu_1(f) \tag{2.23}$$

and

$$\mu_r(h) \le K_5^r A\mu_r(f) + F_5(M_{r-1}(f)). \tag{2.24}$$

Using (2) above we can lift h to a mapping $\widetilde{h} \in \mathrm{Diff}^1_{[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$ with

$$h \circ \pi_i = \pi_i \circ \widetilde{h} \quad \text{and} \quad \widetilde{h}|_{\{x\in\mathbf{R}^n : pr_i(x)\in\mathbf{Z}\}} = Id.$$

Notice that $\widetilde{h}$ depends continuously on h, $\mu_r(\widetilde{h}) = \mu_r(h)$ and $\mu_{r,\alpha}(\widetilde{h}) = \mu_{r,\alpha}(h)$. Now choose a periodic function $\xi \in \mathrm{C}^\infty(\mathbf{R},[0,1])$ of period 1 with

$$\xi|_{[m-\frac{1}{6},m+\frac{1}{6}]} \equiv 0, \quad \xi|_{[m+\frac{2}{6},m+\frac{4}{6}]} \equiv 1 \qquad \forall m \in \mathbf{Z}$$

and let

$$\widetilde{h}_- := (\xi \circ pr_i)(\widetilde{h} - Id) + Id, \quad \widetilde{h}_+ := \widetilde{h}\widetilde{h}_-^{-1}.$$

Thus $\widetilde{h}_-|_{\mathbf{R}^{i-1}\times[m-\frac{1}{6},m+\frac{1}{6}]\times\mathbf{R}^{n-i}} = Id$ and $\widetilde{h}_+|_{\widetilde{h}(\mathbf{R}^{i-1}\times[m+\frac{2}{6},m+\frac{4}{6}]\times\mathbf{R}^{n-i})} = Id$, $\forall m \in \mathbf{Z}$. If we choose U^n_A sufficiently small then $\widetilde{h}(\mathbf{R}^{i-1}\times[m+\frac{5}{12},m+\frac{7}{12}]\times\mathbf{R}^{n-i}) \subseteq \mathbf{R}^{i-1}\times[m+\frac{2}{6},m+\frac{4}{6}]\times\mathbf{R}^{n-i}$ and $\widetilde{h}_+|_{\mathbf{R}^{i-1}\times[m+\frac{5}{12},m+\frac{7}{12}]\times\mathbf{R}^{n-i}} = Id$, $\forall m \in \mathbf{Z}$. By shrinking U^n_A we may further assume $\widetilde{h}_\pm \in \mathrm{Diff}^1_{[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$. In order to estimate $\mu_{r,\alpha}(\widetilde{h}_\pm)$ in terms of $\mu_{r,\alpha}(\widetilde{h})$ we start with

$$\begin{aligned}
\mu_{r,\alpha}(\widetilde{h}_-) &= \sup_{x\neq y\in\mathbf{R}^n} \frac{\|\Big(D^r((\xi\circ pr_i)(\widetilde{h}-Id))\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
&\le \sup_{x\neq y\in\mathbf{R}^n} \sum_{k=0}^r \binom{r}{k} \frac{\|\Big(D^k(\xi\circ pr_i)D^{r-k}(\widetilde{h}-Id)\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
&\le \sum_{k=0}^r \binom{r}{k} \Bigg\{ \sup_{x\neq y\in\mathbf{R}^n} \frac{\|D^k(\xi\circ pr_i)\Big|_y^x\|\|D^{r-k}(\widetilde{h}-Id)(x)\|}{\alpha(\|x-y\|)} \\
&\qquad + \sup_{x\neq y\in\mathbf{R}^n} \frac{\|D^k(\xi\circ pr_i)(y)\|\|D^{r-k}(\widetilde{h}-Id)\Big|_y^x\|}{\alpha(\|x-y\|)} \Bigg\} \\
&\le \sum_{k=0}^r \binom{r}{k} \Big\{ \|\xi\circ pr_i\|_{k,\alpha}\mu_{r-k}(\widetilde{h}) + \|\xi\circ pr_i\|_k \mu_{r-k,\alpha}(\widetilde{h}) \Big\} \\
&\le C_2\mu_{r,\alpha}(\widetilde{h})
\end{aligned}$$

where C_2 is a constant independent of A. If $\mu_{r,\alpha}(\widetilde{h})$ is small we thus have

$$\mu_{r,\alpha}(\widetilde{h}_-^{-1}) \;\le\; 2\mu_{r,\alpha}(\widetilde{h}_-)$$

(cf. Lemma 2.30), and hence there exists a constant $C_3 > 0$ independent of A, such that

$$\begin{aligned}
\mu_{r,\alpha}(\widetilde{h}_+) &= \mu_{r,\alpha}(\widetilde{h}\widetilde{h}_-^{-1}) \\
&\le \mu_{r,\alpha}(\widetilde{h}) + \mu_{r,\alpha}(\widetilde{h}_-^{-1}) + \underbrace{C_4\mu_{r,\alpha}(\widetilde{h})}_{\le 1}\mu_{r,\alpha}(\widetilde{h}_-^{-1}) \\
&\le (1+2C_2)\mu_{r,\alpha}(\widetilde{h}) \le (1+2C_2)\mu_{r,\alpha}(h) \le C_3\mu_{r,\alpha}(\Gamma_{i,A}(f))
\end{aligned}$$

provided $\mu_{r,\alpha}(f)$ is small. For the last inequality we used (2.22). Further we have

$$\begin{aligned}
\mu_r(\widetilde{h}_-) &= \|(\xi\circ pr_i)(\widetilde{h}-Id)\|_r \\
&\le \sum_{k=0}^{r}\binom{r}{k}\sup_{x\in\mathbf{R}^n}\|D^k(\xi\circ pr_i)(x)\|\|D^{r-k}(\widetilde{h}-Id)(x)\| \\
&\le \sum_{k=0}^{r}\binom{r}{k}\|\xi\circ pr_i\|_k\mu_{r-k}(\widetilde{h}) \\
&= \mu_r(\widetilde{h})+\sum_{k=1}^{r-1}\binom{r}{k}\|\xi\circ pr_i)\|_k\mu_{r-k}(\widetilde{h})+\|\xi\circ pr_i)\|_r\underbrace{\mu_0(\widetilde{h})}_{\le\mu_1(\widetilde{h})} \\
&\le \mu_r(h)+\widetilde{K}M_{r-1}(h),
\end{aligned}$$

where $\widetilde{K}$ is a constant depending on r. Here we used $\mu_r(\widetilde{h})=\mu_r(h)$ if $r\ge 1$ and $\mu_0(\widetilde{h})\le\mu_1(\widetilde{h})$, since $\widetilde{h}-Id$ is periodic with period 1, and $\widetilde{h}-Id|_{\mathbf{R}^{i-1}\times\mathbf{Z}\times\mathbf{R}^{n-i}}=0$. Together with (2.23) and (2.24) and Lemma 2.29(4) this yields

$$\mu_r(\widetilde{h}_\pm)\le K_6^rA\mu_r(f)+F_6(M_{r-1}(f))$$

where F_6 is an admissible polynomial of one variable, and K_6 is a constant independent of r and A. Summing up we have

(1) $f=Id\Longrightarrow\widetilde{h}_\pm=Id$

(2) $\widetilde{h}=\widetilde{h}_+\widetilde{h}_-$

(3) If $m\in\mathbf{Z}$ then

$$\widetilde{h}_-|_{\mathbf{R}^{i-1}\times[m-\frac{1}{6},m+\frac{1}{6}]\times\mathbf{R}^{n-i}}=Id,\qquad \widetilde{h}_+|_{\mathbf{R}^{i-1}\times[m+\frac{5}{12},m+\frac{7}{12}]\times\mathbf{R}^{n-i}}=Id$$

(4) $f\in U_A^n\cap\mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i-1}\times[-2A,2A]^{n-(i-1)}}(\mathbf{R}^n)\Rightarrow\widetilde{h}_\pm\in\mathrm{Diff}^{r,[\alpha]}_{[-2,2]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$ and $\widetilde{h}_\pm$ depend continuously on f with respect to C^r-topologies.

(5) There exist $\delta_3>0$ depending on n, r, α, A and $C_3>0$ depending on n, r, α but independent of A, such that

$$\mu_{r,\alpha}(\widetilde{h}_\pm)\le C_3\mu_{r,\alpha}(\Gamma_{i,A}(f)) \tag{2.25}$$

$\forall f\in U_A^n\cap\mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f)\le\delta_3$.

(6) There exists a constant K_6 independent of r and A and an admissible polynomial of one variable F_6 such that

$$\mu_r(\widetilde{h}_\pm)\le K_6^rA\mu_r(f)+F_6(M_{r-1}(f)) \tag{2.26}$$

We let

$$E_- := \{x \in \mathbf{R}^n : -1 \le pr_i(x) \le 0\}, \qquad E_+ := \{x \in \mathbf{R}^n : \frac{1}{2} \le pr_i(x) \le \frac{3}{2}\}$$

and define $\Psi_{i,A}(f)$ by

$$\Psi_{i,A}(f)|_{E_+} = \widetilde{h}_+|_{E_+}, \qquad \Psi_{i,A}(f)|_{E_-} = \widetilde{h}_-|_{E_-}, \qquad \Psi_{i,A}|_{\mathbf{R}^n\setminus(E_-\cup E_+)} = Id.$$

(1) and (2) of Lemma 2.38 follow immediately. To see (3) notice that there exist $h_\pm \in \mathrm{Diff}^1(C_i)$ with $h_\pm \circ \pi_i = \pi_i \circ \widetilde{h}_\pm$. Since $\widetilde{h_+h_-} = \widetilde{h}_+\widetilde{h}_- = \widetilde{h}$ we get $h_+h_- = h$, hence $\Gamma_{i,A}(\Psi_{i,A}(f)) = h_+h_- = h$ and thus

$$\Gamma_{i,A}(f)\Gamma_{i,A}(\Psi_{i,A}(f))^{-1} = \Gamma_{i,A}(f)h^{-1} = g \in G_i^{r,[\alpha]}$$

$\forall f \in U_A^n \cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n)$. By shrinking U_A^n we may assume $f, \Psi_{i,A}(f) \in U'$ (cf. Lemma 2.36), and we can apply Lemma 2.36. Consequently $\tau_{i,A}f$ and $\tau_{i,A}\Psi_{i,A}(f)$ are conjugated in $\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ$ and thus $[f] = [\Psi_{i,A}(f)] \in H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ)$. (2.25), Lemma 2.33(4) and the obvious fact

$$\mu_{r,\alpha}(\Psi_{i,A}(f)) \le \mu_{r,\alpha}(\widetilde{h}_-) + \mu_{r,\alpha}(\widetilde{h}_+)$$

together yield (4) of Lemma 2.38. Similarly we obtain (5) of Lemma 2.38 from

$$\mu_r(\Psi_{i,A}(f)) = \max\{\mu_r(h_-), \mu_r(h_+)\}$$

and (2.26). □

For the following construction we fix $g_A \in \mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ$ with $g_A|_{[-2,2]^n} = A \cdot Id$.

2.39. Corollary. *For $A \ge 1$ there exists a neighborhood V_A^n of $Id \in \mathrm{Diff}^1_{[-2,2]^n}(\mathbf{R}^n)_\circ$ such that*

$$\begin{aligned} \vartheta_{f,A} : V_A^n &\to \mathrm{Diff}^1_{[-2,2]^n}(\mathbf{R}^n)_\circ \\ h &\mapsto \psi_{n,A}\dots\psi_{1,A}(g_A f h g_A^{-1}) \end{aligned}$$

is well defined $\forall f \in V_A^n$, and has the following properties:

(1) If α is a modulus of continuity, $1 \le r \le \infty$ and $f \in V_A^n \cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n)$ then

$$\vartheta_{f,A}|_{V_A^n\cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n)} : V_A^n \cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n) \to \mathrm{Diff}^{r,[\alpha]}_{[-2,2]^n}(\mathbf{R}^n)_\circ$$

is continuous with respect to C^r-topologies.

(2) $f, h \in V_A^n \cap \mathrm{Diff}^{r,[\alpha]}(\mathbf{R}^n) \Longrightarrow [\vartheta_{f,A}(h)] = [fh] \in H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ)$

Proof. Since $\Psi_{i,A}$ are continuous (cf. Lemma 2.38) and since $g_A f h g_A^{-1}$ depends continuously on (f,h), there exists a neighborhood V_A^n of $Id \in \mathrm{Diff}^1_{[-2,2]^n}(\mathbf{R}^n)_\circ$, such that $\psi_{n,A}\dots\psi_{1,A}(g_A f h g_A^{-1})$ is defined $\forall f, h \in V_A^n$. Since $\mathrm{supp}(fh) \subseteq [-2,2]^n$ we have

$\mathrm{supp}(g_A f h g_A^{-1}) \subseteq [-2A, 2A]^n$ and thus (1) follows immediately from Lemma 2.38(2). From Lemma 2.38(3) we obtain

$$[\vartheta_{f,A}(h)] = [\Psi_{n,A} \cdots \Psi_{1,A}(g_A f h g_A^{-1})] = [g_A f h g_A^{-1}] = [fh] \in H_1(\mathrm{Diff}^{r,[\alpha]}_c(\mathbf{R}^n)_\circ).$$

□

2.40. Lemma. *Let α be a modulus of continuity and suppose $n+2 \leq r < \infty$. Then there exists $A_0 \geq 1$ and $\epsilon_0 > 0$, such that*

$$\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \leq \epsilon \Longrightarrow f, h \in V^n_{A_0}, \quad \mu_{r,\alpha}(\vartheta_{f,A_0}(h)) \leq \epsilon$$

$\forall 0 < \epsilon \leq \epsilon_0$ *and* $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$.

Proof. Since $n+2 \leq r$ we can choose $A_0 \geq 1$ such that $3C^n A_0^{1-r+n} \leq 1$, where C is the constant in Lemma 2.38(4). Using Lemma 2.26(4)&(5) one sees, that there exists $\epsilon_0 > 0$, such that

$$f \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n), \quad \mu_{r,\alpha}(f) \leq \epsilon_0 \Longrightarrow f \in V^n_{A_0}.$$

By Lemma 2.30(1) we can shrink ϵ_0 such that

$$\mu_{r,\alpha}(fh) \leq \mu_{r,\alpha}(f) + \mu_{r,\alpha}(h) + \underbrace{C_1 \mu_{r,\alpha}(f)}_{\leq 1} \mu_{r,\alpha}(h) \leq 3\epsilon,$$

$\forall 0 < \epsilon \leq \epsilon_0$ and $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \leq \epsilon$. In order to estimate $\mu_{r,\alpha}(g_{A_0} f h g_{A_0}^{-1})$, we first calculate

$$\begin{aligned} D^r(g_{A_0} f h g_{A_0}^{-1}) &= D^r\Big((A_0 \cdot Id) f h (\frac{1}{A_0} \cdot Id)\Big) \\ &= D^{r-1}\Big(A_0 \cdot (D(fh) \circ (\frac{1}{A_0} Id)) \cdot \frac{1}{A_0}\Big) \\ &= D^{r-2}\Big((D^2(fh) \circ (\frac{1}{A_0} Id)) \cdot \frac{1}{A_0}\Big) \\ &= \ldots = A_0^{1-r}\Big(D^r(fh) \circ \frac{1}{A_0} Id\Big). \end{aligned} \tag{2.27}$$

Thus we have

$$\begin{aligned} \mu_{r,\alpha}(g_{A_0} f h g_{A_0}^{-1}) &= \sup_{x \neq y \in \mathbf{R}^n} \frac{\|D^r(g_{A_0} f h g_{A_0}^{-1})(x) - D^r(g_{A_0} f h g_{A_0}^{-1})(y)\|}{\alpha(\|x-y\|)} \\ &= A_0^{1-r} \sup_{x \neq y \in \mathbf{R}^n} \frac{\|D^r(fh)(\frac{1}{A_0}x) - D^r(fh)(\frac{1}{A_0}y)\|}{\alpha(A_0 \|\frac{1}{A_0}x - \frac{1}{A_0}y\|)} \\ &\leq A_0^{1-r} \sup_{x \neq y \in \mathbf{R}^n} \frac{\|D^r(fh)(\frac{1}{A_0}x) - D^r(fh)(\frac{1}{A_0}y)\|}{\alpha(\|\frac{1}{A_0}x - \frac{1}{A_0}y\|)} \\ &\leq A_0^{1-r} \mu_{r,\alpha}(fh) \leq 3A_0^{1-r}\epsilon, \end{aligned}$$

$\forall 0 < \epsilon \le \epsilon_0$ and $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \le \epsilon$. By shrinking ϵ_0 once again we may assume $\epsilon_0 \le \delta$, where δ is the constant in Lemma 2.38(4) corresponding to C above. Since $A_0, C \ge 1$ we have $3C^i A_0^{1-r+i}\epsilon_0 \le \delta$ for $0 \le i \le n$, and hence using Lemma 2.38(4) we obtain inductively

$$\begin{aligned}\mu_{r,\alpha}(\vartheta_{f,A_0}(h)) &= \mu_{r,\alpha}(\Psi_{n,A_0}\cdots\Psi_{1,A_0}(g_{A_0} f h g_{A_0}^{-1}))\\ &\le C^n A_0^n \mu_{r,\alpha}(g_{A_0} f h g_{A_0}^{-1})\\ &\le \underbrace{C^n A_0^n 3 A_0^{1-r}}_{\le 1}\epsilon \le \epsilon\end{aligned}$$

$\forall 0 < \epsilon \le \epsilon_0$ and $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \le \epsilon$. □

2.41. Lemma. *Let α be a modulus of continuity, $1 \le r < \infty$ and let $a > 0$. If $\epsilon > 0$ is sufficiently small, then*

$$L := \{h \in \mathrm{Diff}^{r,\alpha}_{[-a,a]^n}(\mathbf{R}^n) : \mu_{r,\alpha}(h) \le \epsilon\}$$

equipped with the C^r-topology has the fixed-point property, i.e. every continuous mapping $L \to L$ has at least one fixed point.

Proof. Recall that $\big(\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n), \|\cdot\|_r\big)$ is a Banach space, and look at the subspace

$$L' := \{h \in \mathrm{C}^{r,\alpha}_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n) : \|h\|_{r,\alpha} \le \epsilon\} \subseteq \big(\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n), \|\cdot\|_r\big).$$

Since the topology induced from $\|\cdot\|_r$ on $\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n)$ is exactly the C^r-topology (cf. Lemma 2.26(4)), $h \mapsto h - Id$ is a homeomorphism from L onto L', provided $\epsilon > 0$ is small (cf. Example 1.15).

Next we show L' is closed in $\big(\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n), \|\cdot\|_r\big)$. If $(f_i)_{i\in\mathbf{N}} \in L'$ is a sequence that converges to $f_\infty \in \big(\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n), \|\cdot\|_r\big)$, then $\lim_{i\to\infty} D^r f_i(x) = D^r f_\infty(x)$, $\forall x \in \mathbf{R}^n$. Since $f_i \in L'$ we have for arbitrary $x \ne y \in \mathbf{R}^n$

$$\frac{\|D^r f_i(x) - D^r f_i(y)\|}{\alpha(\|x-y\|)} \le \|f_i\|_{r,\alpha} \le \epsilon, \qquad \forall i \in \mathbf{N},$$

and thus

$$\frac{\|D^r f_\infty(x) - D^r f_\infty(y)\|}{\alpha(\|x-y\|)} \le \epsilon.$$

So we have $\|f_\infty\|_{r,\alpha} \le \epsilon$, hence $f_\infty \in L'$ and thus L' is closed.

Next consider the linear mapping

$$\begin{array}{ccc}\big(\mathrm{C}^r_{[-a,a]^n}(\mathbf{R}^n, \mathbf{R}^n), \|\cdot\|_r\big) & \stackrel{T}{\longrightarrow} & \big(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^r(\mathbf{R}^n, \mathbf{R}^n)), \|\cdot\|_0\big)\\ h & \mapsto & D^r h\end{array}$$

T is an isometry since

$$\|Th\|_0 = \sup_{x\in\mathbf{R}^n} \|D^r h(x)\| = \|h\|_r.$$

Now look at $L'' := T(L')$. L'' is closed in $\left(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^r(\mathbf{R}^n,\mathbf{R}^n)), \|\cdot\|_0\right)$, for L' is closed and T is an isometry. From Lemma 2.26(5) one obtains immediately that L'' is bounded. Further we have

$$\begin{aligned} \|D^r h(x) - D^r h(y)\| &= \underbrace{\frac{\|D^r h(x) - D^r h(y)\|}{\alpha(\|x-y\|)}}_{\le\epsilon} \alpha(\|x-y\|) \\ &\le \epsilon\alpha(\|x-y\|), \end{aligned}$$

$\forall x,y \in \mathbf{R}^n \forall h \in L''$, and hence L'' is equicontinuous. Since $[-a,a]^n$ is compact, we can apply Ascoli-Arzela's theorem, and so L'' is relative compact. But L'' is closed and hence $L' \cong L''$ is compact.

Since L' is a convex, compact subset of a Banach space we obtain by the fixed-point theorem of Schauder-Tychonoff, that every continuous map $L' \to L'$ has a fixed point. Since $L \cong L'$ the same is true for L. □

Proof of Theorem 2.19 in the case $n+2 \le r < \infty$. By Remark 2.22 we only have to show

$$\mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n) \subseteq [\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ] \subseteq \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n) \tag{2.28}$$

Given n, r and α, we fix A_0 and ϵ_0 by Lemma 2.40, and we choose $0 < \epsilon \le \epsilon_0$ such that

$$L := \{h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n) : \mu_{r,\alpha}(h) \le \epsilon\}$$

has the fixed-point property (cf. Lemma 2.41). By Lemma 2.40 and Corollary 2.39(1) we have $L \subseteq V^n_{A_0}$, and for all $f \in L$

$$\mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n) \supseteq L \xrightarrow{\vartheta_{f,A_0}|_L} L \subseteq \mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$$

is continuous with respect to C^r-topology. Hence for every $f \in L$ there exists $h_f \in L$ with $\vartheta_{f,A_0}(h_f) = h_f$. Together with Corollary 2.39(2) this yields

$$[f][h_f] = [fh_f] = [\vartheta_{f,A}(h_f)] = [h_f] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ),$$

hence $[f] = [Id] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$ for all $f \in L$, and thus we have shown

$$L \subseteq [\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ].$$

$\mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$ equipped with the $\mathrm{C}^{r,\alpha}$-topology is a connected topological group (cf. Corollary 2.31), and L is a neighborhood of Id. So L generates $\mathrm{Diff}^{r,\alpha}_{[-2,2]^n}(\mathbf{R}^n)$, and this yields (2.28). □

2.42. Lemma. *If $A \ge 1$ and $r \ge 2$, then there exists a constant $K > 1$, independent of A and r, and an admissible polynomial F depending on n, r, α and A, such that*

$$\mu_r(\vartheta_{f,A}(h)) \le A^{n+1-r} K^{rn}\Big(\mu_r(f) + \mu_r(h)\Big) + F(M_{r-1}(f) + M_{r-1}(h))$$

$\forall f,h\in \mathrm{Diff}^r(\mathbf{R}^n)\cap V^n_A$, where V^n_A is the C^1-neighborhood of Id defined in Corollary 2.39.

Proof. From (2.27) we obtain

$$\begin{aligned}\mu_r(g_A f h g_A^{-1}) &= \sup_{x\in\mathbf{R}^n} A^{1-r}\|D^r(fh)(\frac{1}{A}x)\| = A^{1-r}\mu_r(fh)\\ &= 2A^{1-r}(\mu_r(f)+\mu_r(h))(1+\mu_1(f)+\mu_1(h))^r\\ &\quad +F_1(M_{r-1}(f)+M_{r-1}(h)),\end{aligned}$$

where F_1 is the admissible polynomial in Lemma 2.29(2). Using Lemma 2.38(5) we obtain inductively

$$\mu_r(\vartheta_{f,A}(h))\le A^nK_1^{rn}\mu_r(g_Afhg_A^{-1})+F_2(M_{r-1}(g_Afhg_A^{-1})),$$

where F_2 is an admissible polynomial of one variable. Recall that $\mu_1(\cdot)$ is bounded on V^n_A. Thus we have

$$\mu_r(\vartheta_{f,A}(h))\le A^{1-r+n}K_2^{rn}\Big(\mu_r(f)+\mu_r(h)\Big)+F_3(M_{r-1}(f)+M_{r-1}(h))$$

$\forall f,h\in\mathrm{Diff}^r(\mathbf{R}^n)\cap V^n_A$, but this is Lemma 2.42. □

2.43. Lemma. *Let $a>0$, $i_0\in\mathbf{N}$ and let $b_{i_0},b_{i_0+1},\dots$ be a sequence of positive numbers. If b_{i_0} is sufficiently small then*

$$L:=\{h\in\mathrm{Diff}^\infty_{[-a,a]^n}(\mathbf{R}^n):\mu_i(h)\le b_i\quad\forall i_0\le i\}$$

equipped with the Whitney C^∞-topology has the fixed-point property, i.e. every continuous mapping $L\to L$ has at least one fixed point.

Proof. Consider

$$L':=\{h\in\mathrm{C}^\infty_{[-a,a]^n}(\mathbf{R}^n,\mathbf{R}^n):\|h\|_i\le b_i\quad\forall i_0\le i\}\subseteq\Big(\mathrm{C}^\infty_{[-a,a]^n}(\mathbf{R}^n,\mathbf{R}^n),(\|\cdot\|_i)_{i\in\mathbf{N}}\Big).$$

Since the topology induced from $(\|\cdot\|_i)_{i\in\mathbf{N}}$ on $\mathrm{C}^\infty_{[-a,a]^n}(\mathbf{R}^n,\mathbf{R}^n)$ is exactly the Whitney-C^∞-topology $h\mapsto h-Id$ is a homeomorphism from L onto L', provided $b_{i_0}>0$ is sufficiently small (cf. Example 1.15 and Lemma 2.26(4)).

L' is closed, since the (semi-)norms $\|\cdot\|_i$ are continuous. For $i\in\mathbf{N}$ look at the linear mapping

$$\begin{array}{ccc}\Big(\mathrm{C}^\infty_{[-a,a]^n}(\mathbf{R}^n,\mathbf{R}^n),(\|\cdot\|_i)_{i\in\mathbf{N}}\Big) & \xrightarrow{S_i} & \Big(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n,L^i(\mathbf{R}^n,\mathbf{R}^n)),\|\cdot\|_0\Big)\\ h & \mapsto & D^ih\end{array}$$

S_i is continuous $\forall i\in\mathbf{N}$, since

$$\|S_ih\|_0=\sup_{x\in\mathbf{R}^n}\|D^ih(x)\|=\|h\|_i. \tag{2.29}$$

Further $S_i(L')$ is bounded. If $i \ge n+2$ this is obvious from (2.29) and the definition of L', if $i < n+2$ one additionally needs the estimation in Lemma 2.26(4). Since we have

$$\|S_i(h)(x) - S_i(h)(y)\| = \|D^i h(x) - D^i h(y)\| \le \|h\|_{i+1}\|x-y\|$$

$S_i(L')$ is equicontinuous $\forall i \in \mathbf{N}$. Hence, by a theorem of Ascoli-Arzela, $S_i(L')$ is relative compact in $\left(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^i(\mathbf{R}^n,\mathbf{R}^n)), \|\cdot\|_0\right)$. Let $(h_i)_{i\in\mathbf{N}}$ be an arbitrary sequence in L'. Since $S_1(L')$ is relative compact there exists a subsequence $(h_i^1)_{i\in\mathbf{N}}$ of $(h_i)_{i\in\mathbf{N}}$, such that $(S_1(h_i^1))_{i\in\mathbf{N}}$ converges in $\left(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^1(\mathbf{R}^n,\mathbf{R}^n)), \|\cdot\|_0\right)$. Now, since $S_2(L')$ is relative compact, we obtain a subsequence $(h_i^2)_{i\in\mathbf{N}}$ of $(h_i^1)_{i\in\mathbf{N}}$, such that $(S_2(h_i^2))_{i\in\mathbf{N}}$ converges in $\left(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^2(\mathbf{R}^n,\mathbf{R}^n)), \|\cdot\|_0\right)$. We continue inductively and obtain subsequences $(h_i^j)_{i\in\mathbf{N}}$ of $(h_i^{j-1})_{i\in\mathbf{N}}$, such that $(S_j(h_i^j))_{i\in\mathbf{N}}$ converges in $\left(\mathrm{C}^0_{[-a,a]^n}(\mathbf{R}^n, L^j(\mathbf{R}^n,\mathbf{R}^n)), \|\cdot\|_0\right)$, $\forall j \in \mathbf{N}$. Now consider the sequence $(h_i^i)_{i\in\mathbf{N}}$. Then $(S_j(h_i^i))_{i\in\mathbf{N}}$ converges for all j, and hence by (2.29) $(h_i^i)_{i\in\mathbf{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_j$ on $\mathrm{C}^\infty(\mathbf{R}^n,\mathbf{R}^n)$, $\forall j \in \mathbf{N}$. Since L' is closed subspace of a Fréchet space, $(h_i^i)_{i\in\mathbf{N}}$ converges in L' and thus L' is compact. Since L' is a convex and compact subset of a Fréchet space we obtain by a theorem of Schauder-Tychonoff that every continuous map $L' \to L'$ has at least one fixed point. Since $L \cong L'$ the same is true for L. □

Proof of Theorem 2.19 in the case $r=\infty$. Recall that the constant K in Lemma 2.42 is independent of A and define $A_0 := \max\{K^n, 4K^{(n+2)n}\}$. If $r \ge n+2$ we thus have

$$A_0^{n+1-r}K^{rn} = \underbrace{A_0^{n+1-(n+2)}K^{n(n+2)}}_{A_0^{-1}K^{n(n+2)}\le\frac14}\underbrace{\left(\frac{K^n}{A_0}\right)^{r-(n+2)}}_{\le 1} \le \frac14.$$

By Lemma 2.26(4) we can choose $\epsilon_{n+2} > 0$, such that $f \in V^n_{A_0}$, $\forall f \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{n+2}(f) \le \epsilon_{n+2}$. Using again the estimation of Lemma 2.26(4) and Lemma 2.42 we obtain

$$\begin{aligned}\mu_{n+2}(\vartheta_{f,A_0}(h)) \;\le\; & \underbrace{A_0^{n+1-r}K^{rn}}_{\le\frac14}\underbrace{(\mu_{n+2}(f)+\mu_{n+2}(h))}_{\le 2\epsilon_{n+2}} \\ & + \underbrace{F(M_{n+1}(f)+M_{n+1}(h))}_{\le\frac{\epsilon_{n+2}}{2}} \le \epsilon_{n+2},\end{aligned}$$

$\forall f, h \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{n+2}(f), \mu_{n+2}(h) \le \epsilon_{n+2}$, provided ϵ_{n+2} is chosen sufficiently small. Here it is essential, that F doesn't have any linear or constant terms. We further assume that ϵ_{n+2} satisfies the condition on b_{i_0} in Lemma 2.43.

Sublemma. *If $f \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ and $\mu_{n+2}(f) \le \epsilon_{n+2}$, then there exist $\epsilon_{f,i} > 0$, $n+3 \le i < \infty$, such that*

$$\mu_i(h) \le \epsilon_{f,i} \quad \forall n+2 \le i < \infty \Longrightarrow \mu_i(\vartheta_{f,A_0}(h)) \le \epsilon_{f,i} \quad \forall n+2 \le i < \infty,$$

where $\epsilon_{f,n+2} := \epsilon_{n+2}$.

Proof of the Sublemma. We will choose $\epsilon_{f,i}$ inductively. Suppose we have chosen $\epsilon_{f,n+2},\dots,\epsilon_{f,i-1}$ with

$$\mu_j(h) \le \epsilon_{f,j} \quad \forall n+2 \le j \le i-1 \Longrightarrow \mu_j(\vartheta_{f,A_0}(h)) \le \epsilon_{f,j} \quad \forall n+2 \le j \le i-1.$$

Then there exists a constant $a_i > 0$ depending on f, i and $\epsilon_{f,n+2},\dots,\epsilon_{f,i-1}$, such that $F(M_{i-1}(h)+M_{i-1}(f)) \le a_i$ for all $h \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_j(h) \le \epsilon_{f,j}$, $n+2 \le j \le i-1$. If we let $\epsilon_{f,i} := \mu_i(f) + 2a_i$, Lemma 2.42 yields

$$\begin{aligned}\mu_i(\vartheta_{f,A_0}(h)) &\le \underbrace{A_0^{n+1-i}K^{in}}_{\le\frac14}\underbrace{(\mu_i(f)+\mu_i(h))}_{\le 2\mu_i(f)+2a_i} + \underbrace{F(M_{i-1}(h)+M_{i-1}(f))}_{\le a_i}\\ &\le \frac12\mu_i(f) + \frac32 a_i \le \epsilon_{f,i}\end{aligned}$$

for all $h \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_j(h) \le \epsilon_{f,j}$, $n+2 \le j \le i$. □

Next consider

$$L_f := \{h \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n) : \mu_i(h) \le \epsilon_{f,i} \quad \forall n+2 \le i < \infty\}.$$

From the Sublemma above and Lemma 2.39(1) we obtain, that ϑ_{f,A_0} restricts to a continuous mapping

$$\mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n) \supseteq L_f \overset{\vartheta_{f,A_0}|_{L_f}}{\longrightarrow} L_f \subseteq \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n),$$

$\forall f \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)$ with $\mu_{n+2}(f) \le \epsilon_{n+2}$. By Lemma 2.43 there exists $h_f \in L_f$ with $\vartheta_{f,A_0}(h_f) = h_f$. From Lemma 2.39(2), we get

$$[f][h_f] = [fh_f] = [\vartheta_{f,A}(h_f)] = [h_f] \in H_1(\mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ),$$

hence $[f] = [Id] \in H_1(\mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ)$ for all $f \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)_\circ$ with $\mu_{n+2}(f) \le \epsilon_{n+2}$, and thus

$$\{f \in \mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n) : \mu_{n+2}(f) \le \epsilon_{n+2}\} \subseteq [\mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ].$$

But the set on the left side generates $\mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n)_\circ$ and consequently

$$\mathrm{Diff}^\infty_{[-2,2]^n}(\mathbf{R}^n) \subseteq [\mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ].$$

By Remark 2.22 this is all we have to show. This completes the proof of Theorem 2.19. □

2.44. Corollary. *Let M be a connected, smooth manifold, $n = \dim(M)$, $n+2 \le r \le \infty$. Then $\mathrm{Diff}^r_c(M)_\circ$ is perfect and hence simple by Corollary 1.17.*

Proof. This is an immediate consequence of Theorem 2.19 and Lemma 2.17. □

3. The Simplicity of $\mathrm{Diff}^r_c(M)_\circ$, $1 \le r \le \dim(M)$

3.1. Lemma (Mather). *[8] Let $A \geq 1$ and let α be a modulus of continuity. Then there exists a neighborhood $\widetilde{U}^n_A$ of $\mathrm{Id} \in \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ$ and mappings*

$$\widetilde{\Psi}_{i,A} : \widetilde{U}^n_A \to \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ \qquad 1 \leq i \leq n$$

with the following properties:

(1) $\widetilde{\Psi}_{i,A}(Id) = Id$

(2)

$$\widetilde{\Psi}_{i,A} : \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n) \to \mathrm{Diff}^{r,\alpha}_{[-2,2]^{i-1} \times [-2A,2A]^{n-(i-1)}}(\mathbf{R}^n)_\circ$$

is continuous with respect to C^r*-topologies.*

(3) $f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \Longrightarrow [f] = [\widetilde{\Psi}_{i,A}(f)] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$

(4) There exists $\delta > 0$ depending on n, r, α, A and $C > 0$ depending on n, r, α, but independent of A, with

$$\mu_{r,\alpha}(\widetilde{\Psi}_{i,A}(f)) \leq \frac{C}{A} \mu_{r,\alpha}(f)$$

$\forall f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n)$ *with* $\mu_{r,\alpha}(f) \leq \delta$.

Proof. As we did in the proof of Lemma 2.38 we choose $\widetilde{U}^n_A$ sufficiently small, such that for $f \in \widetilde{U}^n_A$ we can define $g \in \mathrm{Diff}^1_{[-2A,2A]^{i-1} \times \mathrm{S}^1 \times [-2A,2A]^{n-i}}(C_i)_\circ$ by

$$g(x_1, \ldots, \vartheta, \ldots, x_n) := \Gamma_{i,A}(f)(x_1, \ldots, 0, \ldots, x_n) + (0, \ldots, \vartheta, \ldots, 0).$$

Further we let $h := g^{-1}\Gamma_{i,A}(f) \in \mathrm{Diff}^1_{[-2A,2A]^{i-1} \times \mathrm{S}^1 \times [-2A,2A]^{n-i}}(C_i)_\circ$, and obtain similarly to the proof of Lemma 2.38:

(1) $f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \Longrightarrow g \in G^{r,\alpha}_i$

(2) $h|_{\{\theta \in C_i : pr_i(\theta) = 0\}} = Id$

(3) $f = Id \Longrightarrow h = Id$

(4) $f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n) \Longrightarrow h \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^{i-1} \times \mathrm{S}^1 \times [-2A,2A]^{n-i}}(C_i)_\circ$ and h depends continuously on f with respect to C^r-topologies.

(5) There exists $\delta_1 > 0$ depending on n, r, α and A, such that

$$\mu_{r,\alpha}(h) \leq 5\mu_{r,\alpha}(\Gamma_{i,A}(f)) \tag{3.1}$$

$\forall f \in \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \cap \widetilde{U}^n_A$ with $\mu_{r,\alpha}(f) \leq \delta_1$

Using (2) above we may lift h to a mapping $\widetilde{h} \in \mathrm{Diff}^1_{[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$ with

$$h \circ \pi_i = \pi_i \circ \widetilde{h}, \qquad \widetilde{h}|_{\{x\in\mathbf{R}^n : pr_i(x)\in\mathbf{Z}\}} = Id. \tag{3.2}$$

Let a be the least half-integer with $-2A \le a$, and for $j \in \mathbf{N}$ define

$$\begin{aligned} E^j_- &:= \{x \in \mathbf{R}^n : a + 3j - 3 \le pr_i(x) \le a + 3j - 2\} \\ E^j_+ &:= \{x \in \mathbf{R}^n : a + 3j - \frac{3}{2} \le pr_i(x) \le a + 3j - \frac{1}{2}\}. \end{aligned}$$

Further let B be the greatest integer such that $a + 3B - \frac{1}{2} \le 2A$ and notice that $B \ge \frac{A}{2}$. So we have at least A disjoint stripes $E^1_-, E^1_+, E^2_-, \dots, E^B_-, E^B_+$, all contained in $\mathbf{R}^{i-1} \times [-2A, 2A] \times \mathbf{R}^{n-i}$. Now we define

$$\begin{aligned} \widetilde{h}_1 &:= \frac{1}{B}(\widetilde{h} - Id) + Id \\ \widetilde{h}_2 &:= \frac{1}{B-1}(\widetilde{h}\widetilde{h}_1^{-1} - Id) + Id \\ &\vdots \\ \widetilde{h}_j &:= \frac{1}{B-(j-1)}(\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-1}^{-1} - Id) + Id \\ &\vdots \\ \widetilde{h}_B &:= (\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{B-1}^{-1} - Id) + Id \end{aligned}$$

By shrinking $\widetilde{U}^n_A$ we may arrange $\widetilde{h}_1, \dots, \widetilde{h}_B \in \mathrm{Diff}^1_{[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$. In order to estimate $\mu_{r,\alpha}(\widetilde{h}_j)$ in terms of $\mu_{r,\alpha}(\widetilde{h})$ we need the following Sublemma, since the estimations in Lemma 2.30 do not suffice for this purpose.

Sublemma. *Let* $0 < \lambda < 1$, $1 \le r < \infty$, α *a modulus of continuity and let* $f \in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^{i-1}\times\mathbf{R}\times[-2A,2A]^{n-i}}(\mathbf{R}^n)$. *If* $g := \lambda(f - Id) + Id$ *then there exists* $C' > 0$ *and* $\delta' > 0$ *with*

$$\mu_{r,\alpha}(fg^{-1}) = (1-\lambda)\mu_{r,\alpha}(f) + C'\mu_{r,\alpha}(f)^2.$$

provided $\mu_{r,\alpha}(f) \le \delta'$.

Proof of the Sublemma. First consider the case $r = 1$. We have $Dg = \lambda(Df - Id) + Id$, hence $\mu_{1\alpha}(g) = \lambda\mu_{1,\alpha}(f)$ and

$$(Dg)^{-1}(x) = \sum_{m=0}^{\infty}\Big(-\lambda(Df(x) - Id)\Big)^m$$

provided $\mu_1(f) < \frac{1}{\lambda}$, but this can be arranged by choosing δ' sufficiently small (cf. Lemma 2.26). Hence we have

$$\begin{aligned} &D(fg^{-1})(g(x)) = \\ &\quad = Df(x)(Dg)^{-1}(x) \end{aligned}$$

$$\begin{aligned}
&= \Big(Id + \big(Df(x) - Id\big)\Big) \sum_{m=0}^{\infty} \Big(-\lambda(Df(x) - Id)\Big)^m \\
&= Id + (1-\lambda)\big(Df(x) - Id\big) + (1 - \frac{1}{\lambda}) \sum_{m=2}^{\infty} \Big(-\lambda(Df(x) - Id)\Big)^m
\end{aligned}$$

and thus for sufficiently small $\mu_{1,\alpha}(f)$

$$\begin{aligned}
&\mu_{1,\alpha}(fg^{-1}) \le \\
&\le \sup_{x\ne y\in\mathbf{R}^n} \frac{\alpha(\|x-y\|)}{\alpha(\|g(x)-g(y)\|)} \sup_{x\ne y\in\mathbf{R}^n} \frac{\|D(fg^{-1})(g(x)) - D(fg^{-1})(g(y))\|}{\alpha(\|x-y\|)} \\
&\le \sup_{x\ne y\in\mathbf{R}^n} \frac{\alpha(\|g^{-1}\|_1\|x-y\|)}{\alpha(\|x-y\|)} \Bigg\{ \sup_{x\ne y\in\mathbf{R}^n} \frac{\|(1-\lambda)(Df - Id)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
&\qquad + (1-\frac{1}{\lambda}) \sum_{m=2}^{\infty} \sup_{x\ne y\in\mathbf{R}^n} \frac{\|(\lambda(Df - Id))^m\Big|_y^x\|}{\alpha(\|x-y\|)} \Bigg\} \\
&\le \underbrace{\big(\mu_1(g^{-1}) + 1\big)}_{\le 2C_1'\mu_{1,\alpha}(g)} \Bigg\{ (1-\lambda)\mu_{1,\alpha}(f) + (1-\frac{1}{\lambda}) \sum_{m=2}^{\infty} m\lambda^m \mu_{1,\alpha}(f)(\underbrace{\mu_1(f)}_{\le C_1'\mu_{1,\alpha}(f)})^{m-1} \Bigg\} \\
&\le \big(C_2'\mu_{1,\alpha}(f) + 1\big) \Bigg\{ (1-\lambda)\mu_{1,\alpha}(f) \\
&\qquad + \underbrace{(1-\frac{1}{\lambda}) \sum_{m=0}^{\infty} (m+2)\lambda^{m+2}(C_1')^{m+1}\mu_{1,\alpha}(f)^m}_{\le C_3'} \mu_{1,\alpha}(f)^2 \Bigg\} \\
&\le (1-\lambda)\mu_{1,\alpha}(f) + C'\mu_{1,\alpha}(f)^2
\end{aligned}$$

Now consider the case $r > 1$. Then we have $D^r g = \lambda D^r f$, $\mu_r(g) = \lambda\mu_r(f)$, $\mu_{r,\alpha}(g) = \lambda\mu_{r,\alpha}(f)$ and from (2.3) we get

$$D^r f \circ g^{-1} = (D^r f \circ g^{-1})D(g^{-1})^r + (Df \circ g^{-1})D^r(g^{-1}) + R_{f,g^{-1}} \tag{3.3}$$

Similarly to (2.10) there exists a constant C_5' such that we have

$$\frac{\|R_{f,g^{-1}}(x) - R_{f,g^{-1}}(y)\|}{\alpha(\|x-y\|)} \le C_4'\mu_{r,\alpha}(f)\mu_{r,\alpha}(g^{-1}) \le C_5'\mu_{r,\alpha}(f)^2 \tag{3.4}$$

provided $\mu_{r,\alpha}(f)$ is sufficiently small. Further we have

$$\begin{aligned}
&\frac{\|\Big((D^r f\circ g^{-1})(D(g^{-1}))^r + (Df\circ g^{-1})D^r(g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \le \\
&\le \frac{\|(D^r f \circ g^{-1})\Big|_y^x \Big(D(g^{-1})(x) - Id\Big)^r\| + \|\Big(D^r f\circ g^{-1} - D^r g \circ g^{-1}\Big)\Big|_y^x\|}{\alpha(\|x-y\|)}
\end{aligned}$$

$$\begin{aligned}
&+\frac{\|D^r f(g^{-1}(y))\Big(D(g^{-1})\Big|_y^x\Big)^r\| + \|(Df\circ g^{-1})\Big|_y^x D^r(g^{-1})(x)\|}{\alpha(\|x-y\|)} \\
&+\frac{\|\Big(D^r g\circ g^{-1} + Df(g^{-1}(y))D^r(g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
\le\ & \mu_{r,\alpha}(f)\Big(\mu_1(g^{-1})+1\Big)\mu_1(g^{-1})^r + (1-\lambda)\mu_{r,\alpha}(f)\Big(\mu_1(g^{-1})+1\Big) \\
&+\|f\|_r\mu_{1,\alpha}(g^{-1})(2\|g^{-1}\|)^{r-1} + \mu_{r,\alpha}(f)\Big(\mu_1(g^{-1})+1\Big)\|g^{-1}\|_r \\
&+\frac{\|\Big(D^r g\circ g^{-1} + Df(g^{-1}(y))D^r(g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
\le\ & (1-\lambda)\mu_{r,\alpha}(f) + C_6'\mu_{r,\alpha}(f)^2 + \frac{\|\Big(D^r g\circ g^{-1} + Df(g^{-1}(y))D^r(g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \qquad (3.5)
\end{aligned}$$

for sufficiently small $\mu_{r,\alpha}(f)$. Here we frequently used the estimations in Lemma 2.26. From (2.3) we get

$$D^r(g^{-1}) = -D(g^{-1})(D^r(g)\circ g^{-1})D(g^{-1})^r - D(g^{-1})R_{g,g^{-1}}.$$

Similarly to (3.4) we have

$$\frac{\|R_{g,g^{-1}}(x) - R_{g,g^{-1}}(y)\|}{\alpha(\|x-y\|)} \le C_7'\mu_{r,\alpha}(g)\mu_{r,\alpha}(g^{-1}) \le C_8'\mu_{r,\alpha}(f)^2$$

and

$$\|R_{g,g^{-1}}(x)\| \le \sum_{\substack{2\le i\le r, 1\le j_l \\ j_1+\cdots+j_i=r}} C_{i,j_1,\dots,j_i}\|g\|_i\|g^{-1}\|_{j_1}\cdots\|g^{-1}\|_{j_i} \le C_9'\mu_{r,\alpha}(f)^2,$$

provided $\mu_{r,\alpha}(f)$ is small. Further we have

$$\begin{aligned}
&\frac{\|\Big(D^r g\circ g^{-1} + Df(g^{-1}(y))D^r(g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} = \\
=\ & \frac{\|\Big(D^r g\circ g^{-1} + Df(g^{-1}(y))D(g^{-1})\Big((D^r g\circ g^{-1})D(g^{-1})^r - R_{g,g^{-1}}\Big)\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
\le\ & +\frac{\|\Big(Id - Df(g^{-1}(y))\Big)\Big(D^r g\circ g^{-1}\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
&+\frac{\|Df(g^{-1}(y))\Big((Id - D(g^{-1}))(D^r g\circ g^{-1})\Big)\Big|_y^x\|}{\alpha(\|x-y\|)} \\
&+\frac{\|Df(g^{-1}(y))\Big(D(g^{-1})(D^r g\circ g^{-1})\Big(Id - D(g^{-1})\Big)^r\Big)\Big|_y^x\|}{\alpha(\|x-y\|)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\|Df(g^{-1}(y))D(g^{-1})R_{g,g^{-1}}\Big|_y^x\|}{\alpha(\|x-y\|)} \\
\leq\ & \mu_1(f)\mu_{r,\alpha}(g)\Big(\mu_1(g^{-1})+1\Big) + \|f\|_1\mu_{1,\alpha}(g^{-1})\|g\|_r \\
&+\|f\|_1\mu_{1,\alpha}(g^{-1})\mu_{r,\alpha}(g)\Big(\mu_1(g^{-1})+1\Big) + \|f\|_1\mu_{1,\alpha}(g^{-1})\|g\|_r\mu_1(g^{-1})^r \\
&+\|f\|_1\|g^{-1}\|_1\mu_{r,\alpha}(g)\Big(\mu_1(g^{-1})+1\Big)\mu_1(g^{-1})^r \\
&+\|f\|_1\|g^{-1}\|_1\|g\|_r r\mu_{1,\alpha}(g^{-1})\mu_1(g^{-1})^{r-1} \\
&+\|f\|_1\mu_{1,\alpha}(g^{-1})C_9'\mu_{r,\alpha}(f)^2 + \|f\|_1\|g^{-1}\|_1C_8'\mu_{r,\alpha}(f)^2 \\
\leq\ & C_{10}'\mu_{r,\alpha}(f)^2
\end{aligned}
\tag{3.6}
$$

provided $\mu_{r,\alpha}(f)$ is sufficiently small. (3.3), (3.4), (3.5) and (3.6) together yield

$$
\begin{aligned}
\mu_{r,\alpha}(fg^{-1}) &= \sup_{x\neq y\in\mathbf{R}^n}\frac{\|D^r(fg^{-1})(x)-D^r(fg^{-1})(y)\|}{\alpha(\|x-y\|)} \\
&\leq (1-\lambda)\mu_{r,\alpha}(f) + \underbrace{(C_5'+C_6'+C_{10}')}_{C':=}\mu_{r,\alpha}(f)^2,
\end{aligned}
$$

and this completes the proof of the Sublemma. □

If $1 \leq j \leq B$ we have

$$
\mu_{r,\alpha}(\widetilde{h}_j) = \frac{1}{B-(j-1)}\|\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-1}^{-1} - Id\|_{r,\alpha} = \frac{1}{B-(j-1)}\mu_{r,\alpha}(\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-1}^{-1}).
$$

Using this and the Sublemma above ($f = \widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-2}^{-1}$, $\lambda = \frac{1}{B-(j-2)}$ and $g = \widetilde{h}_{j-1}$) we obtain

$$
\begin{aligned}
\mu_{r,\alpha}(\widetilde{h}_j) &= \frac{1}{B-(j-1)}\mu_{r,\alpha}(\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-1}^{-1}) \\
&\leq \frac{1}{B-(j-1)}\left\{\frac{B-(j-1)}{B-(j-2)}\underbrace{\mu_{r,\alpha}(\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-2}^{-1})}_{=(B-(j-2))\mu_{r,\alpha}(\widetilde{h}_{j-1})} + C'\mu_{r,\alpha}(\widetilde{h}\widetilde{h}_1^{-1}\cdots\widetilde{h}_{j-2}^{-1})^2\right\} \\
&= \mu_{r,\alpha}(\widetilde{h}_{j-1}) + \frac{\Big(B-(j-2)\Big)^2C'}{B-(j-1)}\mu_{r,\alpha}(\widetilde{h}_{j-1})^2
\end{aligned}
$$

for all $2 \leq j \leq B$, and thus inductively

$$
\begin{aligned}
\mu_{r,\alpha}(\widetilde{h}_j) &\leq \mu_{r,\alpha}(\widetilde{h}_1) + C_2\mu_{r,\alpha}(\widetilde{h}_1)^2 \\
&= \frac{1}{B}\Big(\mu_{r,\alpha}(\widetilde{h}) + \underbrace{\frac{C_2}{B}\mu_{r,\alpha}(\widetilde{h})}_{\leq 1}\mu_{r,\alpha}(\widetilde{h})\Big) \\
&= \frac{2}{B}\mu_{r,\alpha}(\widetilde{h}) \leq \frac{4}{A}\mu_{r,\alpha}(h) \leq \frac{20}{A}\mu_{r,\alpha}(\Gamma_{i,A}(f))
\end{aligned}
$$

provided $\mu_{r,\alpha}(f)$ is sufficiently small. For the last inequalities we used $B \geq \frac{A}{2}$ and (3.1). Summing up we have

(1) $f = Id \Longrightarrow \widetilde{h}_j = Id \quad \forall 1 \le j \le B$

(2) $\widetilde{h} = \widetilde{h}_B \cdots \widetilde{h}_1$

(3) $f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n) \Longrightarrow \widetilde{h}_j \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^{i-1}\times \mathbf{R} \times [-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$, and $\widetilde{h}_j$ depend continuously on f with respect to C^r-topologies.

(4) There exists $\delta_2 > 0$ depending on n, r, α, A such that

$$\mu_{r,\alpha}(\widetilde{h}_j) \le \frac{20}{A}\mu_{r,\alpha}(\Gamma_{i,A}(f)) \qquad 1 \le j \le B \tag{3.7}$$

$\forall f \in \widetilde{U}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f) \le \delta_2$.

Now choose a periodic function $\xi \in \mathrm{C}^\infty(\mathbf{R}, [0,1])$ of period 1 with

$$\xi|_{[m-\frac{1}{6}, m+\frac{1}{6}]} \equiv 0, \quad \xi|_{[m+\frac{2}{6}, m+\frac{4}{6}]} \equiv 1 \qquad \forall m \in \mathbf{Z}$$

and let

$$\widetilde{h}_{j,-} := (\xi \circ pr_i)(\widetilde{h}_j - Id) + Id, \quad \widetilde{h}_{j,+} := \widetilde{h}_j \widetilde{h}^{-1}_{j,-} \qquad 1 \le j \le B.$$

If $\widetilde{U}^n_A$ is chosen sufficiently small, then $\widetilde{h}_{j,\pm} \in \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^{i-1}\times \mathbf{R} \times [-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$. In order to estimate $\mu_{r,\alpha}(\widetilde{h}_{j,\pm})$ in terms of $\mu_{r,\alpha}(\widetilde{h}_j)$ we start with

$$\begin{aligned}
&\mu_{r,\alpha}(\widetilde{h}_{j,-}) = \\
&\quad = \sup_{x \ne y \in \mathbf{R}^n} \frac{\|D^r((\xi \circ pr_i)(\widetilde{h}_j - Id))(x) - D^r((\xi \circ pr_i)(\widetilde{h}_j - Id))(y)\|}{\alpha(\|x-y\|)} \\
&\quad \le \sup_{x \ne y \in \mathbf{R}^n} \sum_{k=0}^r \binom{r}{k} \frac{\|D^k(\xi pr_i) D^{r-k}(\widetilde{h}_j - Id)(x) - D^k(\xi pr_i) D^{r-k}(\widetilde{h}_j - Id)(y)\|}{\alpha(\|x-y\|)} \\
&\quad \le \sum_{k=0}^r \binom{r}{k} \Bigg\{ \sup_{x \ne y \in \mathbf{R}^n} \frac{\|D^k(\xi \circ pr_i)(x) - D^k(\xi \circ pr_i)(y)\| \|D^{r-k}(\widetilde{h}_j - Id)(x)\|}{\alpha(\|x-y\|)} \\
&\qquad + \sup_{x \ne y \in \mathbf{R}^n} \frac{\|D^k(\xi \circ pr_i)(y)\| \|D^{r-k}(\widetilde{h}_j - Id)(x) - D^{r-k}(\widetilde{h}_j - Id)(y)\|}{\alpha(\|x-y\|)} \Bigg\} \\
&\quad \le \sum_{k=0}^r \binom{r}{k} \Big\{ \|\xi \circ pr_i\|_{k,\alpha} \mu_{r-k}(\widetilde{h}_j) + \|\xi \circ pr_i\|_k \mu_{r-k,\alpha}(\widetilde{h}_j) \Big\} \\
&\quad \le C_3 \mu_{r,\alpha}(\widetilde{h}_j)
\end{aligned}$$

where C_3 is a constant independent of A. If $\mu_{r,\alpha}(\widetilde{h}_j)$ is small we thus have

$$\mu_{r,\alpha}(\widetilde{h}^{-1}_{j,-}) \;\le\; 2\mu_{r,\alpha}(\widetilde{h}_{j,-})$$

(cf. Lemma 2.30(2)), and hence there exists a constant $C_4 > 0$ independent of A, such that

$$\begin{aligned}
\mu_{r,\alpha}(\widetilde{h}_{j,+}) &= \mu_{r,\alpha}(\widetilde{h}_j \widetilde{h}_{j,-}^{-1}) \\
&\le \mu_{r,\alpha}(\widetilde{h}_j) + \mu_{r,\alpha}(\widetilde{h}_{j,-}^{-1}) + \underbrace{C_5 \mu_{r,\alpha}(\widetilde{h}_j)}_{\le 1} \mu_{r,\alpha}(\widetilde{h}_{j,-}^{-1}) \\
&\le (1 + 2C_3)\mu_{r,\alpha}(\widetilde{h}_j) \le \frac{C_4}{A} \mu_{r,\alpha}(\Gamma_{i,A}(f)),
\end{aligned}$$

provided $\mu_{r,\alpha}(f)$ is small. For the last inequality we used (3.7). Summing up we have

(1) $f = Id \Longrightarrow \widetilde{h}_{j,\pm} = Id \qquad \forall 1 \le j \le B$

(2) $\widetilde{h} = \widetilde{h}_{B,+}\widetilde{h}_{B,-}\widetilde{h}_{B-1,+}\widetilde{h}_{B-1,-} \cdots \widetilde{h}_{1,+}\widetilde{h}_{1,-}$

(3) If $m \in \mathbf{Z}$ and $1 \le j \le B$ then

$$\widetilde{h}_{j,-}|_{\mathbf{R}^{i-1} \times [m-\frac{1}{6}, m+\frac{1}{6}] \times \mathbf{R}^{n-i}} = Id \qquad \widetilde{h}_{j,+}|_{\mathbf{R}^{i-1} \times [m+\frac{5}{12}, m+\frac{7}{12}] \times \mathbf{R}^{n-i}} = Id$$

(4) $f \in \widetilde{U}_A^n \cap \mathrm{Diff}^{r,\alpha}_{[-2,2]^i \times [-2A,2A]^{n-i}}(\mathbf{R}^n) \Longrightarrow \widetilde{h}_{j,\pm} \in \mathrm{Diff}^{r,\alpha}_{[-2,2]^{i-1} \times \mathbf{R} \times [-2A,2A]^{n-i}}(\mathbf{R}^n)_\circ$, and $\widetilde{h}_{j,\pm}$ depend continuously on f with respect to C^r-topologies.

(5) There exists $\delta_4 > 0$ depending on n, r, α, A and $C_4 > 0$ depending on n, r, α but independent of A, such that

$$\mu_{r,\alpha}(\widetilde{h}_{j,\pm}) \le \frac{C_4}{A} \mu_{r,\alpha}(\Gamma_{i,A}(f)) \qquad 1 \le j \le B \tag{3.8}$$

$\forall f \in \widetilde{U}_A^n \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f) \le \delta_4$.

Using (3) above we may define $\widetilde{\Psi}_{i,A}(f)$ by

$$\widetilde{\Psi}_{i,A}(f)|_{E_+^j} = \widetilde{h}_{j,+}|_{E_+^j}, \quad \widetilde{\Psi}_{i,A}(f)|_{E_-^j} = \widetilde{h}_{j,-}|_{E_-^j} \qquad 1 \le j \le B$$
$$\widetilde{\Psi}_{i,A}(f)|_{\mathbf{R}^n \setminus \bigcup_{j=1}^B (E_-^j \cup E_+^j)} = Id$$

From (2) above and and (3.2) we get $\Gamma_{i,A}(\widetilde{\Psi}_{i,a}(f)) = h$ (cf. Lemma 2.33) and consequently

$$\Gamma_{i,A}(\widetilde{\Psi}_{i,A}(f))\Gamma_{i,A}(f)^{-1} = hh^{-1}g = g \in G_i^{r,\alpha} \qquad \forall f \in \widetilde{U}_A^n \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n).$$

By shrinking $\widetilde{U}_A^n$ if necessary we can apply Lemma 2.36, hence $\tau_{i,A}\widetilde{\Psi}_{i,A}(f)$ and $\tau_{i,A}f$ are conjugated in $\mathrm{Diff}_c^{r,\alpha}(\mathbf{R}^n)_\circ$ and thus

$$[\widetilde{\Psi}_{i,A}(f)] = [f] \in H_1(\mathrm{Diff}_c^{r,\alpha}(\mathbf{R}^n)_\circ) \qquad \forall f \in \widetilde{U}_A^n \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n).$$

(4) of Lemma 3.1 follows immediately from (3.8), Lemma 2.33(4) and

$$\mu_{r,\alpha}(\widetilde{\Psi}_{i,A}(f)) \le 2 \max_{1 \le j \le B} \{\mu_{r,\alpha}(\widetilde{h}_{j,+}), \mu_{r,\alpha}(\widetilde{h}_{j,-})\}.$$

The remaining assertions of Lemma 3.1 are now obvious. □

3.2. Corollary. *Let* $A \ge 1$ *and let* $g_A \in \mathrm{Diff}^\infty_c(\mathbf{R}^n)_\circ$ *with* $g_A|_{[-2,2]^n} = A \cdot Id$ *(cf. page 41). Then there exists a neighborhood* $\widetilde{V}^n_A$ *of* $Id \in \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ$ *such that*

$$\begin{array}{rcl} \widetilde{\vartheta}_{f,A} : \widetilde{V}^n_A & \to & \mathrm{Diff}^1_{[-2A,2A]^n}(\mathbf{R}^n)_\circ \\ h & \mapsto & \widetilde{\Psi}_{1,A} \cdots \widetilde{\Psi}_{n,A}(g_A^{-1} f h g_A) \end{array}$$

is well defined $\forall f \in \widetilde{V}^n_A$*, and the following holds:*

(1) If α *is a modulus of continuity,* $1 \le r \le \infty$ *and* $f \in \widetilde{V}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n)$ *then*

$$\widetilde{\vartheta}_{f,A} : \widetilde{V}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \to \mathrm{Diff}^{r,\alpha}_{[-2A,2A]^n}(\mathbf{R}^n)_\circ$$

is continuous with respect to C^r*-topologies.*

(2) $f, h \in \widetilde{V}^n_A \cap \mathrm{Diff}^{r,\alpha}(\mathbf{R}^n) \Rightarrow [\widetilde{\vartheta}_{f,A}(h)] = [fh] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$

Proof. Similarly to the proof of Corollary 2.39 we find $\widetilde{V}^n_A$ such that $\widetilde{\vartheta}_{f,A}(h)$ is well defined for all $f, h \in \widetilde{V}^n_A$. If $f, h \in \widetilde{V}^n_A$ then $\mathrm{supp}(fh) \subseteq [-2A, 2A]^n$ and thus $\mathrm{supp}(g_A^{-1} f h g_A) \subseteq [-2, 2]^n$. Now (1) follows immediately from Lemma 3.1(2). Using Lemma 3.1(3) we obtain

$$[\widetilde{\vartheta}_{f,A}(h)] = [\widetilde{\Psi}_{1,A} \cdots \widetilde{\Psi}_{n,A}](g_A^{-1} f h g_A)] = [g_A^{-1} f h g_A] = [fh] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$$

and thus Corollary 3.2 is proved. □

3.3. Lemma. *Let* $1 \le r \le n$ *and let* α *be a modulus of continuity. If* $r = n$ *we further assume* $\alpha(tx) \le \sqrt{t}\alpha(x)$ $\forall x \in \mathbf{R}$ *and* $\forall t \ge 1$*. Then there exists* $A_0 \ge 1$ *and* $\epsilon_0 > 0$ *with*

$$\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \le \epsilon \Rightarrow f, h \in \widetilde{V}^n_{A_0} \text{ and } \mu_{r,\alpha}(\widetilde{\vartheta}_{f,A_0}(h)) \le \epsilon.$$

$\forall 0 < \epsilon \le \epsilon_0$ *and* $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$.

Proof. First, using the assumption on r, we choose $A_0 \ge 1$ with

$$3C^n A_0^{r-n} \le 1 \text{ if } r < n \qquad \text{and} \qquad 3C^n A_0^{-\frac{1}{2}} \le 1 \text{ if } r = n,$$

where C is the constant of Lemma 3.1(4). Choosing $\epsilon_0 > 0$ sufficiently small we may arrange that $f \in \widetilde{V}^n_{A_0}$ for all $f \in \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$, with $\mu_{r,\alpha}(f) \le \epsilon_0$. From a calculation similar to (2.27) we obtain (notice that the rôles of g_{A_0} and $g_{A_0}^{-1}$ are reversed)

$$D^r(g_{A_0}^{-1} f h g_{A_0}) = A_0^{r-1}(D^r(fh) \circ (A_0 \cdot Id)),$$

for all $f, h \in \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$, and thus

$$\begin{aligned}
\mu_{r,\alpha}(g_{A_0}^{-1} f h g_{A_0}) &= \sup_{x\neq y\in\mathbf{R}^n} \frac{\|D^r(g_{A_0}^{-1} f h g_{A_0})(x) - D^r(g_{A_0}^{-1} f h g_{A_0})(y)\|}{\alpha(\|x-y\|)} \\
&= A_0^{r-1} \sup_{x\neq y\in\mathbf{R}^n} \frac{\|D^r(fh)(A_0x) - D^r(fh)(A_0y)\|}{\alpha(\frac{1}{A_0}\|A_0x - A_0y\|)} \\
&\le \begin{cases} A_0^r \sup\limits_{x\neq y\in\mathbf{R}^n} \frac{\|D^r(fh)(A_0x)-D^r(fh)(A_0y)\|}{\alpha(\|A_0x-A_0y\|)} & \text{if } 1 \le r < n \\ A_0^{r-1+\frac{1}{2}} \sup\limits_{x\neq y\in\mathbf{R}^n} \frac{\|D^r(fh)(A_0x)-D^r(fh)(A_0y)\|}{\alpha(\|A_0x-A_0y\|)} & \text{if } r = n \end{cases} \\
&\le \begin{cases} A_0^r \mu_{r,\alpha}(fh) & \text{if } 1 \le r < n \\ A_0^{r-\frac{1}{2}} \mu_{r,\alpha}(fh) & \text{if } r = n \end{cases}
\end{aligned}$$

Using Lemma 3.1(4) and Lemma 2.30 we shrink ϵ_0, such that the following estimation holds:

$$\begin{aligned}
\mu_{r,\alpha}(\widetilde{\vartheta}_{f,A_0}(h)) &= \mu_{r,\alpha}(\widetilde{\Psi}_{1,A_0} \cdots \widetilde{\Psi}_{n,A_0}(g_{A_0}^{-1} f h g_{A_0})) \\
&\le \left(\frac{C}{A_0}\right)^n \mu_{r,\alpha}(g_{A_0}^{-1} f h g_{A_0}) \\
&\le \begin{cases} C^n A_0^{r-n} \mu_{r,\alpha}(fh) & \text{if } 1 \le r < n \\ C^n A_0^{r-\frac{1}{2}-n} \mu_{r,\alpha}(fh) & \text{if } r = n \end{cases} \\
&\le \left.\begin{cases} C^n A_0^{r-n} 3\epsilon & \text{if } 1 \le r < n \\ C^n A_0^{-\frac{1}{2}} 3\epsilon & \text{if } r = n \end{cases}\right\} \le \epsilon
\end{aligned}$$

$\forall 0 < \epsilon \le \epsilon_0$ and $\forall f, h \in \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$ with $\mu_{r,\alpha}(f), \mu_{r,\alpha}(h) \le \epsilon$. □

3.4. Theorem (Mather). *[8] Let M be a connected, smooth manifold, $1 \le r \le \dim(M)$ and let α be a modulus of continuity. If $r = \dim(M)$ we further assume that $\alpha(tx) \le \sqrt{t}\alpha(x)$, $\forall x \in [0,\infty)$ and $\forall t \ge 1$. Then $\mathrm{Diff}_c^{r,\alpha}(M)_\circ$ is perfect.*

Proof. By Remark 2.22 it suffices to show

$$\mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n) \subseteq [\mathrm{Diff}_c^{r,\alpha}(\mathbf{R}^n)_\circ, \mathrm{Diff}_c^{r,\alpha}(\mathbf{R}^n)_\circ], \tag{3.9}$$

where A_0 is the constant of Lemma 3.3. Analogously to the proof of Theorem 2.19 (cf. page 44) choose $0 < \epsilon \le \epsilon_0$ such that

$$L := \{h \in \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n) : \mu_{r,\alpha}(h) \le \epsilon\}$$

has the fixed-point property (cf. Lemma 2.41). By Corollary 3.2(1) and Lemma 3.3 we have $L \subset \widetilde{V}^n_{A_0}$, and for all $f \in L$

$$\mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n) \supseteq \overset{\widetilde{\vartheta}_{f,A_0}|_L}{\longrightarrow} L \subseteq \mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$$

is continuous with respect to C^r-topologies. Hence for every $f \in L$ there exists $h_f \in L$ with $\widetilde{\vartheta}_{f,A_0}(h_f) = h_f$. From Corollary 3.2(2) we get

$$[f][h_f] = [fh_f] = [\widetilde{\vartheta}_{f,A_0}(h_f)] = [h_f] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$$

and thus $[f] = [Id] \in H_1(\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ)$. So $f \in [\mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ, \mathrm{Diff}^{r,\alpha}_c(\mathbf{R}^n)_\circ]$ for all $f \in L$ and since L generates $\mathrm{Diff}^{r,\alpha}_{[-2A_0,2A_0]^n}(\mathbf{R}^n)$ (cf. Corollary 2.31) this yields (3.9). □

3.5. Corollary. *Let* M *be a connected, smooth manifold and let* $1 \le r \le \dim(M)$*. Then* $\mathrm{Diff}^{r,\alpha}_c(M)_\circ$ *is perfect and hence simple by Corollary 1.17.*

Proof. This is an immediate consequence of Theorem 3.4, Lemma 2.17 and Remark 2.18. □

4. Isomorphisms between groups of diffeomorphisms

4.1. Definition (Property P). Let M be a smooth, connected manifold and let G be a group of diffeomorphisms, i.e. $G \subseteq \mathrm{Diff}^r(M)$ for some $1 \leq r \leq \infty$. We say G has *Property P* (path transitivity) iff for every smooth path $\sigma : I \to M$ and for every open neighborhood U of $\mathrm{Im}(\sigma)$ there exists $g \in G$ with $\mathrm{supp}(g) \subseteq U$ and $g(\sigma(0)) = \sigma(1)$.

If $x \in M$ we denote by $S_x G$ the isotropy- or stabilizer subgroup at x, i.e.

$$S_x G := \{g \in G : g(x) = x\}$$

and for $g \in G$ we let

$$\mathrm{Fix}(g) := \{x \in M : g(x) = x\}.$$

If we speak of a group of diffeomorphisms $G^r(M)$ we will always assume that M is a smooth, connected manifold, $1 \leq r \leq \infty$ and $G^r(M) \subseteq \mathrm{Diff}^r(M)$ is a subgroup.

4.2. Lemma. *Let $G^r(M)$ be a group of diffeomorphisms that has Property P. If $f, g \in G^r(M) \setminus S_x G^r(M)$ and $f(x), g(x)$ are contained in the same component of $M \setminus \{x\}$ then*

$$f \in (S_x G^r(M)) g (S_x G^r(M)) \tag{4.1}$$

Especially $S_x G^r(M)$ is a maximal subgroup of $G^r(M)$.

Proof. Since $f(x)$, $g(x)$ are contained in the same component of $M \setminus \{x\}$ there exists a smooth path connecting $f(x) \neq x$ and $g(x) \neq x$, that doesn't meat x. By Property P we obtain $h \in S_x G^r(M)$ with $h(f(x)) = g(x)$. Hence $g^{-1} h f \in S_x G^r(M)$ and thus $f \in (S_x G^r(M)) g (S_x G^r(M))$.

In order to show that $S_x G^r(M)$ is a maximal subgroup of $G^r(M)$ notice first that $S_x G^r(M) \neq G^r(M)$ by Property P. Now let $S_x G^r(M) \subset H \subseteq G^r(M)$ and choose $h \in H \setminus S_x G^r(M)$. We have to show $H = G^r(M)$, but since $S_x G^r(M) \subset H$ it suffices to show $G^r(M) \setminus S_x G^r(M) \subseteq H$. If $f \in G^r(M) \setminus S_x G^r(M)$ then either $\{f(x), h(x)\}$ or $\{f(x), h^{-1}(x)\}$ is contained in one component of $M \setminus \{x\}$ and we can apply (4.1). Hence we obtain

$$f \in \underbrace{S_x G^r(M)}_{\in H} \underbrace{h^{\pm 1}}_{\in H} \underbrace{S_x G^r(M)}_{\in H} \in H,$$

and thus $G^r(M) \setminus S_x G^r(M) \subseteq H$. □

4.3. Lemma. *[1] [5] Let $G^r(M)$ and $G^s(N)$ be groups of diffeomorphisms having Property P, and let $\Phi : G^r(M) \to G^s(N)$ be an isomorphism of groups. Notice that we do not consider any topology on $G^r(M)$ and $G^s(N)$ and hence do not require Φ to be continuous. Assume further that there exists $x_0 \in M$ and $y_0 \in N$ with $\Phi(S_{x_0} G^r(M)) = S_{y_0} G^s(N)$. Then there exists a unique homeomorphism $\varphi : M \to N$ inducing Φ, i.e.*

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1} \qquad \forall f \in G^r(M) \tag{4.2}$$

Proof. First we define $\varphi : M \to N$: given $x \in M$, we choose $g \in G^r(M)$ with $g(x_0) = x$ (Property P) and let $\varphi(x) := \Phi(g)(y_0)$. Too see, that this is well defined take another $g' \in G^r(M)$ with $g'(x_0) = x$. Then $g^{-1}g' \in S_{x_0}G^r(M)$ and hence $\Phi(g)^{-1}\Phi(g') = \Phi(g^{-1}g') \in S_{y_0}G^s(N)$. But then $\Phi(g)^{-1}\Phi(g')(y_0) = y_0$ and thus $\Phi(g')(y_0) = \Phi(g)(y_0)$. Using $\Phi^{-1} : G^s(N) \to G^r(M)$ one defines analogously $\varphi^{-1} : N \to M$. Clearly this is the inverse of φ. To check (4.2) take $f \in G^r(M)$, $x \in M$ and $g \in G^r(M)$ with $g(x_0) = x$. Then $(fg)(x_0) = f(x)$ and thus

$$\Phi(f)(\varphi(x)) = \Phi(f)\Phi(g)(y_0) = \Phi(fg)(y_0) = \varphi(f(x)).$$

Since x and f were arbitrary we have shown $\Phi(f) \circ \varphi = \varphi \circ f$ for all $f \in G^r(M)$, but this is (4.2).

Now suppose there exists another bijection $\psi : M \to N$ inducing Φ. Then $\psi^{-1} \circ \varphi : M \to M$ is bijective and for all $f \in G^r(M)$ we have

$$(\psi^{-1} \circ \varphi) f (\psi^{-1} \circ \varphi)^{-1} = \psi^{-1} \circ \Phi(f) \circ \psi = \Phi^{-1}(\Phi(f)) = f. \tag{4.3}$$

If $\varphi \neq \psi$ then there exists $x \in M$ with $\psi^{-1}(\varphi(x)) \neq x$ and by Property P we can choose $f \in G^r(M)$ with $f(x) = x$ and $f(\psi^{-1}(\varphi(x))) \neq \psi^{-1}(\varphi(x))$. Since $f(\psi^{-1}(\varphi(x))) \neq x$ we have

$$\begin{aligned} (\psi^{-1} \circ \varphi) f (\psi^{-1} \circ \varphi)^{-1}(\psi^{-1}(\varphi(x))) &= (\psi^{-1} \circ \varphi)(f(x)) \\ &= \psi^{-1}(\varphi(x)) \neq f(\psi^{-1}(\varphi(x))) \end{aligned}$$

and hence $(\psi^{-1} \circ \varphi) f (\psi^{-1} \circ \varphi)^{-1} \neq f$, a contradiction to (4.3). So $\varphi = \psi$ and we have proven the uniqueness of φ, stated above.

It remains to show, that φ and φ^{-1} are continuous. Therefore look at

$$\begin{aligned} \mathcal{U}_M &:= \{M \setminus \mathrm{Fix}(f) : f \in G^r(M)\} \\ \mathcal{U}_N &:= \{N \setminus \mathrm{Fix}(g) : g \in G^s(N)\}. \end{aligned}$$

Obviously $\mathcal{U}_M$ and $\mathcal{U}_N$ consist of open sets, and by Property P every neighborhood of $x \in M$ contains an open neighborhood $M \setminus \mathrm{Fix}(f)$ of x. Thus $\mathcal{U}_M$ is a basis of the topology on M. Similarly $\mathcal{U}_N$ is a basis of the topology on N. We have

$$\begin{aligned} x \in \mathrm{Fix}(\Phi(f)) &\Leftrightarrow \Phi(f)(x) = x \Leftrightarrow \varphi \circ f \circ \varphi^{-1}(x) = x \\ &\Leftrightarrow \varphi^{-1}(x) \in \mathrm{Fix}(f) \Leftrightarrow x \in \varphi(\mathrm{Fix}(f)), \end{aligned} \tag{4.4}$$

and thus $\mathrm{Fix}(\Phi(f)) = \varphi(\mathrm{Fix}(f))$ for all $f \in G^r(M)$. Hence φ and φ^{-1} induce bijections between $\mathcal{U}_M$ and $\mathcal{U}_N$ and are thus continuous. □

4.4. Lemma. *Let $G^r(M)$ and $G^s(N)$ be groups of diffeomorphisms, which have Property P, and let $\Phi : G^r(M) \to G^s(N)$ be an isomorphism of groups. Suppose there exists $y_0 \in N$ and a closed, nonempty, proper subset A of M (i.e. $A \subseteq M$, $A \neq \emptyset$ and $A \neq M$), such that*

$$g(A) = A \qquad \forall g \in F := \Phi^{-1}(S_{y_0}G^s(N)).$$

Then $A \setminus A^\circ = \{x\}$ *and* $F = S_x G^r(M)$.

Proof. Since M is connected we have $A \setminus A^\circ \neq \emptyset$. Suppose $A \setminus A^\circ = \{x\}$ for some $x \in M$. Then, since g is continuous and $g(A) = A$ for all $g \in F$, we have $g(x) = x$, $\forall g \in F$, and hence $F \subseteq S_x G^r(M)$. By Lemma 4.2 $S_{y_0} G^s(N)$ is a maximal subgroup of $G^s(N)$, so $F = \Phi^{-1}(S_{y_0} G^s(N))$ is a maximal subgroup of $G^r(M)$, and thus $F = S_x G^r(M)$.

Now suppose $A \setminus A^\circ$ contains at least two distinct points x_1 and x_2. We then choose U_1 and U_2, neighborhoods of x_1 and x_2, with $U_1 \cap U_2 = \emptyset$. Using Property P we find $g_i \in G^r(M)$, $i = 1, 2$ with $\mathrm{supp}(g_i) \subseteq U_i$ and $g_i(x_i) \notin A$. Hence $g_i \notin F$ and thus $\Phi(g_i) \notin S_{y_0} G^s(N)$. If necessary we replace g_1 by g_1^{-1} and obtain $g_i \in G^r(M) \setminus F$, such that $\{\Phi(g_1)(y_0), \Phi(g_2)(y_0)\}$ is contained in one component of $N \setminus \{y_0\}$. Further we let $g_3 := g_2 g_1$. Then $g_3 \in G^r(M) \setminus F$ and $\{\Phi(g_i)(y_0) : i = 1, 2, 3\}$ is contained in one component of $N \setminus \{y_0\}$. From Lemma 4.2 we obtain $\Phi(g_i) \in (S_{y_0} G^s(N)) \Phi(g_3) (S_{y_0} G^s(N))$ and thus $g_i \in F g_3 F$ for $i = 1, 2$. Next define $A_i := A \cap g_i^{-1}(M \setminus A)$ for $i = 1, 2, 3$ and notice that $A_i \subseteq U_i$, $A_3 = A_1 \cup A_2$ and $x_2 \in A_2 \neq \emptyset$.

Claim 1. *If* $i \in \{1, 2\}$ *and* $t \in g_3^{-1} F g_i \cap F$, *then* $t(A_i) = A_3$.

Proof of Claim 1. Since $t \in F$ we have $t(A) = A$, and since $t \in g_3^{-1} F g_i$ we find $f \in F$ with $t = g_3^{-1} f g_i$. From $g_3 t(A_i) = f g_i(A_i) \subseteq f(M \setminus A) = M \setminus A$, we obtain

$$t(A_i) \subseteq A \cap g_3^{-1}(M \setminus A) = A_3, \tag{4.5}$$

and since $g_i t^{-1}(A_3) = f^{-1} g_3(A_3) \subseteq f^{-1}(M \setminus A) = M \setminus A$ we have

$$t^{-1}(A_3) \subseteq A \cap g_i^{-1}(M \setminus A) = A_i. \tag{4.6}$$

Combining (4.5) and (4.6) we get $t(A_i) = A_3$ □

Claim 2. *There exist* $s_i, t_i \in g_3^{-1} F g_i \cap F$ *for* $i = 1, 2$ *with* $s_1 s_2 = t_2 t_1$.

Proof of Claim 2. Using Property P we may choose $z \in N$ and $s_1', s_2', t_1' \in G^s(N)$ with

$$\begin{array}{lll} s_2'(y_0) = y_0, & s_2' \Phi(g_2^{-1})(y_0) = \Phi(g_3^{-1})(y_0), & s_2'(z) = \Phi(g_1^{-1})(y_0) \\ t_1'(y_0) = y_0, & t_1' \Phi(g_1^{-1})(y_0) = \Phi(g_3^{-1})(y_0), & t_1'(z) = \Phi(g_2^{-1})(y_0) \end{array}$$

This is possible, even if $\dim(N) = 1$ since $\{\Phi(g_i^{-1})(y_0) : i = 1, 2, 3\}$ is contained in one component of $N \setminus \{y_0\}$. We let $s_1' := t_1'$ and $t_2' := s_1' s_2' t_1'^{-1}$. Then $s_i', t_i' \in S_{y_0} G^s(N)$, and $s_i', t_i' \in \Phi(g_3^{-1})(S_{y_0} G^s(N)) \Phi(g_i)$ since we have

$$\begin{aligned} \Phi(g_3) s_i' \Phi(g_i^{-1})(y_0) &= \Phi(g_3) \Phi(g_3^{-1})(y_0) = y_0 \\ \Phi(g_3) t_1' \Phi(g_1^{-1})(y_0) &= \Phi(g_3) \Phi(g_3^{-1})(y_0) = y_0 \\ \Phi(g_3) t_2' \Phi(g_2^{-1})(y_0) &= \Phi(g_3) s_1' s_2' t_1'^{-1} \Phi(g_2^{-1})(y_0) \\ &= \Phi(g_3) s_1' s_2'(z) \\ &= \Phi(g_3) \Phi(g_3^{-1})(y_0) = y_0. \end{aligned}$$

If we let $t_i := \Phi^{-1}(t_i')$ and $s_i := \Phi^{-1}(s_i')$, Claim 2 follows from $\Phi^{-1}(S_{y_0}G^s(N)) = F$. □
Using Claim 1 we obtain

$$\begin{aligned} s_1 s_2(A_2) &= s_1(A_3) \supseteq s_1(A_1) = A_3 \\ t_2 t_1(A_1) &= t_2(A_3) \supseteq t_2(A_2) = A_3, \end{aligned}$$

but since $A_1 \cap A_2 = \emptyset$ and $A_3 \neq \emptyset$, this is a contradiction to the bijectivity of $s_1 s_2 = t_2 t_1$. So $A \setminus A^\circ$ contains exactly one point, and we are done. □

4.5. Remark. In the situation of Lemma 4.4 assume additionally $\dim(M) \geq 2$. Then $A = \{x\}$, for if $x \neq x' \in A$ we find $f \in S_x G^r(M) = F$ with $f(x') \notin A$ (Property P), a contradiction to $f(A) = A$ for all $f \in F$. If $\dim(M) = 1$, however, this is not true. For example take $M = \mathbf{R}$, $A = [0, \infty)$, $x = 0$ and $G = \mathrm{Diff}_c^\infty(\mathbf{R})_\circ$. Then $f(A) = A$ for all $f \in S_x G$.

4.6. Definition. If $G^r(M)$ is a group of diffeomorphisms, we denote by

$$G_c^r(M) := G^r(M) \cap \mathrm{Diff}_c^r(M)$$

the subgroup of all compactly supported $G^r(M)$-diffeomorphisms. If $U \subseteq M$ is an arbitrary subset we let

$$G_U^r(M) := G^r(M) \cap \mathrm{Diff}_U^r(M),$$

the subgroup of all $G^r(M)$-diffeomorphisms supported on U. A $G^r(M)$-diffeotopy is a C^r-mapping $H : M \times I \to M$, such that $H_t := H(\cdot, t) \in G^r(M)$ and $H_0 = Id$. By $G^r(M)_\circ$ we denote all $G^r(M)$-diffeomorphisms, that are $G^r(M)$-diffeotopic to Id. By $G_c^r(M)_\circ$ we will always mean $(G_c^r(M))_\circ$. Hence $f \in G_c^r(M)_\circ$ iff f is $G_c^r(M)$-diffeotopic to Id. Similarly $G_U^r(M)_\circ = (G_U^r(M))_\circ$. Like we did in the first Chapter, we let

$$\mathcal{U}^r(M) := \{\varphi(\mathbf{B}^n) : \mathbf{R}^n \overset{\varphi}{\hookrightarrow} M \quad C^r\text{-embedding}\}.$$

4.7. Remark. Notice that $G^r(M)_\circ$ is a subgroup of $G^r(M)$, for if $F, H : M \times I \to I$ are $G^r(M)$-diffeotopies with $F_0 = H_0 = Id$ then

$$(x, t) \mapsto F_t(H_{1-t}(H_1^{-1}(x)))$$

is a $G^r(M)$-diffeotopy from $F_1 H_1^{-1}$ to Id. Further, if $g \in G^r(M)$ then

$$(x, t) \mapsto g^{-1}(F_t(g(x)))$$

is a $G^r(M)$-diffeotopy from $g^{-1} F_1 g$ to Id. Thus, analogously to Remark 1.8 we have

$$\begin{aligned} &G_c^r(M)_\circ \lhd G^r(M), \qquad G_c^r(M)_\circ \lhd G^r(M)_\circ, \qquad G_c^r(M)_\circ \lhd G_c^r(M) \\ &G_c^r(M) \lhd G^r(M), \qquad G^r(M)_\circ \lhd G^r(M) \end{aligned}$$

for all groups of diffeomorphisms $G^r(M)$.

4.8. Remark. Since we do not consider any topology on $G^r(M)$ the definition of $G^r(M)_\circ$ is different to the one given in Definition 1.6. But by Corollary 1.14 they coincide in the case $G^r(M) = \mathrm{Diff}^r(M)$. More precisely, for $U \in \mathcal{U}^r(M)$ we have:

$G^r(M)$	$G^r_c(M)$	$G^r_c(M)_\circ$	$G^r_U(M)$	$G^r_U(M)_\circ$
$\mathrm{Diff}^r(M)$	$\mathrm{Diff}^r_c(M)$	$\mathrm{Diff}^r_c(M)_\circ$	$\mathrm{Diff}^r_U(M)$	$\mathrm{Diff}^r_U(M)_\circ$
$\mathrm{Diff}^r_c(M)$	$\mathrm{Diff}^r_c(M)$	$\mathrm{Diff}^r_c(M)_\circ$	$\mathrm{Diff}^r_U(M)$	$\mathrm{Diff}^r_U(M)_\circ$
$\mathrm{Diff}^r_c(M)_\circ$	$\mathrm{Diff}^r_c(M)_\circ$	$\mathrm{Diff}^r_c(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$
$\mathrm{Diff}^r_U(M)$	$\mathrm{Diff}^r_U(M)$	$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)$	$\mathrm{Diff}^r_U(M)_\circ$
$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$	$\mathrm{Diff}^r_U(M)_\circ$

4.9. Definition (Property L). A group of diffeomorphisms $G^r(M)$ is said to have *Property L* (locality), iff for every subgroup $F \subseteq G^r(M)$ and for every open cover $\mathcal{U} \subseteq \mathcal{U}^r(M)$ with $[G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq F \ \forall U \in \mathcal{U}$ we have $[G^r_c(M)_\circ, G^r_c(M)_\circ] \subseteq F$.

4.10. Lemma. *Let $G^r(M)$, $G^s(N)$ be groups of diffeomorphisms, which have Property P and Property L. Assume further, that $G^r_c(M)_\circ$ and $G^r_c(N)_\circ$ are nonabelian, and let $\Phi : G^r(M) \to G^s(N)$ be an isomorphism of groups. For every $y \in N$ we let*

$$\mathcal{C}_y := \{U \in \mathcal{U}^r(M) : [G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq \Phi^{-1}(S_y G^s(N))\}$$

and $C_y := M \setminus \bigcup_{U \in \mathcal{C}_y} U$. Then we have

(1) C_y is closed in M

(2) $f(C_y) = C_y \qquad \forall f \in \Phi^{-1}(S_y G^s(N))$

(3) $C_y \neq \emptyset$

Proof. (1) is obvious, since $\mathcal{C}_y$ consists of open sets. To see (2) take $f \in \Phi^{-1}(S_y G^s(N))$ and $U \in \mathcal{C}_y$. Then $f(U) \in \mathcal{U}^r(M)$ and

$$\begin{aligned}
[G^r_{f(U)}(M)_\circ, G^r_{f(U)}(M)_\circ] &= [f G^r_U(M)_\circ f^{-1}, f G^r_U(M)_\circ f^{-1}] \\
&= f[G^r_U(M)_\circ, G^r_U(M)_\circ] f^{-1} \\
&\subseteq \phi^{-1}(S_y G^s(N)).
\end{aligned}$$

So we have $f(U) \in \mathcal{C}_y$, and hence f induces a bijection on $\mathcal{C}_y$. Thus

$$f(C_y) = f(M \setminus \bigcup_{U \in \mathcal{C}_y} U) = M \setminus f(\bigcup_{U \in \mathcal{C}_y} U) = M \setminus \bigcup_{U \in \mathcal{C}_y} U = C_y.$$

In order to prove (3), suppose conversely $C_y = \emptyset$. Hence $\mathcal{C}_y$ is an open cover of M with the following property:

$$[G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq \Phi^{-1}(S_y G^s(N)) \qquad \forall U \in \mathcal{C}_y.$$

If $z \in N$ we may choose $g \in G^s(N)$ with $g(y) = z$ (Property P), and let $f := \Phi^{-1}(g) \in G^r(M)$. Clearly $(f(U))_{U \in \mathcal{C}_y}$ is an open cover of M by sets of $\mathcal{U}^r(M)$, and for all $U \in \mathcal{C}_y$ we have

$$\begin{aligned} [G^r_{f(U)}(M)_\circ, G^r_{f(U)}(M)_\circ] &= [fG^r_U(M)_\circ f^{-1}, fG^r_U(M)_\circ f^{-1}] \\ &= f[G^r_U(M)_\circ, G^r_U(M)_\circ]f^{-1} \\ &\subseteq f\Phi^{-1}(S_y G^s(N))f^{-1} \\ &= \Phi^{-1}(gS_y G^s(N)g^{-1}) \\ &= \Phi^{-1}(S_z G^s(N)). \end{aligned}$$

Hence, using Property L of $G^r(M)$, we obtain

$$[G^r_c(M)_\circ, G^r_c(M)_\circ] \subseteq \Phi^{-1}(S_z G^s(N)) \qquad \forall z \in N$$

and thus

$$[G^r_c(M)_\circ, G^r_c(M)_\circ] \subseteq \bigcap_{z \in N} \Phi^{-1}(S_z G^s(N)) = \Phi^{-1}(\bigcap_{z \in N} S_z G^s(N)) = \{Id_M\}$$

This is a contradiction, since $G^r_c(M)_\circ$ is nonabelian. □

4.11. Definition (Property B). Let $G^r(M)$ be a group of diffeomorphisms. $G^r(M)$ is said to have *Property B* (ball), iff $\forall U = \varphi(\mathbf{B}^n) \in \mathcal{U}^r(M)$ there exists $f \in G^r(M)$ with $\mathrm{Fix}(f) = (M \setminus U) \cup \{\varphi(0)\}$.

4.12. Lemma. *Let $G^r(M)$ and $G^s(N)$ be groups of diffeomorphisms having Property P and Property B. If $\Phi : G^r(M) \to G^s(N)$ is an isomorphism of groups, then $\forall y \in N$ there exists $Id \neq f \in F := \Phi^{-1}(S_y G^s(N))$, such that* $\mathrm{Fix}(f)^\circ \neq \emptyset$.

Proof. Suppose conversely $\mathrm{Fix}(f)^\circ = \emptyset$, $\forall Id \neq f \in F$. Choose $x_i = \varphi_i(0) \in \varphi_i(\mathbf{B}^n) = U_i \in \mathcal{U}^r(M)$, $i = 1, \dots, 4$, such that $\overline{U_i} \cap \overline{U_j} = \emptyset\ \forall i \neq j$. Since $G^r(M)$ has Property B, we may choose $g_i \in G^r(M)$ such that $\mathrm{Fix}(g_i) = (M \setminus U_i) \cup \{x_i\}$. Since $\mathrm{Fix}(g_i)^\circ \neq \emptyset$ we have $g_i \notin F$, hence $\Phi(g_i) \notin S_y G^s(N)$ and thus we may assume (replace g_i by g_i^{-1} if necessary) that $\{\Phi(g_i)(y) : i = 1, \dots, 4\}$ is contained in one component of $N \setminus \{y\}$. Further let $g_5 := g_3 g_4$ and $U_5 := U_3 \cup U_4$. Similarly to Claim 2 on page 60, we find $s_i, t_i \in g_5^{-1} F g_i \cap F$, $i = 1, 2$, with $s_1 s_2 = t_2 t_1$.

Sublemma. *Let $i \in \{1, 2\}$ and let $t \in g_5^{-1} F g_i \cap F$. Then $M \setminus \overline{U_5} \subseteq t(\overline{U_i})$.*

Proof of the Sublemma. Suppose conversely $M \setminus \overline{U_5} \not\subseteq t(\overline{U_i})$. Then $t^{-1}(M \setminus \overline{U_5}) \cap (M \setminus \overline{U_i}) \neq \emptyset$. Since $t \in g_5^{-1} F g_i$ there exists $s \in F$ with $s g_5 t = g_i$. If $x \in t^{-1}(M \setminus \overline{U_5}) \cap (M \setminus \overline{U_i})$ we have $st(x) = sg_5 t(x) = g_i(x) = x$. Hence

$$\emptyset \neq t^{-1}(M \setminus \overline{U_5}) \cap (M \setminus \overline{U_i}) \subseteq \mathrm{Fix}(st)^\circ$$

and thus $st = Id$, since $st \in F$. But then $t^{-1} g_5 t = g_i$, and thus $\mathrm{Fix}(g_i) = t^{-1}(\mathrm{Fix}(g_5))$. So $\mathrm{Fix}(g_i)$ and $\mathrm{Fix}(g_5)$ are homeomorphic, but

$$\mathrm{Fix}(h_5) = (M \setminus (U_3 \cup U_4)) \cup \{x_3, x_4\}$$

has two discrete points, and

$$\mathrm{Fix}(g_i) = (M \setminus U_i) \cup \{x_i\}$$

has only one discrete point, a contradiction. □

Applying the Sublemma to s_i and t_i above, we obtain

$$s_1 s_2(\overline{U_2}) \supseteq s_1(M \setminus \overline{U_5}) \supseteq s_1(\overline{U_1}) \supseteq M \setminus \overline{U_5},$$

and

$$t_2 t_1(\overline{U_1}) \supseteq t_2(M \setminus \overline{U_5}) \supseteq t_2(\overline{U_2}) \supseteq M \setminus \overline{U_5}.$$

But this is a contradiction, since $M \setminus \overline{U_5} \neq \emptyset$, $\overline{U_1} \cap \overline{U_2} = \emptyset$ and $s_1 s_2 = t_2 t_1$ is bijective. □

4.13. Lemma. *Let $G^r(M)$ and $G^s(N)$ be groups of diffeomorphisms which have Property P and Property B, and let $\Phi : G^r(M) \to G^s(N)$ be an isomorphism of groups. For $y \in N$ consider the set $C_y := M \setminus \bigcup_{U \in \mathcal{C}_y} U$, where*

$$\mathcal{C}_y := \{U \in \mathcal{U}^r(M) : [G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq \Phi^{-1}(S_y G^s(N))\},$$

we have introduced in Lemma 4.10 and for $x \in M$ let $D_x := N \setminus \bigcup_{V \in \mathcal{D}_x} V$, where

$$\mathcal{D}_x := \{V \in \mathcal{U}^s(N) : [G^s_V(N)_\circ, G^s_V(N)_\circ] \subseteq \Phi(S_x G^r(M))\}.$$

Then either there exists $y_0 \in N$ with $C_{y_0} \neq M$ or there exists $x_0 \in M$ with $D_{x_0} \neq N$.

Proof. Choose $y_0 \in N$. Then by Lemma 4.12 there exists $Id \neq f_0 \in \Phi^{-1}(S_{y_0} G^s(N))$ with $A := \mathrm{Fix}(f_0)^\circ \neq \emptyset$. We have $y_0 \in B := \mathrm{Fix}(\Phi(f_0))$, thus $B \neq \emptyset$, and $B \neq N$ since $f_0 \neq Id$. Now consider the following two subgroups of $G^r(M)$:

$$H := \Phi^{-1}(\{g \in G^s(N) : g(B) = B\})$$
$$K := \Phi^{-1}(\{g \in G^s(N) : B \subseteq \mathrm{Fix}(g)\})$$

We then have:

(1) $K \lhd H$

(2) $f_0 \in K$ and thus $K, H \neq \{Id\}$

(3) $K \subseteq \Phi^{-1}(S_{y_0} G^s(N))$

(4) $G^r_A(M) \subseteq H$

Everything is obvious, except (4). If $f \in G^r_A(M)$ we have $f f_0 = f_0 f$ since $\mathrm{supp}(f) \cap \mathrm{supp}(f_0) \subseteq A \cap (M \setminus A) = \emptyset$. Hence

$$\begin{aligned} \Phi(f)(B) &= \Phi(f)(\mathrm{Fix}(\Phi(f_0))) = \mathrm{Fix}(\Phi(f)\Phi(f_0)\Phi(f)^{-1}) \\ &= \mathrm{Fix}(\Phi(f f_0 f^{-1})) = \mathrm{Fix}(\Phi(f_0)) = B, \end{aligned}$$

and thus $f \in H$.

We now distinguish between two possibilities:

Case a) $A \subseteq \text{Fix}(k) \qquad \forall k \in K$

Case b) $\exists k \in K\ \exists x_0 \in A:\ k(x_0) \neq x_0$

In case a) we choose $x_0 \in A$ and $V \in \mathcal{U}^s(N)$ with $V \subseteq N \setminus B$. This is possible since $B \neq N$ and B is closed. If $g \in G^s_V(N)$ then $\text{Fix}(g) \supseteq N \setminus V \supseteq N \setminus (N \setminus B) = B$, hence $\Phi^{-1}(g) \in K$ and thus $\Phi^{-1}(g)(x_0) = x_0$, since $x_0 \in A$. So $\Phi^{-1}(g) \in S_{x_0}G^r(M)$ and thus $g \in \Phi(S_{x_0}G^r(M))$, Therefore we have shown

$$[G^s_V(N)_\circ, G^s_V(N)_\circ] \subseteq G^s_V(N) \subseteq \Phi(S_{x_0}G^r(M)),$$

and consequently $V \in \mathcal{D}_{x_0}$, that is $D_{x_0} \neq N$.

So it remains to prove Lemma 4.13 under the additional assumption of case b), i.e. there exists $k \in K$ and $x_0 \in A$ with $k(x_0) \neq x_0$. Using Property P we find $g_1, g_2 \in G^r_A(M)$ with $x_0 \neq g_i(x_0)$, $g_i(k(x_0)) = k(x_0)$ and $g_1^{-1}(x_0) \neq g_2^{-1}(x_0)$. If we let $k_i := [g_i, k]$ for $i = 1, 2$, we obtain

$$k_i(x_0) = g_i^{-1}k^{-1}g_ik(x_0) = g_i^{-1}k^{-1}k(x_0) = g_i^{-1}(x_0) \neq x_0, \tag{4.7}$$

and $k_i \in K$, since $k \in K \lhd H$ and $g_i \in G^r_A(M) \subseteq H$. From (4.7) we get $x_0 \neq k_i(x_0)$, $k_1(x_0) \neq k_2(x_0)$, and thus $x_0 \neq k_i^{-1}(x_0)$ since k_i are bijections. Using the continuity of k_i we find $U \in \mathcal{U}^r(M)$ with $x_0 \in U \subseteq A$ and such that $k_i(U) \cap U = \emptyset$, $k_i^{-1}(U) \cap U = \emptyset$ and $k_1(U) \cap k_2(U) = \emptyset$. If $h_1, h_2 \in G^r_U(M)$ then $[h_i^{-1}, k_i^{-1}] \in K$ for $i = 1, 2$, since $k_i^{-1} \in K \lhd H$ and $h_i^{-1} \in G^r_U(M) \subseteq G^r_A(M) \subseteq H$. Further

$$[h_i^{-1}, k_i^{-1}] = \begin{cases} k_ih_i^{-1}k_i^{-1} & \text{on } k_i(U) \\ h_i & \text{on } U \\ Id & \text{otherwise} \end{cases}$$

for $i = 1, 2$, and

$$[[h_1^{-1}, k_1^{-1}], [h_2^{-1}, k_2^{-1}]] = \left\{ \begin{array}{ll} h_1^{-1}h_2^{-1}h_1h_2 & \text{on } U \\ Id & \text{on } k_1(U) \\ Id & \text{on } k_2(U) \\ Id & \text{otherwise} \end{array} \right\} = [h_1, h_2],$$

and thus $[h_1, h_2] \in K \subseteq \Phi^{-1}(S_{y_0}G^s(N))$. Since $h_1, h_2 \in G^r_U(M)$ were arbitrary, we have shown

$$[G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq [G^r_U(M), G^r_U(M)] \subseteq \Phi^{-1}(S_{y_0}G^s(N)).$$

So $U \in \mathcal{C}_{y_0}$ and thus $C_{y_0} \neq M$. □

4.14. Theorem (Filipkiewicz, Banyaga). *[5] [1] Let $G^r(M)$ and $G^s(N)$ be groups of diffeomorphisms that have Property P, Property B, Property L and assume that $G^r_c(M)_\circ$, $G^s_c(N)_\circ$ are nonabelian. If $\Phi : G^r(M) \to G^s(N)$ is an isomorphism of* *groups (we do not assume that Φ is continuous with respect to any topology), then there exists an unique homeomorphism $\varphi : M \to N$ with*

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1} \qquad \forall f \in G^r(M).$$

Proof. By Lemma 4.13 there exists $y_0 \in N$ with $C_{y_0} \neq M$ (replace Φ by Φ^{-1} if necessary). From Lemma 4.10 we obtain that C_{y_0} is a nonempty, closed subset of M, invariant under $\Phi^{-1}(S_{y_0}G^s(N))$. So we can apply Lemma 4.4 and get $x_0 \in M$ with

$$\Phi^{-1}(S_{y_0}G^s(N)) = S_{x_0}G^r(M),$$

and thus, by Lemma 4.3, we are finished. □

4.15. Lemma. *Let $\mathcal{U}$ be a cover of $\overline{\mathbf{B}^n}$ by open sets of $\mathbf{R}^n$ and let $0 < r \leq 1$. Then there exist $m \in \mathbf{N}$, $U_1, \dots, U_m \in \mathcal{U}$ and $f_i, g_i \in \mathrm{Diff}^\infty_{U_i}(\mathbf{R}^n)_\circ$ such that*

$$[f_m, g_m] \cdots [f_1, g_1](\overline{\mathbf{B}^n}) \subseteq \overline{\mathbf{B}^n_r(0)}$$

Proof. We fix $\mathcal{U}$ and let

$$R := \{r \in (0,1] : \text{Lemma 4.15 holds for } r\}.$$

Obviously R is an interval containing 1, and thus we are done if we show $r_0 := \inf R = 0$. Suppose conversely $r_0 > 0$ and choose $\epsilon > 0$, such that $\forall x \in \overline{\mathbf{B}^n}$, $\mathbf{B}^n_\epsilon(x)$ is contained in some $U \in \mathcal{U}$. Now choose $x_1, \dots, x_k \in \partial \mathbf{B}^n_{r_0}(0)$ with

$$\partial \mathbf{B}^n_{r_0}(0) \subseteq \bigcup_{i=1}^k \mathbf{B}^n_{\frac{\epsilon}{2}}(x_i)$$

and choose $g_i \in \mathrm{Diff}^\infty_{\mathbf{B}^n_\epsilon(x_i)}(\mathbf{R}^n)_\circ$ such that

$$g_i(\mathbf{B}^n_{\frac{\epsilon}{2}}(x_i)) \subseteq \mathbf{R}^n \setminus \mathbf{B}^n_{r_0+\frac{\epsilon}{2}}(0) \quad \text{and} \quad g_i^{-1}(\mathbf{B}^n_{\frac{\epsilon}{2}}(x_i)) \subseteq \mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0).$$

By the choice of ϵ there exist $U_1, \dots, U_k \in \mathcal{U}$ with $\mathbf{B}^n_\epsilon(x_i) \subseteq U_i$ and hence $g_i \in \mathrm{Diff}^\infty_{U_i}(\mathbf{R}^n)_\circ$. Further we choose $0 < \epsilon' < \frac{\epsilon}{2}$ with

$$\mathbf{B}^n_{r_0+\epsilon'}(0) \setminus \mathbf{B}^n_{r_0-\epsilon'}(0) \subseteq \bigcup_{i=1}^k \mathbf{B}^n_{\frac{\epsilon}{2}}(x_i)$$

and $f \in \mathrm{Diff}^\infty_{\mathbf{B}^n_{r_0+\epsilon'}(0)\setminus\mathbf{B}^n_{r_0-\epsilon'}(0)}(\mathbf{R}^n)_\circ$ with

$$f^{-1}(\mathbf{B}^n_{r_0+\frac{\epsilon'}{2}}(0)) \subseteq \mathbf{B}^n_{r_0-\frac{\epsilon'}{2}}(0).$$

By Claim 2 on page 10 we find $m \in \mathbf{N}$ and $f_j \in \mathrm{Diff}^\infty_{\mathbf{B}^n_{\frac{\epsilon}{2}}(x_{i(j)})}(\mathbf{R}^n)_\circ$ with $f = f_1 \cdots f_m$. Hence $f_j \in \mathrm{Diff}^\infty_{U_{i(j)}}(\mathbf{R}^n)_\circ$ and we have

$$h_j := [f_j, g_{i(j)}] = \begin{cases} f_j^{-1} & \text{on } \mathbf{B}^n_{\frac{\epsilon}{2}}(x_{i(j)}) \\ g_{i(j)}^{-1} f_j g_i & \text{on } g_{i(j)}^{-1}(\mathbf{B}^n_{\frac{\epsilon}{2}}(x_{i(j)})) \\ Id & \text{otherwise} \end{cases}.$$

Especially

$$h_m \cdots h_1(\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)) \subseteq \mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0) = f^{-1}(\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)). \tag{4.8}$$

Since $\operatorname{supp}(f_j) \subseteq \mathbf{B}^n_{\frac{\epsilon}{2}}(x_{i(j)})$ we have

$$h_j|_{\mathbf{B}^n_{r_0+\frac{\epsilon}{2}}(0)\setminus\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)} = f_j^{-1}|_{\mathbf{B}^n_{r_0+\frac{\epsilon}{2}}(0)\setminus\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)}$$

and thus

$$h_m \cdots h_1|_{\mathbf{B}^n_{r_0+\frac{\epsilon}{2}}(0)\setminus\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)} = f^{-1}|_{\mathbf{B}^n_{r_0+\frac{\epsilon}{2}}(0)\setminus\mathbf{B}^n_{r_0-\frac{\epsilon}{2}}(0)}. \tag{4.9}$$

(4.8) and (4.9) together yield

$$h_m \cdots h_1(\mathbf{B}^n_{r_0+\frac{\epsilon'}{2}}(0)) \subseteq f^{-1}(\mathbf{B}^n_{r_0+\frac{\epsilon'}{2}}(0)) \subseteq \mathbf{B}^n_{r_0-\frac{\epsilon'}{2}}(0),$$

a contradiction to $r_0 = \inf R$. □

4.16. Lemma. *[5] Let M be a connected, smooth manifold, $1 \leq \dim(M) < \infty$, $1 \leq r \leq \infty$ and let $G^r(M)$ be one of the groups $\mathrm{Diff}^r(M)$, $\mathrm{Diff}^r_c(M)$, $\mathrm{Diff}^r_c(M)_\circ$. Then $G^r_c(M)_\circ$ is nonabelian and $G^r(M)$ has Property P, Property B and Property L.*

Proof. Obviously $G^r_c(M)_\circ$ is nonabelian (cf. Remark 4.8). From Lemma 1.11 we obtain $\mathrm{Diff}^r_c(M)_\circ \subseteq G^r(M)$ has Property P, and hence $G^r(M)$ has it too. Given $U = \varphi(\mathbf{B}^n) \in \mathcal{U}^r(M)$ we choose $\lambda \in C^\infty([0,\infty),[0,\infty))$ with $\lambda(x) = 0 \Leftrightarrow x \in [1,\infty)$ and consider the vector field $X(x) := \lambda(\|x\|^2)\cdot x$ on $\mathbf{R}^n$. Then $f := \mathrm{Fl}^X_1 \in \mathrm{Diff}^\infty_{\overline{\mathbf{B}^n}}(\mathbf{R}^n)_\circ$ and $\mathrm{Fix}(f) = (\mathbf{R}^n \setminus \mathbf{B}^n) \cup \{0\}$. If we let

$$\widetilde{f} := \begin{cases} \varphi \circ f \circ \varphi^{-1} & \text{on } \varphi(\mathbf{R}^n) \\ Id & \text{otherwise} \end{cases}$$

then $\widetilde{f} \in \mathrm{Diff}^r_c(M)_\circ \subseteq G^r(M)$ and $\mathrm{Fix}(\widetilde{f}) = (M \setminus U) \cup \{\varphi(0)\}$. Thus $G^r(M)$ has Property B.

In order to show that $G^r(M)$ has Property L, recall that $G^r_U(M)_\circ = \mathrm{Diff}^r_U(M)_\circ$ for all $U \in \mathcal{U}^r(M)$ (cf. Remark 4.8). Let F be a subgroup of $G^r(M)$ and let $\mathcal{U} \subseteq \mathcal{U}^r(M)$ be an open cover of M, such that

$$[\mathrm{Diff}^r_U(M)_\circ, \mathrm{Diff}^r_U(M)_\circ] = [G^r_U(M)_\circ, G^r_U(M)_\circ] \subseteq F \qquad \forall U \in \mathcal{U}.$$

Now consider the following subgroup of $\mathrm{Diff}^r(M)$:

$$H := \langle [\mathrm{Diff}^r_V(M)_\circ, \mathrm{Diff}^r_V(M)_\circ] : V \in \mathcal{U}^r(M)\rangle_{\mathrm{Diff}^r(M)}$$

Obviously we have

$$\{Id\} \neq H \subseteq [\mathrm{Diff}^r_c(M)_\circ, \mathrm{Diff}^r_c(M)_\circ],$$

and $H \lhd \mathrm{Diff}^r(M)$, for if $V \in \mathcal{U}^r(M)$, $h_1, h_2 \in \mathrm{Diff}^r_V(M)_\circ$ and $f \in \mathrm{Diff}^r(M)$ then $f(V) \in \mathcal{U}^r(M)$ and thus

$$f[h_1,h_2]f^{-1} = [fh_1f^{-1}, fh_2f^{-1}] \in \underbrace{[f\mathrm{Diff}^r_V(M)_\circ f^{-1}, f\mathrm{Diff}^r_V(M)_\circ f^{-1}]}_{=[\mathrm{Diff}^r_{f(V)}(M)_\circ, \mathrm{Diff}^r_{f(V)}(M)_\circ]} \subseteq H.$$

Since $[\mathrm{Diff}_c^r(M), \mathrm{Diff}_c^r(M)]$ is simple (cf. Corollary 1.17) we obtain

$$H = [\mathrm{Diff}_c^r(M)_\circ, \mathrm{Diff}_c^r(M)_\circ],$$

and it thus remains to show

$$[\mathrm{Diff}_V^r(M)_\circ, \mathrm{Diff}_V^r(M)_\circ] \subseteq F \qquad \forall V \in \mathcal{U}^r(M), \tag{4.10}$$

for then by Remark 4.8

$$[G_c^r(M)_\circ, G_c^r(M)_\circ] = [\mathrm{Diff}_c^r(M)_\circ, \mathrm{Diff}_c^r(M)_\circ] = H \subseteq F.$$

In order to prove (4.10) let $V = \varphi(\mathbf{B}^n) \in \mathcal{U}^r(M)$. Then, since $\mathcal{U}$ covers M, $(\varphi^{-1}(U))_{U\in\mathcal{U}}$ is an open cover of $\mathbf{R}^n$, and we can apply Lemma 4.15. Hence there exist $U_0, \ldots, U_m \in \mathcal{U}$ and $f_i, g_i \in \mathrm{Diff}^\infty_{\varphi^{-1}(U_i)}(\mathbf{R}^n)_\circ$ with

$$[f_m, g_m] \cdots [f_1, g_1](\overline{\mathbf{B}^n}) \subseteq \varphi^{-1}(U_0).$$

If we let

$$\widetilde{f}_i := \begin{cases} \varphi \circ f_i \circ \varphi^{-1} & \text{on } \varphi(\mathbf{R}^n) \\ Id & \text{otherwise} \end{cases}$$

$$\widetilde{g}_i := \begin{cases} \varphi \circ g_i \circ \varphi^{-1} & \text{on } \varphi(\mathbf{R}^n) \\ Id & \text{otherwise} \end{cases}$$

we obtain $\widetilde{f}_i, \widetilde{g}_i \in \mathrm{Diff}_{U_i}^r(M)_\circ$, $\widetilde{h} := [\widetilde{f}_m, \widetilde{g}_m] \cdots [\widetilde{f}_1, \widetilde{g}_1] \in F$, $\widetilde{h}(V) \subseteq U_0 \in \mathcal{U}$ and thus

$$\begin{aligned} [\mathrm{Diff}_V^r(M)_\circ, \mathrm{Diff}_V^r(M)_\circ] &\subseteq [\mathrm{Diff}^r_{\widetilde{h}^{-1}(U_0)}(M)_\circ, \mathrm{Diff}^r_{\widetilde{h}^{-1}(U_0)}(M)_\circ] \\ &= \widetilde{h}^{-1}[\mathrm{Diff}_{U_0}^r(M)_\circ, \mathrm{Diff}_{U_0}^r(M)_\circ]\widetilde{h} \subseteq F. \end{aligned}$$

This completes the proof of Lemma 4.16. □

4.17. Theorem (Filipkiewicz). *[5] Let M and N be connected, smooth manifolds, let $G^r(M)$ be one of the groups $\mathrm{Diff}^r(M)$, $\mathrm{Diff}_c^r(M)$, $\mathrm{Diff}_c^r(M)_\circ$ and let $G^s(N)$ be $\mathrm{Diff}^s(N)$, $\mathrm{Diff}_c^s(N)$ or $\mathrm{Diff}_c^s(N)_\circ$. If $\Phi : G^r(M) \to G^s(N)$ is an isomorphism of groups (we do not assume that Φ is continuous with respect to any topologies) then $r = s$ and there exists an unique C^r-diffeomorphism $\varphi : M \to N$ inducing Φ, i.e.*

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1} \qquad \forall f \in G^r(M).$$

Proof. By Lemma 4.16 we can apply Theorem 4.14 and obtain an unique homeomorphism $\varphi : M \to N$ inducing Φ. Hence it remains to show $r = s$ and φ is a $\mathrm{C}^r - diffeomorphism$.

Sublemma. (cf. p. 77-78 in [2]) *If $m_0 \in M$ and $n = \dim(M)$ then there exists an C^∞-embedding $\iota : \mathbf{R}^n \hookrightarrow M$ with $\iota(0) = m_0$,*

$$\begin{aligned} l : \mathbf{R}^n \times M &\to M \\ (x, m) &\mapsto \begin{cases} \iota(\iota^{-1}(m) + x) & \textit{on } \iota(\mathbf{R}^n) \\ m & \textit{otherwise} \end{cases} \end{aligned}$$

is a C^∞*-action of the Lie group* $(\mathbf{R}^n, +)$ *on* M*, and* $l_x := l(x, \cdot) \in \mathrm{Diff}_c^\infty(M)_\circ$ *for all* $x \in \mathbf{R}^n$*.*

Proof of the Sublemma. Choose $\lambda \in \mathrm{Diff}^\infty([0,1),[0,\infty))$ with

$$\lambda(t) := \begin{cases} t & 0 \le t \le \frac{1}{3} \\ e^{\frac{1}{(1-t)^2}} & \frac{2}{3} \le t < 1 \end{cases}.,$$

and define $\sigma \in \mathrm{Diff}^\infty(\mathbf{B}^n, \mathbf{R}^n)$ by

$$\begin{aligned} \sigma : \mathbf{B}^n &\to \mathbf{R}^n \\ x &\mapsto \frac{\lambda(\|x\|)}{\|x\|} \cdot x \end{aligned}$$

If $y \in \mathbf{R}^n$ we denote by ∂_y the vector field constant y on $\mathbf{R}^n$ and define a vector field X_y on $\mathbf{R}^n$ by

$$X_y(x) := \begin{cases} \sigma^*(\partial_y) & \text{on } \mathbf{B}^n \\ 0 & \text{otherwise} \end{cases}.$$

An elementary calculation, using the special choice of λ, shows that X_y is C^∞ for all $y \in \mathbf{R}^n$. Choose a C^∞-chart ψ on M with $\psi(m_0) = 0$ and $\mathrm{Im}(\psi) = \mathbf{R}^n$. Then the vector field

$$\widetilde{X_y} := \begin{cases} \psi^*(X_y) & \text{on } \mathrm{dom}(\psi) \\ 0 & \text{otherwise} \end{cases}$$

is C^∞ and $\mathrm{Fl}_1^{\widetilde{X_y}} \in \mathrm{Diff}_c^\infty(M)_\circ$ for all $y \in \mathbf{R}^n$. On the other side we have

$$\begin{aligned} \mathrm{Fl}_1^{\widetilde{X_y}}(m) &= \begin{cases} \mathrm{Fl}_1^{\psi^*\sigma^*(\partial_y)}(m) & \text{on } (\sigma\circ\psi)^{-1}(\mathbf{R}^n) \\ \mathrm{Fl}_1^0(m) & \text{otherwise} \end{cases} \\ &= \begin{cases} ((\sigma\circ\psi)^*\mathrm{Fl}_1^{(\partial_y)})(m) & \text{on } (\sigma\circ\psi)^{-1}(\mathbf{R}^n) \\ m & \text{otherwise} \end{cases} \\ &= \begin{cases} (\sigma\circ\psi)^{-1}((\sigma\circ\psi)(m) + y) & \text{on } (\sigma\circ\psi)^{-1}(\mathbf{R}^n) \\ m & \text{otherwise} \end{cases}, \end{aligned}$$

and thus $\iota := (\sigma\circ\psi)^{-1}$ is a C^∞-embedding with $\iota(0) = m_0$ and l defined above is C^∞. Further we have $l_0 = Id_M$, and

$$\begin{aligned} l_x(l_y(m)) &= \begin{cases} \iota(\iota^{-1}(\iota(\iota^{-1}(m) + y)) + x) & \forall m \in \iota(\mathbf{R}^n) \\ l_x(m) & \text{otherwise} \end{cases} \\ &= \begin{cases} \iota(\iota^{-1}(m) + y + x) & \forall m \in \iota(\mathbf{R}^n) \\ m & \text{otherwise} \end{cases} \\ &= l_{x+y}(m), \end{aligned}$$

and thus l is a C^∞-action of $(\mathbf{R}^n, +)$ on M. □

Let $m_0 \in M$ and consider the C^∞-action l constructed in the Sublemma above. Since ϕ is a homeomorphism

$$\begin{aligned} \widetilde{l} : \mathbf{R}^n \times N &\rightarrow N \\ (x, n) &\mapsto \varphi(l_x(\varphi^{-1}(n))) \end{aligned}$$

is a continuous action of $(\mathbf{R}^n, +)$ on N. If we fix $x \in \mathbf{R}^n$, then $\widetilde{l}_x = \Phi(l_x) \in G^s(N)$, since $l_x \in \mathrm{Diff}_c^r(M)_\circ \subseteq G^r(M)$. Hence, by a Theorem of Montgomery and Zippin (chapter 5.2 of [10]) $\widetilde{l}$ is a C^s-action. We have

$$\widetilde{l}_x(\varphi(m_0)) = \varphi(l_x(\varphi^{-1}(\varphi(m_0)))) = \varphi(l_x(m_0)) = \varphi(\iota(x))$$

and thus $\varphi \circ \iota : \mathbf{R}^n \rightarrow N$ is C^s. But since ι^{-1} is a chart around m_0, and since m_0 was arbitrary in M we obtain φ is C^s. Replacing Φ by Φ^{-1} we get φ^{-1} is C^r. Let p be maximal in $\mathbf{N} \cup \{0, \infty\}$ with φ is a C^p-diffeomorphism. What we have done above shows $\min\{r, s\} \leq p$. Now suppose $r \neq s$ and assume without loss of generality $r < s$. Then $p < \min\{r, s\} = r$, for otherwise φ would induce an isomorphism $G^r(M) \rightarrow G^s(N) \cap \mathrm{Diff}^r(N) \neq G^s(N)$. But then $p < \min\{r, s\} \leq p$, a contradiction. So we must have $r = s \leq p$, and thus we have finished the proof of Theorem 4.17. □

Appendix

A. Isomorphisms between permutation groups

One could ask whether Theorem 4.17 remains valid if we replace the manifolds M, N by finite sets X, Y and $\mathrm{Diff}^r_c(M)$, $\mathrm{Diff}^s_c(N)$ by $\Sigma(X)$, $\Sigma(Y)$ (cf. Definition A.3). The answer is 'yes', except in the case $|X| = 6$ (cf. Corollary A.8 and Theorem A.10). The existence of this exceptional case made me present this elementary group theoretical result here. For references see [12].

A.1. Definition. If G is a group we denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, i.e. the group of isomorphisms $G \to G$. If $g \in G$ we let $\mathrm{conj}_g \in \mathrm{Aut}(G)$ be given by $\mathrm{conj}_g(h) := ghg^{-1}$. The group of *inner automorphisms* of G is

$$\mathrm{Inn}(G) := \{\mathrm{conj}_g : g \in G\}.$$

For an arbitrary subset $A \subseteq G$ the *centralizer of A in G* is

$$\mathrm{C}_G(A) := \{g \in G : ga = ag \quad \forall a \in A\},$$

and the *center of G* is $\mathrm{C}(G) := \mathrm{C}_G(G)$.

If X is a set we denote by $|X|$ the cardinality of X. $|G|$ is called the *order of the group G*. If $|G| < \infty$ and $g \in G$ then the minimal integer $n \geq 1$ with $g^n = e$ is called the *order of g*. For a subgroup $H \subseteq G$ we denote by G/H the set of right cosets of H in G. The index of H in G is $|G/H|$.

A.2. Remark. If G is a group, $g \in G$ and $\varphi \in \mathrm{Aut}(G)$ we have

$$(\varphi \mathrm{conj}_g \varphi^{-1})(x) = \varphi(g\varphi^{-1}(x)g^{-1}) = \varphi(g)x\varphi(g)^{-1} = \mathrm{conj}_{\varphi(g)}(x),$$

and thus $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$. So $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is as well a group. $\mathrm{C}_G(A)$ is a subgroup of G and $\mathrm{C}(G) \lhd G$. The mapping $G \to \mathrm{Inn}(G) : g \mapsto \mathrm{conj}_g$ is a surjective homomorphism with kernel $\mathrm{C}(G)$.

A.3. Definition. If X is a set, we denote by $\Sigma(X)$ the *group of permutations* of X, i.e. the group of bijective mappings $X \to X$. Especially we let $\Sigma_n := \Sigma(\{1, \ldots, n\})$. In analogy to Definition 4.1 we denote by

$$S_x\Sigma(X) := \{\pi \in \Sigma(X) : \pi(x) = x\}$$

the *isotropy group of $x \in X$*. If $a_1, \ldots, a_r \in \{1, \ldots, n\}$ with $a_i \neq a_j$ for $i \neq j$ we denote by $(a_1, \ldots, a_r)$ the permutation in Σ_n given by

$$\begin{array}{rcll} a_i & \mapsto & a_{i+1} & \forall 1 \leq i \leq r-1 \\ a_r & \mapsto & a_1 & \\ m & \mapsto & m & \text{otherwise,} \end{array}$$

and call it an *r-cycle*. 2-cycles are also called *transpositions*. Two cycles $(a_1, \dots, a_r)$ and $(b_1, \dots, b_s)$ are called *disjoint* iff $\{a_1, \dots, a_r\} \cap \{b_1, \dots, b_s\} = \emptyset$.

A.4. Remark. It is well known that $|\Sigma_n| = n!$ and that every permutation in Σ_n can be written as a composition of disjoint cycles. Moreover $\langle\{(1,i) : i = 2, \dots, n\}\rangle_{\Sigma_n} = \Sigma_n$, and hence every permutation can be written as composition of transpositions. Such a representation is not unique, but if $\pi = \tau_r \cdots \tau_1 = \sigma_s \cdots \sigma_1$, where τ_i, σ_j are transpositions, then $r \equiv s \pmod 2$. If r is even resp. odd we call π an *even* resp. *odd permutation* and let $\mathrm{sgn}(\pi) := 1$ resp. $\mathrm{sgn}(\pi) := -1$. $\mathrm{sgn} : \Sigma_n \to (\pm 1, \cdot)$ is a homomorphism, and it is onto iff $n \geq 2$. We call $A_n := \ker(\mathrm{sgn})$, the *alternating group* of order n. We have $A_n \lhd \Sigma_n$ and $A_n = \langle\{3-\text{cycles}\}\rangle_{\Sigma_n}$.

If $n \geq 3$ and $Id \neq \sigma \in \Sigma_n$ then there exists a transposition which does not commute with σ (let $\tau := (i,j)$ where $\sigma(i) \neq i$ and $j \neq i, \sigma(i)$; then $\tau\sigma\tau^{-1}(i) \neq \tau\sigma\tau^{-1}(j) = \sigma(i)$) and thus $\mathrm{C}(\Sigma_n) = \{Id\}$.

A.5. Lemma. *Let $\Phi \in \mathrm{Aut}(\Sigma_n)$ and assume that Φ maps transpositions to transpositions. Then $\Phi \in \mathrm{Inn}(\Sigma_n)$.*

Proof. Since $\Phi((1,2))$ is a transposition there exist $a_1 \neq a_2 \in \{1, \dots, n\}$ with $\Phi((1,2)) = (a_1, a_2)$. Similarly there exist $b \neq a_3 \in \{1, \dots, n\}$ with $\Phi((1,3)) = (b, a_3)$. Since the order of an element of a group is invariant under isomorphisms, and since

$$(b, a_3)(a_1, a_2) = \Phi((1,3))\Phi((1,2)) = \Phi((1,3)(1,2)) = \Phi((1,2,3))$$

is of order 3, $\{a_1, a_2\}$ and $\{b, a_3\}$ must have exactly one element in common, say $a_1 = b$. Analogously we find $a_1, \dots, a_n \in \{1, \dots, n\}$ with $\Phi((1,i)) = (a_1, a_i)$ for $i = 2, \dots, n$. Since Φ is bijective we have $a_i \neq a_j$ for $i \neq j$ and hence we may define $\varphi \in \Sigma_n$ by $\varphi(i) := a_i$. We then have

$$\Phi((1,i)) = (a_1, a_i) = \varphi(1,i)\varphi^{-1} = \mathrm{conj}_\varphi((1,i))$$

and thus $\Phi = \mathrm{conj}_\varphi \in \mathrm{Inn}(\Sigma_n)$, since $\{(1,i) : i = 2, \dots, n\}$ generates Σ_n. $\square$

A.6. Theorem. *If $n \neq 6$ then* $\mathrm{Inn}(\Sigma_n) = \mathrm{Aut}(\Sigma_n)$.

Proof. Let $\Phi \in \mathrm{Aut}(\Sigma_n)$ and let $\tau \in \Sigma_n$ be a transposition. Counting the elements of Σ_n that commute with τ one obtains

$$|\mathrm{C}_{\Sigma_n}(\{\tau\})| = (n-2)! \cdot 2.$$

Clearly, since $\Phi \in \mathrm{Aut}(\Sigma_n)$ the order of $\Phi(\tau)$ is as well 2, and thus $\Phi(\tau)$ can be written as a composition of disjoint 2-cycles, i.e.

$$\Phi(\tau) = (a_1, b_1) \cdots (a_r, b_r)$$

where $a_1, b_1, \dots, a_r, b_r$ are all distinct, and $1 \leq r \leq \frac{n}{2}$. We then have

$$|\mathrm{C}_{\Sigma_n}(\{\Phi(\tau)\})| = (n-2r)! 2^r r!,$$

but since $\Phi \in \mathrm{Aut}(\Sigma_n)$ we must have

$$(n-2)! \cdot 2 = |\mathrm{C}_{\Sigma_n}(\{\tau\})| = |\mathrm{C}_{\Sigma_n}(\{\Phi(\tau)\})| = (n-2r)!2^r r!. \tag{A.1}$$

If $r = 1$ (A.1) holds $\forall n \geq 2r$. If $r = 2$ there is no $2r \leq n \in \mathbf{N}$, satisfying (A.1). If $r = 3$ then $n = 6$ is the unique solution of (A.1) with $2r \leq n \in \mathbf{N}$. If $r \geq 4$ then there is no $2r \leq n \in \mathbf{N}$ satisfying (A.1), for if $2r \leq n$ and $r \geq 4$ then

$$\begin{aligned}
&(n-2r+1)\underbrace{(n-2r+2)}_{\geq 2}\cdots(n-3)\underbrace{(n-2)}_{\geq 2(r-1)} \geq \\
\geq\ & (n-2r+1)(n-2r+3)\cdots\underbrace{(n-3)}_{\geq 2r-3}\cdot\underbrace{2\cdot 4\cdot 6\cdots 2(r-1)}_{=2^{r-1}(r-1)!} \\
\geq\ & (2r-3)2^{r-1}(r-1)! \\
>\ & r2^{r-1}(r-1)! = 2^{r-1}r!,
\end{aligned}$$

and thus

$$\begin{aligned}
(n-2)! \cdot 2 &= (n-2r)! \cdot 2 \cdot (n-2r+1) \cdot (n-2r+2) \cdots (n-2) \\
&> (n-2r)! \cdot 2 \cdot 2^{r-1} r! = (n-2r)! 2^r r!,
\end{aligned}$$

a contradiction to (A.1). Since we have assumed $n \neq 6$ we obtain $r = 1$ and hence Φ maps transpositions to transpositions. So we can apply Lemma A.5 and obtain $\mathrm{Aut}(\Sigma_n) = \mathrm{Inn}(\Sigma_n)$. □

A.7. Corollary to the proof of Theorem A.6. *If $\Phi \in \mathrm{Aut}(\Sigma_6)$ then Φ maps transpositions either to transpositions or to compositions of exactly three disjoint transpositions.*

Proof. If $n = 6$ we found $r = 1$ or $r = 3$ (cf. proof of Theorem A.6), but this is Corollary A.7. □

A.8. Corollary. *Let X and Y be two finite, nonempty sets, and let $\Phi : \Sigma(X) \to \Sigma(Y)$ be an isomorphism of groups. Then $|X| = |Y|$ and, if we assume additionally $|X| \neq 6$, there exists a bijection $\varphi : X \to Y$ inducing Φ, i.e.*

$$\Phi(\pi) = \varphi \circ \pi \circ \varphi^{-1} \qquad \forall \pi \in \Sigma(X).$$

Moreover φ is unique, provided $|X| \geq 3$.

Proof. Since $|X|! = |\Sigma(X)| = |\Sigma(Y)| = |Y|!$, we obtain $|X| = |Y|$, and we may choose a bijection $\psi : X \to Y$. Now consider the isomorphism of groups $\Psi : \Sigma(X) \to \Sigma(Y)$ given by $\Psi(\pi) := \psi \circ \pi \circ \psi^{-1}$. We then have $\Psi^{-1}\Phi \in \mathrm{Aut}(\Sigma(X))$, and hence by Theorem A.6 there exists $\sigma \in \Sigma(X)$ with $\Psi^{-1}\Phi = \mathrm{conj}_\sigma$. Thus, for every $\pi \in \Sigma(X)$ we have

$$\Phi(\pi) = \Psi(\mathrm{conj}_\sigma(\pi)) = \Psi(\sigma\pi\sigma^{-1}) = \psi \circ \sigma\pi\sigma^{-1} \circ \psi^{-1} = \varphi \circ \pi \circ \varphi^{-1},$$

where $\varphi := \psi \circ \sigma : \Sigma(X) \to \Sigma(Y)$.

Suppose $\varphi' : X \to Y$ is another bijection inducing Φ. Then $\varphi^{-1} \circ \varphi' \in \Sigma(X)$ and $\mathrm{conj}_{\varphi^{-1}\circ\varphi'} = Id_{\Sigma(X)}$, for

$$\pi = \Phi^{-1}(\Phi(\pi)) = \Phi^{-1}(\varphi' \circ \pi \circ \varphi'^{-1}) = (\varphi^{-1} \circ \varphi')\pi(\varphi'^{-1} \circ \varphi) = \mathrm{conj}_{\varphi^{-1}\circ\varphi'}(\pi).$$

Since $|X| \geq 3$, $\Sigma(X)$ has trivial center (cf. Remark A.4) and hence $\mathrm{conj}_{\varphi^{-1}\circ\varphi'} = Id_{\Sigma(X)}$ yields $\varphi^{-1} \circ \varphi' = Id_X$ (cf. Remark A.2). Thus ϕ is unique. □

A.9. Lemma. *If $n \neq 4$ then $\{Id\}$, A_n and Σ_n are all normal subgroups of Σ_n.*

Proof. Assume $n \geq 3$ and let $\{Id\} \neq N \lhd \Sigma_n$. Now choose $Id \neq \sigma \in N$ and a transposition $\tau \in \Sigma_n$, that does not commute with σ (cf. Remark A.4). We then have $Id \neq \rho := [\sigma, \tau] \in N$, since $N \lhd \Sigma_n$, and ρ is a composition of two transpositions, namely $\sigma^{-1}\tau^{-1}\sigma$ and τ. Thus ρ is either a 3-cycle or a composition of two disjoint transpositions. Conjugating ρ we obtain that N either contains all 3-cycles or N contains all compositions of two disjoint transpositions. In the second case we must have $n \geq 5$ by assumption. So given any 3-cycle (a, b, c) we find (d, e) disjoint to (a, b, c). Then

$$(a, b, c) = \underbrace{(a, c)(d, e)}_{\in N}\,\underbrace{(d, e)(a, b)}_{\in N},$$

and thus, even in the second case, N contains all 3-cycles. Since the 3-cycles generate A_n we have $\mathrm{A}_n \subseteq N$ and consequently $N = \mathrm{A}_n$ or $N = \Sigma_n$, since $\mathrm{A}_n \subseteq \Sigma_n$ has index 2. □

A.10. Theorem. *We have*

$$\mathrm{Out}(\Sigma_6) := \mathrm{Aut}(\Sigma_6)/\mathrm{Inn}(\Sigma_6) \cong \mathbf{Z}/2\mathbf{Z},$$

and hence Corollary A.8 does not hold in the case $|X| = 6$.

Proof. Suppose there exist $\Phi, \Psi \in \mathrm{Aut}(\Sigma_6) \setminus \mathrm{Inn}(\Sigma_6)$. Then, by Corollary A.7 and Lemma A.5, Φ and Ψ map transpositions to compositions of exactly three disjoint transpositions. In Σ_6 there are $\binom{6}{2} = 15$ transpositions and $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15$ compositions of three disjoint transpositions. Hence $\Psi^{-1}\Phi$ maps transpositions to transpositions and thus, by Lemma A.5, we obtain $\Psi^{-1}\Phi \in \mathrm{Inn}(\Sigma_6)$. Summing up we have

$$|\mathrm{Aut}(\Sigma_6)/\mathrm{Inn}(\Sigma_6)| \leq 2.$$

We finish the proof by showing $\mathrm{Aut}(\Sigma_6) \setminus \mathrm{Inn}(\Sigma_6) \neq \emptyset$. Consider Σ_5 and denote by S_5 the set of those subgroups of Σ_5, that have order 5. Since every subgroup in Σ_5 of order 5 consists of $\{Id\}$ and four 5-cycles, and since $U \cap V = \{Id\}$ $\forall U \neq V \in S_5$, and since there are $\frac{5!}{5} = 4!$ 5-cycles in Σ_5, we get $|S_5| = \frac{4!}{4} = 6$. Now consider the following homomorphism:

$$\begin{array}{rcl} \Sigma_5 & \xrightarrow{\phi} & \Sigma(S_5) \\ \pi & \mapsto & (U \mapsto \pi U \pi^{-1} \qquad \forall U \in S_5) \end{array}$$

Since $A_5 \ni (123) \notin \ker(\phi) \lhd \Sigma_5$, we obtain from Lemma A.9, $\ker(\phi) = \{Id\}$ and thus ϕ is injective. So $H := \phi(\Sigma_5)$ is a subgroup of index 6, but H is no isotropy group, since Σ_5 acts (via ϕ) transitively on S_5. Now consider $M := \Sigma(S_5)/H$ and the homomorphism

$$\begin{array}{rcl} \Sigma(S_5) & \stackrel{\Psi}{\to} & \Sigma(M) \\ \pi & \mapsto & (\sigma H \mapsto \pi\sigma H \qquad \forall \sigma \in \Sigma(S_5)). \end{array}$$

Then $|M| = 6$ and $\ker(\Psi) \subseteq H$. From Lemma A.9 we obtain $\ker(\Psi) = \{Id\}$, and so Ψ is an isomorphism. Hence $\Psi(H) \subseteq \Sigma(M)$ is a subgroup of index 6, that fixes $H \in M$ and thus $\Psi(H) = S_H\Sigma(M)$. Using isomorphisms $\Sigma(S_5) \cong \Sigma_6 \cong \Sigma(M)$ induced from bijections $S_5 \cong \{1, \ldots, 6\} \cong M$ we obtain $\Psi' \in \mathrm{Aut}(\Sigma_6)$ and a subgroup $H' \subseteq \Sigma_6$, such that $\Psi'(H')$ is an isotropy group, but H' is not. So Ψ' can't be inner, and thus $|\mathrm{Out}(\Sigma_6)| = 2$. $\square$

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Lebenslauf

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