

Strategy

- 1) Replace β by $\beta_K(r) = \max\{-k, \min\{\beta(r), k\}\}$
- 2) Solve the problem (P_n) with β_n
- 3) Pass to the limit as $K \rightarrow +\infty$

O Problem (P_n) find $u_n \in V$: $Au_n + Bu_n = \ell$

We solve this by a fixed-point argument:

Let $\tilde{K} = \{v \in H : \|v\|_H \leq M\}$, solve
 $Au + Bu_n = \ell$ by Lax-Milgram-Lions.

This defines a map

$$S: K \rightarrow V \subset H$$

We would like to use Schauder's fixed point theorem to find $u \in K$: $Su = u$
and therefore solve Problem (P_n) .

Schauder theorem Let $S: K \rightarrow K \subset H$
continuous, K convex and closed, and $S(K)$
compact. Then S has a fixed point.

Application

- 1) Call $u = S\tilde{u}$. Then $Au + B_k^{\tilde{u}} = f$. Test on u and get

$$\begin{aligned} \alpha \|u\|_V^2 &\leq \langle Au, u \rangle = \langle f, u \rangle - \langle B_k^{\tilde{u}}, u \rangle \\ &\leq \|f\|_{V^*} \|u\|_V + \|B_k^{\tilde{u}}\|_H \|u\|_H \\ &\leq \|f\|_{V^*} \|u\|_V + k(\gamma)^{1/2} c_{V \hookrightarrow H} \|u\|_V \\ \Rightarrow \|u\|_H &\leq c_{V \hookrightarrow H} \|u\|_V \leq \\ &\leq \frac{c_{V \hookrightarrow H}}{\alpha} \left(\|f\|_{V^*} + k(\gamma)^{1/2} c_{V \hookrightarrow H} \right) =: M \end{aligned}$$

Hence S maps K into K

- 2) Since $\|\tilde{u}\|_V \leq \frac{M}{c_{V \hookrightarrow H}} =: M'$ one has that
 $S(K)$ is bounded in V hence compact in H
- 3) The set K is obviously convex and closed, for it is a closed ball.
- 4) We just need to check for continuity in H . To this end, let $\tilde{u}_n \rightarrow \tilde{u}$ in H and define $u = S\tilde{u}$ and $u_n = S\tilde{u}_n$. We need to show that $u_n \rightarrow u$ in H .

Use again the estimate to show that $\|u_n\|_V \leq M$.
Hence one can use the compactness of the embedding $V \hookrightarrow H$ in order to extract a subsequence such that $u_n \xrightarrow{V} v$ and $u_n \xrightarrow{H} v$.
One now just needs to check that indeed $v = u$.
In order to do so, we will prove that v solves $Av + B_k^* \tilde{u} = l$ and that the latter has just one solution, namely $u = S\tilde{u}$.

O 1) v solves $Av + B_k^* \tilde{u} = l$ We have that

$$\tilde{u}_n \xrightarrow{H} \tilde{u} \Rightarrow B_k^* \tilde{u}_n \xrightarrow{H} B_k^* \tilde{u}$$

$$\left[\int_R |\beta_k(\tilde{u}_n) - \beta_k(\tilde{u})|^2 \leq L_B^2 \int_R \|\tilde{u}_n - \tilde{u}\|^2 \rightarrow 0 \right]$$

$$u_n \xrightarrow{V} v \Rightarrow Au_n \xrightarrow{V'} Av$$

$$\begin{aligned} \left[\langle Au_n, w \rangle &= \int_R Du_n \cdot Dw \, dx = \\ &= (u_n, w) \rightarrow (v, w) = \langle Av, w \rangle \right] \end{aligned}$$

We can pass to the limit in

$$Au_n + B_k^* \tilde{u}_n = l$$

$$\downarrow V' \quad \downarrow H \quad \downarrow$$

$$Av + B_k^* \tilde{u} = l$$

So v solves the equation.

2) Uniqueness for $Au + B\tilde{u} = \ell$.

Let u_1 and u_2 be two solutions of the equation. Then

$$Au_1 - Au_2 = 0$$

Test with $u_1 - u_2$ and get

$$\alpha \|u_1 - u_2\|_V^2 \leq \langle Au_1 - Au_2, u_1 - u_2 \rangle = 0$$

So that $u_1 = u_2$. Namely, the solution is unique.

Estimates independent of κ

Tent $Au_n + Bu_n = l$ on u_n . We get

$$\begin{aligned} \alpha \|u_n\|_V^2 + 0 &\leq \langle Au_n, u_n \rangle + \langle Bu_n, u_n \rangle = \langle l, u_n \rangle \\ &\leq \|l\|_V \|u_n\|_V \end{aligned}$$

Hence $\|u_n\|_V \leq \|l\|_V / \alpha$ independently of κ .

○ Here we used that

$$\langle B_\kappa u_n, u_n \rangle = \int_{\Omega} B_\kappa(u_n) u_n \geq 0.$$

Limit one can now extract a subsequence u_{K_n} such that $u_{K_n} \xrightarrow{V} u$, $u_{K_n} \xrightarrow{H} u$. As before $Au_{K_n} \xrightarrow{V} Au$.

As $u_{K_n} \xrightarrow{H} u$ one has that $u_{K_n} \rightarrow u$ pointwise a.e.

Hence, $B_\kappa(u_{K_n}) \rightarrow B(u)$ pointwise a.e. as well. On the other hand

$$\text{A. } |\beta_{K_n}(u_{K_n})| \leq |\beta(u_{K_n})| \leq L_\beta \|u_{K_n}\|_V \leq \alpha$$

so that $\beta_{K_n}(u_{K_n}) \xrightarrow{H} \beta(u)$ by generalized dominated convergence. We can hence pass to the limit:

$$\begin{array}{c} \lim_{n \rightarrow \infty} \|A u_{K_n} + B u_{K_n}\|_V = \|l\|_V \\ \downarrow V \quad \downarrow H \\ A u + B u = l. \end{array}$$

Chwignen let u_1, u_2 be two solutions
of $Au + Bu = \ell$. Then

$$Au_1 - Au_2 + Bu_1 - Bu_2 = 0$$

Test on $u_1 - u_2$ and get that

$$\alpha \|u_1 - u_2\|_V^2 + 0 \leq \langle Au_1 - Au_2, u_1 - u_2 \rangle$$

$$+ \underbrace{\langle Bu_1 - Bu_2, u_1 - u_2 \rangle}_0 = 0$$

$$\int_{\Omega} (\beta(u_1) - \beta(u_2))(u_1 - u_2) \geq 0$$

Hence $u_1 = u_2$

Another extension: the case f not Lipschitz

$$\begin{cases} -\Delta u + \beta(u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

$\Omega \subset \mathbb{R}^d$ nonempty, open, smooth, bounded
 $f \in L^2(\Omega)$, $\beta: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing, continuous, $0 = \beta(0)$
 and linearly bounded $|\beta(r)| \leq C(1 + |r|) \quad \forall r$

Variational formulation:

(P) Find $u \in V$: $Au + Bu = l$ in V'

with $V = H_0^1(\Omega)$, $A: V \rightarrow V'$: $\langle Au, v \rangle = \int_{\Omega} Du \cdot Dv \, dx$
 $B: H \rightarrow H = L^2(\Omega)$: $\langle Bu, v \rangle = \int_{\Omega} \beta(u) v \, dx$,
 $\langle l, v \rangle = \int_{\Omega} f v \, dx$.

We proceed by regularization: for all $\varepsilon > 0$ define

(Q) $\beta_{\varepsilon}(r) = \frac{r - J_{\varepsilon}(r)}{\varepsilon} \quad (\text{Moreau-Yosida approx.})$

with $J_{\varepsilon}(r) = (\text{id} + \varepsilon \beta(\cdot))^{-1}(r)$ (resolvent)

Properties

- 1) $\beta_{\varepsilon}(0) = 0$ (because $\beta(0) = 0$)
- 2) β_{ε} is $\frac{1}{\varepsilon}$ -Lipschitz
- 3) $|\beta_{\varepsilon}(r)| \leq |\beta(r)| \quad \forall r$
- 4) $\beta_{\varepsilon}(r) = \beta(J_{\varepsilon}(r)) \quad \forall r$
- 5) J_{ε} is a 1-contraction
- 6) $J_{\varepsilon}(r) \rightarrow r$ as $\varepsilon \rightarrow 0$ for all r

with β_ε we define $B_\varepsilon : H \rightarrow H$ and consider problem

$$(P_\varepsilon) \text{ find } u_\varepsilon \in V: \quad Au_\varepsilon + B_\varepsilon u_\varepsilon = l \text{ in } V'$$

We have already proved that Problem (P_ε) has a unique solution u_ε . Our aim is to check that $u_\varepsilon \rightarrow u$, where u solves Problem (P) .

Step 1: estimate Test the equation in (P_ε)

○ by u_ε getting

$$\begin{aligned} \alpha \|u_\varepsilon\|_V^2 &\leq \langle Au_\varepsilon, u_\varepsilon \rangle + \langle B_\varepsilon u_\varepsilon, u_\varepsilon \rangle \\ &= \langle l, u_\varepsilon \rangle \leq \frac{\alpha}{2} \|u_\varepsilon\|_V^2 + \frac{1}{2\alpha} \|l\|_V^2, \end{aligned}$$

We hence have that $\|u_\varepsilon\|_V \leq C$ indep. of ε .

Step 2: limit We hence extract a not relabeled subsequence u_ε such that

$$u_\varepsilon \xrightarrow[V]{} u \quad \text{and} \quad u_\varepsilon \xrightarrow[H]{} u$$

As A is linear and continuous we have that $Au_\varepsilon \rightarrow Au$ in V' . On the other hand

$$\begin{aligned} \|J_\varepsilon u_\varepsilon - u\|_H &\leq \|J_\varepsilon u_\varepsilon - J_\varepsilon u\|_H + \|J_\varepsilon u - u\|_H \\ &\stackrel{\text{Prop.5}}{\leq} \|u_\varepsilon - u\|_H + \|J_\varepsilon u - u\|_H \xrightarrow{\text{Prop.6}} 0 \end{aligned}$$

Hence $\beta_\varepsilon(u_\varepsilon) \xrightarrow{\text{prop. 4}} \beta(J_\varepsilon u_\varepsilon) \rightarrow \beta(u)$ a.e.

We can now apply the Generalized Dominated Convergence Theorem in order to deduce that

$\beta_\varepsilon(u_\varepsilon) \rightarrow \beta(u)$ in $L^2(\Omega)$. Indeed

$$\| \beta_\varepsilon(u_\varepsilon) \|_H \stackrel{\text{prop. 3}}{\leq} \| \beta(u_\varepsilon) \|_H \leq C(1 + \| u_\varepsilon \|_H).$$

We conclude that $B_\varepsilon u_\varepsilon \rightarrow Bu$ in L^2 and can then pass to the limit

$$\begin{array}{l} Au_\varepsilon + B_\varepsilon u_\varepsilon = \ell \\ \downarrow V' \quad \downarrow H \quad \downarrow \\ Au + Bu = \ell \end{array}$$

Step 3: Uniqueness. Let u_1 and u_2 be two solutions of Problem (P). Then

$$Au_1 - Au_2 + Bu_1 - Bu_2 = 0 \text{ in } V'$$

Test on $u_1 - u_2$ and get:

$$\alpha \| u_1 - u_2 \|_V^2 \leq \langle A(u_1 - u_2), u_1 - u_2 \rangle = 0$$

so that $u_1 = u_2$.

In particular, the convergence in Step 2 holds for the whole subspace.

Troubles may occur if the nonnegativity of
B fails:

$V = V' = \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ positive definite (coercive!)

$Bu = -\lambda u$ for $\lambda \in \mathbb{R}$, $l \in \mathbb{R}^d$. Then, the

equation has a solution (any l) if either

1) λ is not an eigenvalue of A

2) λ is an eigenvalue of A but
 l is in the range of $A - \lambda I_d$

B is nondecreasing $\Rightarrow \lambda \leq 0 \Rightarrow$

$\Rightarrow \lambda$ not an eigenvalue of $A \Rightarrow$ case 1)

B is increasing $\Rightarrow \lambda > 0 \Rightarrow$ there
is no solution if λ is an eigenvalue
and l is not in the range, case 2)

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\lambda = 1$, $l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

There exists no $(u_1, u_2)'$ such that

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{u} + \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{Bu} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_l.$$