

The weakly coercive case

The case $\langle Au, u \rangle \geq -\gamma \|u\|_H^2 + \alpha \|u\|_V^2 \quad \forall u \in V$
 can be treated by means of the same
 time-discretization technique:

1) The elliptic problems

$$(I + \tau A)u = l \in V'$$

are unique solvable for $\tau \leq \gamma^{-1}$ as

$$\begin{aligned} \langle (I + \tau A)u, u \rangle &= \|u\|_H^2 + \tau \langle Au, u \rangle \\ &\geq \|u\|_H^2 - \tau \gamma \|u\|_H^2 + \tau \gamma \|u\|_V^2 \\ &= (1 - \tau \gamma) \|u\|_H^2 + \tau \gamma \|u\|_V^2 \\ &\geq \tau \gamma \|u\|_V^2. \end{aligned}$$

2) The estimate is a little more involved as

$$\begin{aligned} \frac{1}{2} \|u_m\|_H^2 + \frac{\alpha}{2} \sum_{i=1}^m \tau \|u_i\|_V^2 &\leq \frac{1}{2} \|u_0\|_H^2 \\ + \frac{1}{2\alpha} \sum_{i=1}^m \tau \|l_i\|_{V'}^2 + \gamma \sum_{i=1}^m \tau \|u_i\|_H^2 \end{aligned}$$

Take now $\tau < \frac{1}{4\gamma}$. Then

$$\frac{1}{4} \|u_m\|_H^2 \leq \gamma \sum_{i=1}^{m-1} \tau \|u_i\|_H^2 + C$$

and one can use the discrete Gronwall lemma

$$0 \leq \varphi_m \leq \varphi_0 + \gamma \sum_{i=1}^{m-1} \varphi_i$$

$$\Rightarrow \varphi_m \leq \varphi_0 \prod_{i=1}^{m-1} (1 + \gamma) \leq \varphi_0 e^{(m-1)\gamma}$$

Another possible trick: change variables to $v(t) = e^{-\gamma t} u(t)$. Indeed $u' + Au = l$ iff

$$e^{\gamma t} v' + \gamma e^{\gamma t} v + A(e^{\gamma t} v) = l. \text{ Multiply everything by } e^{-\gamma t} \text{ and you get}$$

$$v' + (\gamma I + A)v = e^{-\gamma t} l$$

Now $(\gamma I + A)$ is coercive and one can solve with the new datum $e^{-\gamma t} l$ and the initial condition $v(0) = e^{-\gamma \cdot 0} u(0) = u^0$.

Application: existence and uniqueness for

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega & \leftarrow \text{without conditions!} \\ \partial_n u = g & \text{on } \partial\Omega & \leftarrow \\ u(0) = u^0 & \text{in } \Omega \end{cases}$$

A Lipschitz perturbation: Assume to be given

$F: H \rightarrow H$ Lipschitz. Consider the problem

$$(P) \begin{cases} u' + Au = l + F(u) \text{ in } V', \text{ a.e. in } (0, T), \\ u(0) = u^0, \end{cases}$$

with $A: V \rightarrow V'$ linear, continuous, and coercive
 $l \in L^2(0, T; V')$, $u^0 \in H$.

We can solve Problem (P) via a fixed point argument: fix a tentative $\tilde{u} \in C([0, T]; H)$.

Then, $F(\tilde{u})$ is continuous in H as well.

In particular $F(\tilde{u}) \in L^2(0, T; V')$.

We can use the existence result for the linear parabolic problem to show that there exists a unique $u = S(\tilde{u}) \in H^1(0, T; V') \cap L^2(0, T; V) \subset C([0, T]; H)$ such that

$$\begin{cases} u' + Au = l + F(\tilde{u}) \text{ in } V', \text{ a.e. in } (0, T), \\ u(0) = u^0. \end{cases}$$

We shall now prove that S has a fixed point in $C([0, T]; H)$. Take \tilde{u}_1, \tilde{u}_2 and let $u_1 = S(\tilde{u}_1)$ and $u_2 = S(\tilde{u}_2)$. By taking the difference of the respective equations, testing on $u_1 - u_2$, and integrating on $(0, t)$ we get

$$\begin{aligned}
\frac{1}{2} \|u_1(t) - u_2(t)\|_H^2 &\leq \int_0^t (F(\tilde{u}_1) - F(\tilde{u}_2), u_1 - u_2) \\
&\leq L_F \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|_H \|u_1 - u_2\|_H \\
&\leq \frac{L_F}{2} \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|_H^2 + \frac{L_F}{2} \int_0^t \|u_1 - u_2\|_H^2
\end{aligned}$$

By Gronwall we have that

$$\|u_1(t) - u_2(t)\|_H^2 \leq C \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|_H^2 \quad (*)$$

Denote now by $S^{(k)}$ the k -iterate of S .

We would like to prove by induction on k that

$$\begin{aligned}
\sup_{[0,t]} \|S^{(k)}(\tilde{u}_1) - S^{(k)}(\tilde{u}_2)\|_H^2 \\
\leq \frac{C^k t^k}{k!} \sup_{[0,t]} \|\tilde{u}_1 - \tilde{u}_2\|_H^2 \quad \forall t \quad (E_k)
\end{aligned}$$

Indeed (E_1) follows from $(*)$ as

$$\begin{aligned}
\sup_{[0,t]} \|u_1 - u_2\|_H^2 &\leq C \int_0^t \sup_{[0,t]} \|\tilde{u}_1 - \tilde{u}_2\|_H^2 \\
&= Ct \sup_{[0,t]} \|\tilde{u}_1 - \tilde{u}_2\|_H^2.
\end{aligned}$$

Assume now (E_k) and prove (E_{k+1}) as follows

$$\begin{aligned}
& \| S^{(k+1)}(\tilde{u}_1)(t) - S^{(k+1)}(\tilde{u}_2)(t) \|_H^2 \\
& \stackrel{(*)}{\leq} C \int_0^t \| S^{(k)}(\tilde{u}_1) - S^{(k)}(\tilde{u}_2) \|_H^2 \\
& \stackrel{(E_k)}{\leq} C \int_0^t \frac{C^k t^k}{k!} \sup_{[0,t]} \| \tilde{u}_1 - \tilde{u}_2 \|_H^2 \\
& = \frac{C^{k+1} t^{k+1}}{(k+1)!} \sup_{[0,t]} \| \tilde{u}_1 - \tilde{u}_2 \|_H^2
\end{aligned}$$

Take now $t = T$. For k large enough

$$\frac{C^k T^k}{k!} < 1$$

and $S^{(k)}$ is a contraction. We can then use the following lemma:

$S^{(k)}$ is a contraction for some $k \in \mathbb{N} \implies$
 $\implies S$ has a unique fixed point.

which gives us the existence.

As for the uniqueness we argue as follows:
 take the difference of the equations for two solutions u_1, u_2 , test it on $u_1 - u_2$, and integrate on $(0, t)$ getting.

$$\begin{aligned} \frac{1}{2} \|(u_1 - u_2)(t)\|_H^2 &\leq \int_0^t (F(u_1) - F(u_2), u_1 - u_2) \\ &\leq L_F \int_0^t \|u_1 - u_2\|_H^2 \end{aligned}$$

Then $u_1 - u_2 = 0$ by Gronwall.

With this technology we can again solve the weakly coercive case $\langle Au, u \rangle \geq \alpha \|u\|_V^2 - \delta \|u\|_H^2$ by replacing

$$\begin{aligned} u' + Au &= l && \text{by} \\ u' + \underbrace{Au + \delta u}_{\text{coercive}} &= l + \underbrace{\delta u}_{\text{Lipschitz}} \end{aligned}$$

Proof of $S^{(k)}$ contraction $\Rightarrow S$ has a unique fixed point.

By the Contraction Mapping principle $S^{(k)}$ has a unique fixed point. Call it u . Then

$$S(u) = S(S^{(k)}(u)) = S^{(k+1)}(u) = S^{(k)}(S(u))$$

which means that $S(u)$ is a fixed point of $S^{(k)}$ hence necessarily $S(u) = u$, namely u is a fixed point for S .

Assume now that there exist u_1 and u_2 fixed points of S . Then u_1 and u_2 are also fixed points of $S^{(k)}$ hence $u_1 = u_2$.

Another case that can be studied

$$\begin{cases} u' + Au + B(u) = \ell & \text{in } V' \text{ a.e. } (0, T) \\ u(0) = u^0 \end{cases}$$

With $B: H \rightarrow H$ continuous and monotone

$$(B(u_1) - B(u_2), u_1 - u_2) \geq 0 \quad \forall u_1, u_2 \in H$$

○ Strategy: Moreau - Yosida approximations B_ε , use the Lipschitz argument, estimate and remove $\varepsilon \rightarrow 0$

Yet another even more general case

$$\begin{cases} u' + Au + B(u) = \ell + F(u) & \text{in } V' \text{ a.e. } (0, T) \\ u(0) = u^0 \end{cases}$$

○ $B: H \rightarrow H$ continuous, monotone

$F: H \rightarrow H$ Lipschitz

The maximum principle

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{0, -u\}$$

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{in } \partial\Omega \times (0, T) \\ u(0) = u^0 & \text{on } \Omega \end{cases}$$

○ thm $u^0 \geq 0$ a.e. $\Rightarrow u \geq 0$ a.e.

✱ Multiply the equation by $-u^-$ and integrate in space and time

$$\underbrace{\int_0^t \int_{\Omega} \partial_t u (-u^-)}_{= \int_0^t \frac{d}{dt} \frac{\|u^-\|_H^2}{2}} - \underbrace{\int_0^t \int_{\Omega} \Delta u (-u^-)}_{= \int_0^t \int_{\Omega} \nabla u \cdot \nabla (-u^-)} = 0$$

○ Then,

$$\frac{1}{2} \|u^-(t)\|_H^2 - \frac{1}{2} \|u^-(0)\|_H^2 + \underbrace{\int_0^t \int_{\Omega} |\nabla(u^-)|^2}_{\geq 0} = 0$$

We conclude that

$$\frac{1}{2} \|u^-(t)\|_H^2 \leq \frac{1}{2} \|u^-(0)\|_H^2 = \frac{1}{2} \|(u^0)^-\|_H^2 = 0$$

so that $u^- = 0$ a.e. and $u \geq 0$ a.e. \square

An antimonotone case

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } \Omega \times (0, T) \\ \partial_n u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(\cdot, 0) = u^0 & \text{in } \Omega \\ u^0 \in L^\infty(\Omega), \end{cases}$$

f non decreasing, continuous, locally bounded
 $f(0) \geq 0$, $\exists M > 0: f(M) \leq \frac{M - \|u^0\|_{L^\infty}}{T}$

○ Strategy: For all $\tilde{u} \in L^2(0, T; H)$ solve for $u = S(\tilde{u})$ such that

$$\begin{cases} \partial_t u - \Delta u = f(\tilde{u}) & \text{in } \Omega \times (0, T) \\ \partial_n u = 0 & \text{in } \partial\Omega \times (0, T) \\ u(\cdot, 0) = u^0 & \text{in } \Omega \end{cases}$$

Check that S has a fixed point.

○ Crucial remark: $\tilde{u}_1 \leq \tilde{u}_2$ a.e. \Rightarrow
 $f(\tilde{u}_1) \leq f(\tilde{u}_2)$ a.e. $\Rightarrow S(\tilde{u}_1) \leq S(\tilde{u}_2)$ a.e.
 \uparrow maximum principle

\rightarrow use a fixed point theorem for nonlinearity mappings!

Example:

$f: [0, 1] \rightarrow [0, 1]$ non decreasing
 $\Rightarrow f$ has a fixed point.

(E, \leq) ordered set (non total), $F \subseteq E$

$\bar{f} \in F$ is maximal iff $f \leq \bar{f} \quad \forall f \in F \Rightarrow f = \bar{f}$

$f \in F$ is maximum iff $f \leq \bar{f} \quad \forall f \in F$

$e \in E$ upper bound for F iff $f \leq e \quad \forall f \in F$

$e = \sup F$ iff it is the minimum of the upper bounds

○ F is a chain iff it is totally ordered

(E, \leq) is completely inductive iff all bounded chains have sup and inf.

Lemma (Zorn) (E, \leq) completely inductive
 $\Rightarrow E$ has a maximal element

○ Fixed point (Kakutani). (E, \leq) ordered.

$I = \{ u_* \leq u \leq u^* \} \subseteq E$ completely inductive

$S: (I, \leq) \rightarrow (I, \leq)$ non decreasing. Then

S has a fixed point in I .

Application

$$E = L^2(0, T; H)$$

$$u_1 \leq u_2 \stackrel{\text{def}}{\iff} u_1 \leq u_2 \text{ a.e.}$$

$$u_x = 0 \quad \rightarrow \quad S(u_x) \geq 0$$

$$u^* = M \quad \Rightarrow \quad S(u^*) \leq M$$

This means that $S: I \rightarrow I$

S is nondecreasing.

Then, there exists $u \in L^2(0, T; H)$ such that
 $u = S(u)$ and $0 \leq u \leq M$ a.e.

No uniqueness: take $u^0 = 0$

and $f(u) = \frac{2}{3}(u^+)^{3/2}$. Note that, for all $T > 0$

one can find $M > 0$ such that

$$f(M) \leq \frac{M - \|u^0\|_{L^\infty}}{T}$$

Nonetheless $u = 0$ and $u(t) = (t - t_x)^2$
 are solutions, for all $t_x > 0$.