

# Applied Analysis - Variational methods

WS 18

The concept of "well-posedness"

$D \subset D$  (top.) space of data

$\mathcal{F}$  (top.) space of solutions

$T: D \rightarrow \mathcal{F}$  solution operator

Hadamard's well-posedness:

- 1) Existence:  $\forall d \in D$  one can def.  $T(d) \in \mathcal{F}$
- 2) Uniqueness: the latter  $T(d)$  is unique
- 3) Continuous dependence:  $T$  is continuous

Example: Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega \subset \mathbb{R}^d \text{ nonempty, open, bounded,} \\ u = g & \text{on } \partial\Omega \text{ with smooth boundary} \end{cases}$$

$$D = \{(f, g) \in C(\bar{\Omega}) \times C(\partial\Omega)\}$$

$$\mathcal{F} = \{u \in C^2(\bar{\Omega}), \cap C(\bar{\Omega})\}$$

A different view  $S = T^{-1}$

$$S : \mathcal{F} \rightarrow D \subset \mathcal{D}$$

$$S(u) = (-\Delta u, u|_{\partial\Omega})$$

- 1) Existence:  $S$  is onto  $D$
- 2) Uniqueness:  $S$  is injective
- 3) Cont. dependence:  $S^{-1}$  is continuous



Remark: Continuous dependence  $\Rightarrow$  uniqueness  
 $\Leftarrow$

$$\int_{\Omega} \cdot \cdot \cdot$$

$$\text{Ex: } D = \mathcal{D} = \mathcal{F} = \mathbb{R}, T(d) = \begin{cases} 0 & \text{if } d \in \mathbb{Q} \\ 1 & \text{if } d \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Example Neumann problem

$$\begin{cases} -\Delta u = 0 \text{ on } \Omega \\ \frac{\partial u}{\partial n} = \gamma \text{ on } \partial\Omega \end{cases}$$

$$D = \{ g \in C(\partial\Omega) : \int_{\partial\Omega} g \, ds = 0 \}$$

$$\mathcal{F} = \{ u \in C^2(\Omega) : \frac{\partial u}{\partial n} \in C(\partial\Omega) \}$$

$$\text{No uniqueness: } S^{-1}(g) = u + R.$$

## A first variational idea

$\Omega \subset \mathbb{R}^d$  smooth and bounded domain and  $f \in C(\bar{\Omega})$ ,  
and define  $\mathcal{F} = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) : v=0 \text{ on } \partial\Omega\}$  and

$$E(u) = \int_{\Omega} \frac{1}{2} |\nabla u(x)|^2 - f(x) u(x) dx \quad \forall u \in \mathcal{F}.$$

Then the following are equivalent:

$$1) \begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$2) u \in \mathcal{F} \text{ and } E(u) \leq E(v) \quad \forall v \in \mathcal{F}$$

If Ad 1)  $\Rightarrow$  2). Fix  $v \in \mathcal{F}$ , letting  $w = v-u$  we have

$$\begin{aligned} E(v) &= E(u+w) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \nabla u \cdot \nabla w + \frac{1}{2} |\nabla w|^2 dx \\ &\quad - \int_{\Omega} fu dx - \int_{\Omega} fw dx = \\ &= E(u) + \underbrace{\int_{\Omega} (-\Delta u - f) w dx}_{=0 \text{ by 1)}} + \underbrace{\int_{\Omega} \frac{1}{2} |\nabla w|^2 dx}_{\geq 0} \geq 0. \end{aligned}$$

Ad 2)  $\Rightarrow$  1) Fix  $w \in \mathcal{F}$  and define  $h \in (-1, 1) \mapsto \varphi(h)$ ,  
 $\varphi(h) = E(u+hw)$ . As  $u$  is a minimizer  $\varphi'(0) = 0$   
In particular:

$$0 = \varphi'(0) = \int_{\Omega} \nabla u \cdot \nabla w - fw dx = \int_{\Omega} (-\Delta u - f) w dx.$$

Since this holds for all  $w \in \mathcal{F}$  we conclude that  $-\Delta u = f$ .  $\square$

## The Direct Method

$f: (X, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $f \neq +\infty$  (proper)

$f$  is coercive iff  $\exists K \subset X$  s.t.  $\inf_X f = \inf_K f$

$f$  is lower semicontinuous iff

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n) \quad \forall x_n \rightarrow x.$$

○ thm (Direct Method) The problem  $\min_f$  has a sol.

+ Let  $x_m$  be an infimizing sequence:

$f(x_m) \rightarrow \inf_X f$ . As  $f$  is coercive we can assume w.o.l.o.g. that  $x_m \in K$ . As  $K$  is compact we can extract  $x_{m_k}$  such that  $x_{m_k} \rightarrow x$ . The lower semicontinuity implies that

$$f(x) \leq \liminf_{k \rightarrow +\infty} f(x_{m_k}) = \lim_{k \rightarrow +\infty} f(x_{m_k}) = \inf_X f.$$

This entails that  $f(x) \equiv \inf_X f$  which is actually a minimum.

Cor (Weierstrass theorem)  $f: K \subset \mathbb{R}^d \rightarrow \mathbb{R}$  continuous. Then  $f$  attains its minimum.

$f$  continuous  $\Rightarrow$   $f$  lower semicontinuous.  $\square$

Remark without coercivity or lower semicontinuity (l.s.c.)  
the statement does not hold

Remark coercivity and l.s.c. are not necessary in  
order to find a minimum

Cor (Direct method, even easier) let  $f: X, d \rightarrow \mathbb{R} \cup \{+\infty\}$   
proper with compact sublevels. Then  $\min_X f$  is solvable

- p let  $\bar{x} \in X$  such that  $f(\bar{x}) < +\infty$ . Then  $\inf_X f = \inf_K f$  with  $K = \{x \in X : f(x) < f(\bar{x})\} \subset X$ .  
Hence,  $f$  is coercive

Assume by contradiction that  $\exists x_n \rightarrow x$  with  
 $f(x) \geq l + 2\varepsilon$ , with  $l = \liminf f(x_n)$ ,  $\varepsilon > 0$ .  
A subsequence of  $x_n$  belongs to  $K = \{f \leq l + \varepsilon\}$ .  
Since this is compact, its limit  $x$  has to  
belong to  $K$  as well. Hence,

$$f(x) \leq l + \varepsilon < l + 2\varepsilon \leq f(x)$$

a contradiction.  $\square$

Remark compactness of sublevels  $\Rightarrow$   $\begin{cases} \text{coercivity} \\ \text{l.s.c.} \end{cases} \Leftrightarrow$

example  $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Remark  $f$  l.s.c.  $\Leftrightarrow \{f \leq l\}$  closed  $\forall l$

In order to apply the Direct Method to our Dirichlet problem we hence have to discuss (separable) topologies on spaces of functions. We hence need some:

### Results from functional analysis

○ Banach spaces, dual spaces, duality and dual norms, canonical injection.

Hilbert spaces, projections, Riesz representation, bases,

$L^p$ ,  $W^{k,p}$  spaces

Weak topologies, convexity; Banach-Alaoglu-Banach-

○ Separability,  $E'_{\text{sep}} \Rightarrow E_{\text{sep}}$ , metrizability of weak top.  $E'$

Reflexivity, Representation in  $L^p$ , uniform convexity,  $E_{\text{sep. reflex}} \Leftrightarrow E'_{\text{sep. reflex}}$ , enhancement of weak convergence

Nemytskii operators