

A hyperbolic problem

$\Omega \subset \mathbb{R}^d$ nonempty, open, connected, and smooth
 $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^\infty(\Omega \times (0, T))$

Problem find $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ such

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega \\ u_t(\cdot, 0) = u_1(\cdot) & \text{in } \Omega \end{cases}$$

Variational formulation: given $u_0 \in H$, $u_1 \in V$
 and $\ell \in L^\infty(0, T; H)$, to find

$u \in W^{2, \infty}(0, T; V') \cap W^{1, \infty}(0, T; H) \cap L^\infty(0, T; V)$
 such that

$$\begin{cases} u'' + Au = \ell & \text{in } V' \\ u(0) = u_0 & \text{in } V \\ u'(0) = u_1 & \text{in } H \end{cases}$$

where

$$V = H_0^1(\Omega)$$

$$H = L^2(\Omega)$$

$$A : V \rightarrow V' \quad \langle Au, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$\langle \ell, v \rangle = \int_{\Omega} f v \, dx$$

Remark: The initial conditions make sense as $u \in W^{1,\infty}(0,T; H)$ implies that $u(0)$ is well-defined (in H) and $u' \in W^{1,\infty}(0,T; V')$ implies that $u'(0)$ is well-defined (in V').

Remark 2: The stated regularities are coherent as

$$u \in L^\infty(0,T; V) \Rightarrow Au \in L^\infty(0,T; V')$$

$$u \in W^{2,\infty}(0,T; V') \Rightarrow u'' \in L^\infty(0,T; V')$$

○ and $\ell \in L^\infty(0,T; H)$

Galerkin approximation

Let λ_m be the eigenvalues of the operator A and call e_m the corresponding eigenfunctions with $\|e_m\|_V = 1$.

○ Call $V_m = \text{span} \{e_1, \dots, e_n\}$ and solve the problem restricted to V_m , namely to those functions u of the form

$$u^{(n)}(t) = \sum_{i=1}^m u_i(t) e_i$$

for $u_i : [0,T] \rightarrow \mathbb{R}$, $i = 1, \dots, n$.

In order to do so we need to define

$$u_0^{(m)} = \sum_{i=1}^n \underbrace{(u_0, e_i)}_v e_i$$

$$\ell^{(n)}(t) = \sum_{i=1}^n \underbrace{\langle \ell(t), e_i \rangle}_v e_i$$

$$u_1^{(m)} = \sum_{i=1}^n \underbrace{(u_1, e_i)}_v e_i$$

↗
 u_{1i}

Watch out! In order to possibly consider this last scalar product in V we need to momentarily assume $u_1 \in V$. This will be removed later on.

Now the problem reads

$$\begin{cases} u_i''(t) + \lambda_i u_i(t) = \ell_i(t) & t \in (0, T) \\ u_i(0) = u_{0i} \\ u_i'(0) = u_{1i} \end{cases}$$

for $i = 1, \dots, n$.

This is a system of n linear ODEs. It hence has a unique solution vector $(u_1(t), \dots, u_n(t))$. In particular, we have that

$$\begin{cases} (u^{(n)})'' + A u^{(n)} = \ell^{(n)} \text{ in } V' \text{ for all } t \in (0, T) \\ u^{(n)}(0) = u_0^{(n)} \\ (u^{(n)})'(0) = u_1^{(n)} \end{cases}$$

H 4

A priori estimate: test on $(u^{(n)})'$ and integrate on $(0, t)$, $t \in (0, T)$.

$$\frac{1}{2} \|(u^{(n)})'(t)\|_H^2 + \frac{1}{2} \langle Au^{(n)}(t), u^{(n)}(t) \rangle$$

$$= \frac{1}{2} \|u_1^{(n)}\|_H^2 + \frac{1}{2} \langle Au_0^{(n)}, u_0^{(n)} \rangle$$

$$+ \int_0^t \langle \ell^{(n)}, (u^{(n)})' \rangle dt$$

$$\textcircled{O} \leq \frac{1}{2} \|u_1^{(n)}\|_H^2 + \frac{1}{2} \langle Au_0^{(n)}, u_0^{(n)} \rangle$$

$$+ \frac{1}{2} \int_0^t \|\ell^{(n)}\|_H^2 + \frac{1}{2} \int_0^t \|(u^{(n)})'\|_H^2$$

By Gronwall we get

$$\|(u^{(n)})'(t)\|_H^2 + \|u^{(n)}(t)\|_V^2$$

$$\textcircled{O} \leq C \left(\|u_0^{(n)}\|_V^2, \|u_1^{(n)}\|_H^2, \|\ell^{(n)}\|_{L^2(0,T;H)} \right)$$

$$\leq C$$

Then, by comparison

$$\|(u^{(n)})''\|_{L^\infty(0,T;V')} \leq$$

$$\|\ell^{(n)}\|_{L^\infty(0,T;V')} + \|Au^{(n)}\|_{L^\infty(0,T;V')} \leq C.$$

Passage to the limit: by extracting a subsequence
(not relabelled)

$$u^{(n)} \rightarrow u \text{ in } W^{2,\infty}(V') \cap W^{1,\infty}(H) \cap L^\infty(V)$$

which implies

$$\begin{aligned} u^{(n)}(0) &\rightarrow u(0) \text{ in } H \\ u_0^{(n)} &\xrightarrow{\parallel} u_0 \quad \text{in } V \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow u(0) = u_0.$$

$$\begin{aligned} (u^{(n)})'(0) &\rightarrow u'(0) \text{ in } V' \\ u_1^{(n)} &\xrightarrow{\parallel} u_1 \quad \text{in } V \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow u'(0) = u_1. \\ \text{this because } u_1 &\in V! \end{aligned}$$

$$l^{(n)} \rightarrow l \text{ in } L^\infty(0,T; V')$$

$$A u^{(n)} \rightarrow A u \text{ in } L^\infty(0,T; V').$$

We have hence proved that u solves the problem

Removal of the assumption $u_1 \in V$.

Let u_1 be just in H and define $u_1^{(n)} \in V$ such that $u_1^{(n)} \rightarrow u_1$ in H . Then one can find a (unique) $u^{(n)}$ such that

$$\begin{cases} (u^{(n)})'' + Au^{(n)} = \ell \\ u^{(n)}(0) = u_0 \\ (u^{(n)})'(0) = u_1^{(n)} \end{cases}$$

- The solution for $u'(0) = u_1$ is obtained by passing to the limit.

Uniqueness Let $u^{(1)}$ and $u^{(2)}$ be two solutions with the same data. Take the difference in order to find their $u = u^{(1)} - u^{(2)}$ solves

$$\begin{cases} u'' + Au = 0 \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

Test on u' and integrate on $(0, t)$ to get

$$\frac{1}{2} \|u(t)\|_H^2 + \frac{1}{2} \langle Au(t), u(t) \rangle = 0$$

from which we deduce $u \equiv 0$.

A nonlinear problem (for $p > 1$) not

Find $u \in W^{1,\infty}(0,T; L^2(\mathbb{R})) \cap L^\infty(0,T; L^p(\mathbb{R}) \cap H_0^1(\mathbb{R}))$
 $\cap W^{2,\infty}(0,T; L^{p'}(\mathbb{R}) + H^{-1}(\mathbb{R}))$ such that

$$\begin{cases} u_{tt} - \Delta u + |u|^{p-2}u = 0 \\ u(\cdot, 0) = u_0 \\ u_t(\cdot, 0) = u_1 \end{cases}$$

(This is the semilinear wave equation)

1) Use the Galerkin scheme to find an approximate solution $u^{(m)}$.

2) Estimate a priori: test on $u_t^{(n)}$

$$\frac{1}{2} \|u_t^{(m)}(t)\|_H^2 + \frac{1}{2} \langle Au^{(m)}(t), u^{(m)}(t) \rangle$$

$$+ \int_{\mathbb{R}} \frac{|u^{(n)}(t)|^p}{p} = \frac{1}{2} \|u_1\|_H^2 + \frac{1}{2} \langle Au_0, u_0 \rangle + \int_{\mathbb{R}} \frac{|u_0|^p}{p} < C$$

3) Passage to the limit $u^{(m)} \rightarrow u$ in $L^2(\mathbb{R} \times (0,T))$ can be deduced (for some subsequence). Then

$$|u^{(n)}|^{p-2} u^{(n)} \rightarrow |u|^{p-2} u \text{ p.a.e.}$$

and $|u^{(m)}|^{p-2} u^{(m)}$ is bounded in $L^\infty(0,T; L^{p'}(\mathbb{R}))$

The dominated convergence entails that

$$|u^{(n)}|^{p-2} u^{(n)} \rightarrow |u|^{p-2} u \text{ in } L^{1/\varepsilon}(0,T; L^{p'-\varepsilon}(\mathbb{R}))$$

$$\forall \varepsilon > 0$$

The latter convergence is enough in order to check that u solves the problem!

a) Uniqueness, take the difference

$$(u^{(1)} - u^{(2)})'' + A(u^{(1)} - u^{(2)}) + |u^{(1)}|^{p-2}u^{(1)} - |u^{(2)}|^{p-2}u^{(2)} = 0$$

and test on $(u^{(1)} - u^{(2)})'$ in order to get that $u^{(1)} = u^{(2)}$.

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