

Lax-Milgram-Lemma

Then $a: H \times H \rightarrow \mathbb{R}$ bilinear continuous and coercive. For all $f \in H$ there exists a unique $u \in H$ such that $a(u, v) = (f, v) \forall v \in H$. If a is symmetric, u is the unique minimizer of

$$J(u) = \frac{1}{2} a(u, u) - (f, u).$$

Proof ① For $a \in H$ define the linear and continuous operator $v \mapsto a(u, v) \in H'$ and let Au be its representative, namely $(Au, v) = a(u, v)$ for all $v \in H$. The first assertion of the theorem corresponds to the statement

$$\forall f \in H \exists u \in H: Au = f.$$

We hence have to check that $A: H \rightarrow H$ is bijective.

1) A is injective: Let $Au_1 = Au_2$, then

$$\alpha \|u_1 - u_2\|^2 \leq (Au_1 - Au_2, u_1 - u_2) = 0$$

and $u_1 = u_2$

2) A is onto:

2.1) $Rg(A)$ is closed: Let $Au_m \rightarrow \bar{A}$. Since

$$\alpha \|u_m\| \leq M \|Au_m\| \text{ one can extract and find}$$

$u_{m_k} \rightarrow u$. Then $Au_m \rightarrow Au = \bar{A}$. Hence $\bar{A} \in Rg(A)$.

2.2) $Rg(A)$ is dense: Let $v \in Rg(A)^\perp$, namely

$$(Au, v) = 0 \quad \forall u \in H. \text{ By choosing } u = v \text{ we get } \alpha \|v\|^2 \leq (Av, v) = 0, \text{ namely } v = 0.$$

Let now a be symmetric and $a(u, v) = \langle f, v \rangle$ $\forall f \in H$.
 One has that

$$\begin{aligned} J(v) &= J(u + w) \stackrel{w=v-u}{=} \frac{1}{2} a(u+w, u+w) - \langle f, u+w \rangle \\ &= \frac{1}{2} a(u, u) + a(u, w) + \frac{1}{2} a(w, w) - \langle f, u \rangle - \langle f, w \rangle \\ &= J(u) + a(w, w) \geq J(u) \end{aligned}$$

↑

with equality iff $w=0$,

namely iff $u=v$.

On the other hand, let u minimize J . Then, there exists the function $t \in (-1, 1) \mapsto g(t) := J(u + tv)$ is a quadratic polynomial in t and minimized for $t=0$. In particular

$$\begin{aligned} 0 &= g'(0) = \left(\frac{1}{2} a(u, u) + t a(u, v) + \frac{t^2}{2} a(v, v) \right) \Big|_{t=0} = \\ &\quad - \langle f, u \rangle - t \langle f, v \rangle \\ &= a(u, v) - \langle f, v \rangle. \end{aligned}$$

Note that the minimizer of J is unique, for J is uniformly convex. In particular, assume $u_1 \neq u_2$ to minimize J . Then

$$\begin{aligned} J\left(\frac{u_1+u_2}{2}\right) &= \frac{1}{8} a(u_1, u_1) + \frac{1}{4} a(u_1, u_2) + \frac{1}{8} a(u_2, u_2) \\ &\quad - \frac{1}{2} \langle f, u_1 \rangle - \frac{1}{2} \langle f, u_2 \rangle = \frac{1}{2} J(u_1) + \frac{1}{2} J(u_2) - \frac{1}{8} \|u_1 - u_2\|^2 \\ &< \frac{1}{2} J(u_1) + \frac{1}{2} J(u_2) = \min J \end{aligned}$$

a contradiction. \square

Proof (Galerkin method, conformal finite el)
 Assume additionally that H is separable
 and let $\{e_n\}$ denote an orthonormal basis for H . For all $n \in \mathbb{N}$, define the finite-dimensional $H_n = \text{span} \{e_1, \dots, e_n\}$, and consider the finite-dimensional problem:

$$\text{find } u_n \in H_n : a(u_n, v_n) = (f, v_n) \quad \forall v_n \in H_n \quad (1)$$

- Step 1: Problem (1) is solvable. Rewrite everything in terms of coordinates $\{e_1, \dots, e_n\}$. Namely,

$$u_n = \sum_{i=1}^n \underbrace{(u, e_i)}_{\vec{u}_i} e_i \xrightarrow{\text{associate}} \vec{u}$$

$$v_n = \sum_{i=1}^n \underbrace{(v, e_i)}_{\vec{v}_i} e_i \xrightarrow{\text{associate}} \vec{v}$$

- Then, $a(u_n, v_n) = (f, v_n) \quad \forall v_n \in H_n$ iff

$$\vec{u} \cdot A \vec{v} = \vec{F} \cdot \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n \quad (2)$$

where

$$A_{ij} = a(e_i, e_j), \quad F_i = (f, e_i)$$

Problem (2) is then equivalent to

$$A^T \vec{u} = \vec{F} \quad (3)$$

Problem (3) has a unique solution since A is positive definite:

$$(A\vec{U}, \vec{U}) = a(u_n, u_n) \geq \alpha \|u_n\|^2 = \alpha \|\vec{U}\|$$

for all $\vec{U} \in \mathbb{R}^n$, where $u_n = \sum_{i=1}^n U_i e_i$.

Step (2): Estimate We have that

$$\begin{aligned} \alpha \|u_n\|^2 &\leq a(u_n, u_n) = (f, u_n) \\ &\leq \|f\| \|u_n\| \end{aligned}$$

In particular, u_n is bounded in H , independently of n .

Step (3): Convergence. By extracting a subsequence we have that $u_{n_k} \rightarrow u$.

Let now $v \in H$ be fixed and define

$$v_n = \sum_{i=1}^m (v, e_i) e_i, \text{ so that } v_n \rightarrow v.$$

We have that

$$\begin{aligned} a(u, v) &= \lim_{k \rightarrow \infty} a(u_{n_k}, v) = \\ &= \lim_{k \rightarrow \infty} \left(a(u_{n_k}, v_{n_k}) + a(u_{n_k}, v - v_{n_k}) \right) = \\ &= \lim_{k \rightarrow +\infty} (f, v_{n_k}) = (f, v). \end{aligned}$$

Step (4): uniqueness Let u_1 and u_2 be two solutions. Then $u_1 - u_2$ is a solution with $f = 0$, by linearity, namely $a(u_1 - u_2, v) = 0$ $\forall v \in V$. Choose now $v = u_1 - u_2$ and get

$$\alpha \|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) = 0$$

Hence, $u_1 = u_2$.

Step (5): Direct proof in the symm. case. Define

$$E_n(v_n) = \begin{cases} \frac{1}{2}a(v_n, v_n) - (f, v_n) & \text{for } v_n \in H_n \\ +\infty & \text{elsewhere in } H/H_n. \end{cases}$$

In coordinates, we can rewrite $E_n(v_n) = G(\vec{v})$ for all $v_n \in H_n$, $\vec{v} = \sum_{i=1}^n (v_n, e_i) e_i \in \mathbb{R}^n$ as

$$G(\vec{v}) = \frac{1}{2} \vec{v} \cdot A \vec{v} - F \cdot \vec{v}$$

The latter is a quadratic polynomial in \vec{v} , with positive definite Hessian A . It hence has a unique minimum point \vec{u} . It follows that E_n has the unique minimum point u_n . By arguing as in steps (2) - (3), we find a subsequence $u_n \xrightarrow{w_k} u$. The sublevels of E are convex and strongly closed, hence weakly closed. This implies l.s.c. of E . For every v let $v_n = \sum_{i=1}^n (v, e_i) e_i$ so that $v_n \xrightarrow{w_k} v$. We have

$$\begin{aligned}
 E(u) &\leq \liminf_{K \rightarrow \infty} E(u_{n_k}) = \\
 &= \liminf_{K \rightarrow \infty} E_{n_k}(u_{n_k}) \leq \\
 &\leq \liminf_{K \rightarrow \infty} E_{n_k}(v_{n_k}) = E(v)
 \end{aligned}$$

So that u is a minimizer of E . \square

A semilinear elliptic problem

Let's consider $\begin{cases} -\Delta u + \beta(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

where $\Omega \subset \mathbb{R}^d$ nonempty, open, smooth, and bounded
 $\beta: \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz continuous, non-decreasing,
with $\beta(0) = 0$, and $f \in L^2(\Omega) =: H$.

- We recast this problem variationally by letting $V = H_0^1(\Omega)$, $A, B: V \rightarrow V^*$, and $l \in V^*$ defined as

$$\langle Au, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$\langle Bu, v \rangle = \int_{\Omega} \beta(u) v \, dx$$

$$\langle l, v \rangle = \int_{\Omega} fv \, dx.$$

Note that $\|B u\|_{V^*} = \sup_{\|v\|_V=1} |\langle Bu, v \rangle|$

$$\leq \sup_{\|v\|_V=1} \|\beta(u)\|_H \|v\|_H \leq$$

$$\leq \sup_{\|v\|_V=1} \|\beta(u)\|_H c_{V \hookrightarrow L^2} \|v\|_V = c_{V \hookrightarrow H} \left(\int_{\Omega} |\beta(u)|^2 \right)^{1/2}$$

$$\leq c_{V \hookrightarrow H} L_{\beta} \|u\|_H \leq c_{V \hookrightarrow H}^2 L_{\beta} \|u\|_V < \infty$$

for all $u \in V$.