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# Asymptotic Development by Γ-Convergence

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**Abstract.** A description of the asymptotic development of a family of minimum problems is proposed by a suitable iteration of Γ-limit procedures. An example of asymptotic development for a family of functionals related to phase transformations is also given.

**Key Words.** Γ-convergence, Asymptotic developments, *BV* functions.

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## 0. Introduction

It is very common, both in pure and applied mathematics, to have to deal with a family of problems depending on a parameter $\varepsilon > 0$, being interested in the asymptotic behavior of the problems as $\varepsilon \to 0$. Typically, in the calculus of variations we are given a family of minimum problems

$$\min\{\mathscr{F}_\varepsilon(u): u \in X\} \tag{0.1}$$

and we are interested in finding a "limit problem"

$$\min\{\mathscr{F}(u): u \in X\}. \tag{0.2}$$

Of course, from the point of view of the calculus of variations, the limit problem must be such that its minimizers are closely related to the possible limit points of minimizers $\{u_\varepsilon\}_\varepsilon$ of problem (0.1).

A notion of convergence for functionals, which is very well suited to the variational setting, is the well-known Γ-convergence, introduced by De Giorgi [DF]. In fact, if a functional $\mathscr{F}$ is the Γ-limit of the $\mathscr{F}_\varepsilon$ and if $u_\varepsilon$ are minimizers of $\mathscr{F}_\varepsilon$ and $u_\varepsilon \to u$, then $u$ is a minimizer of $\mathscr{F}$ (see Section 1 for a precise statement).

Roughly speaking, we have

$$\{\text{limits of minimizers}\} \subset \{\text{minimizers of the } \Gamma\text{-limit}\}, \tag{0.3}$$

where the inclusion may well be proper, as we can see by very simple and natural examples. Hence the Γ-limit, though very useful, fails in general to characterize *completely* the asymptotic behavior of the family $\mathscr{F}_\varepsilon$.

Our remark is that in fact the Γ-limit is only the first step toward the description of the asymptotic behavior of $\mathscr{F}_\varepsilon$, and that we may try to pursue further the description looking for an asymptotic development

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon\mathscr{F}^{(1)} + \varepsilon^2\mathscr{F}^{(2)} + \cdots + \varepsilon^k\mathscr{F}^{(k)} + o(\varepsilon^k), \tag{0.4}$$

where the first-order term $\mathscr{F}^{(0)}$ is just the Γ-limit $\mathscr{F}$ of the family $\mathscr{F}_\varepsilon$ and each one of the higher-order terms $\mathscr{F}^{(i)}$ is a functional defined by a natural recursive procedure on the space $\mathscr{U}^{(i-1)}$ of the minimizers of $\mathscr{F}^{(i-1)}$ (see the following section). When a development as in (0.4) holds, then we have the following situation:

$$\begin{aligned}\{\text{limits of minimizers}\} &\subset \{\text{minimizers of } \mathscr{F}^{(k)}\}\\ &\subset \{\text{minimizers of } \mathscr{F}^{(k-1)}\} \subset \cdots \subset \{\text{minimizers of } \mathscr{F}^{(0)}\}.\end{aligned} \tag{0.5}$$

This may provide a considerable improvement of (0.3), and in some cases may give a complete characterization of the asymptotic behavior of $\mathscr{F}_\varepsilon$.

In Section 1 we give the general definitions and theorems about the notion of asymptotic development of a family of functionals. In Section 2 we illustrate the general theory by a simple but not completely obvious example, related to the well-known example by Modica and Mortola [MM]. It was in fact in the context of this example that we first got the idea of characterizing the asymptotic behavior through a sequence of functionals $\mathscr{F}^{(k)}$ defined on nested spaces. The idea that the functionals $\mathscr{F}^{(k)}$ could be thought formally as an asymptotic development (written as in (0.4)) was suggested by an incidental remark by De Giorgi. On the other hand, the fact that some notion of asymptotic development by Γ-convergence could be useful, must have been known more or less explicitly by many people working in the field. For instance, something which is close to an asymptotic development can be found in a work by Buttazzo and Percivale [BP], and a first attempt at a definition can be found at the end of a paper by Modica [M]. After all, even the scaling of the functionals in the first paper by Modica and Mortola [MM] may be thought of as an unconscious order-one development.

## 1. The Asymptotic Development of a Family of Functionals by Γ-Convergence

Let $X$ be a topological space for which the first axiom of countability holds, and let

$$\mathscr{F}_\varepsilon : X \to \bar{\mathbf{R}}$$

be a family of functionals, with $\varepsilon$ a positive parameter.

For such a family the following definition of the Γ-limit is well known (see [DF] and [DM]).

**Definition 1.1.** A functional $\mathscr{F}^{(0)}: X \to \bar{\mathbf{R}}$ is said to be the $\Gamma(X^-)$-limit of the family $\mathscr{F}_\varepsilon$ at a point $\bar{u} \in X$ iff the following statements are fulfilled for each sequence $\varepsilon_j \downarrow 0$:

(i) For each sequence $\{u_j\} \subset X$ with $u_j \to \bar{u}$ we have

$$\liminf_{j \to +\infty} \mathscr{F}_{\varepsilon_j}(u_j) \geq \mathscr{F}^{(0)}(\bar{u}).$$

(ii) There exists a sequence $\{u_j\} \subset X$ with $u_j \to \bar{u}$ and

$$\limsup_{j \to +\infty} \mathscr{F}_{\varepsilon_j}(u_j) \leq \mathscr{F}^{(0)}(\bar{u}).$$

In this situation we write

$$\Gamma(X^-) \lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\bar{u}) = \mathscr{F}^{(0)}(\bar{u}).$$

**Definition 1.2.** We write

$$\Gamma(X^-) \lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon = \mathscr{F}^{(0)} \qquad \text{in} \quad E \subset X$$

if $\mathscr{F}^{(0)}$ is the $\Gamma(X^-)$-limit of $\mathscr{F}_\varepsilon$ at each point $\bar{u} \in E$.

When no confusion may arise we often omit the specification of the space $X$. The introduction of Γ-convergence in the calculus of variations is justified by the following well-known result, whose easy proof can be found in the papers quoted above.

**Theorem 1.1.** *Let $\varepsilon_j \downarrow 0$ be a fixed sequence, let $\{u_j\} \subset X$ be such that $\mathscr{F}_{\varepsilon_j}(u_j) = \min\{\mathscr{F}_{\varepsilon_j}(u): u \in X\}$. If $\mathscr{F}^{(0)}$ is the Γ-limit of $\mathscr{F}_\varepsilon$ on the whole space $X$ and $u_j \to \bar{u}$ in $X$, then $\bar{u}$ is a minimizer of $\mathscr{F}^{(0)}$ and we have*

$$\lim_{j \to +\infty} \mathscr{F}_{\varepsilon_j}(u_j) = \mathscr{F}^{(0)}(\bar{u}).$$

Unfortunately, we may well have many minimizers of the Γ-limit, which are not limit points of minimizers of the functionals $\mathscr{F}_\varepsilon$. A trivial example of such a phenomenon is the following one.

**Example 1.1.** Consider the case $X = \mathbf{R}$ and

$$\mathscr{F}_\varepsilon(u) = \varepsilon |u|.$$

We can easily check that the functions $\mathscr{F}_\varepsilon(u)$ Γ-converge to the constant function $\mathscr{F}^{(0)} \equiv 0$. Clearly, every point in $\mathbf{R}$ is a minimum point of $\mathscr{F}^{(0)}$, while the only limit point of the minimizers of $\mathscr{F}_\varepsilon$ is the point $u = 0$.

Now, the idea is that of introducing a notion of "asymptotic expansion"

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon \mathscr{F}^{(1)} + \cdots + \varepsilon^k \mathscr{F}^{(k)} + o(\varepsilon^k)$$

of a family $\mathscr{F}_\varepsilon(u)$ in such a way that the knowledge of the functionals $\mathscr{F}^{(k)}$ gives additional information on the limit points of minimizers. Precisely: any limit point of a sequence of minimizers $u_\varepsilon$ will also be a minimizer of each one of the functionals $\mathscr{F}^{(k)}$ appearing in the development above. Just by construction, the sequence of sets

$$\mathscr{U}^k = \{\text{minimizers of } \mathscr{F}^{(k)}\}$$

will be nonincreasing, and we may hope that in some cases the minimizers of some term $\mathscr{F}^{(k)}$ of the development are exactly *all* the possible limit points of minimizers $u_\varepsilon$.

Now, let us discuss how to define a suitable notion of asymptotic development. For instance, suppose we want to give meaning to the expression

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon\mathscr{F}^{(1)} + o(\varepsilon). \tag{1.1}$$

Naively, we should like to say that (0.1) is equivalent to

$$\lim_{\varepsilon\to 0} \frac{\mathscr{F}_\varepsilon - \mathscr{F}^{(0)}}{\varepsilon} = \mathscr{F}^{(1)},$$

where the limit should be taken in the sense of $\Gamma$-convergence. Unfortunately such a definition makes little sense, as we can convince ourselves by trying to apply it to simple situations. For example, it may happen that $\mathscr{F}_\varepsilon$ and $\mathscr{F}^{(0)}$ are finite on disjoint domains (see the case in [MM]). However, it turns out that we can give a simple and very good substitute for the naive definition:

Set

$$\mathscr{F}^{(0)} = \Gamma - \lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon \qquad \text{in } X$$

(we assume the the $\Gamma$-limit exists) and also set

$$m_0 = \inf_X \mathscr{F}^{(0)}, \qquad \mathscr{U}_0 = \{u \in X : \mathscr{F}^{(0)}(u) = m_0\}.$$

**Definition 1.3.** With the notation above, assume that $m_0 < +\infty$ and $\mathscr{U}_0 \neq \emptyset$. We say that the first-order asymptotic development

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon\mathscr{F}^{(1)} + o(\varepsilon) \tag{1.2}$$

holds, if we have

$$\Gamma - \lim_{\varepsilon\to 0} \frac{\mathscr{F}_\varepsilon - m_0}{\varepsilon} = \mathscr{F}^{(1)} \qquad \text{in } \mathscr{U}_0.$$

From now on we use the notation

$$\mathscr{F}_\varepsilon^{(1)} = \frac{\mathscr{F}_\varepsilon - m_0}{\varepsilon}.$$

The definition (1.2) is motivated mainly by the following very simple results.

**Theorem 1.2.** *Suppose the first-order asymptotic development* (1.2) *holds, and let $\varepsilon_j \downarrow 0$ be a sequence for which there exists a sequence $\{u_j\} \subset X$, $u_j \to \bar{u}$ in $X$, and $\mathscr{F}_{\varepsilon_j}(u_j) = \min\{\mathscr{F}_{\varepsilon_j}(v): v \in X\}$. Then $\bar{u} \in \mathscr{U}^0$ and $\bar{u}$ minimizes $\mathscr{F}^{(1)}$ in $\mathscr{U}^0$. Moreover, if $m_\varepsilon$ denotes the infimum of $\mathscr{F}_\varepsilon$ on $X$ and $m_1$ denotes the infimum of $\mathscr{F}^{(1)}$ on $\mathscr{U}^0$ we have*

$$m_{\varepsilon_j} = m_0 + \varepsilon_j m_1 + o(\varepsilon_j). \tag{1.3}$$

*Proof.* The fact that $\bar{u} \in \mathscr{U}^0$ is a consequence of Theorem 1.1. Let $v \in \mathscr{U}^0$ and let $\{v_j\} \subset X$ be a sequence converging to $v$ in $X$ such that $\mathscr{F}^{(1)}_{\varepsilon_j}(v_j) \to \mathscr{F}^{(1)}(v)$: from the Γ-convergence of $\mathscr{F}^{(1)}_\varepsilon$ it follows that such a sequence exists. For each fixed index $j$ we have that $\mathscr{F}^{(1)}_{\varepsilon_j}(v_j) \geq \mathscr{F}^{(1)}_{\varepsilon_j}(u_j)$, and so the Γ-convergence yields

$$\mathscr{F}^{(1)}(v) = \lim_{j \to +\infty} \mathscr{F}^{(1)}_{\varepsilon_j}(v_j) \geq \liminf_{j \to +\infty} \mathscr{F}^{(1)}_{\varepsilon_j}(u_j) \geq \mathscr{F}^{(1)}(\bar{u}).$$

In particular we have

$$\lim_{j \to +\infty} \mathscr{F}^{(1)}_{\varepsilon_j}(u_j) = \mathscr{F}^{(1)}(\bar{u}).$$

This reads $\lim_{j \to +\infty}[(m_{\varepsilon_j} - m_o)/\varepsilon_j - m_1] = 0$, which implies (1.3). □

**Remark.** Note that (1.3) in general is true only for sequences $\varepsilon_j \downarrow 0$ for which there is compactness for minimizers: in particular, it is *false* that

$$m_\varepsilon = m_0 + \varepsilon m_1 + o(\varepsilon), \tag{1.4}$$

as the following example shows.

Let $X = \mathbf{R}$, and consider the following family of functionals

$$\mathscr{F}_\varepsilon(u) = \begin{cases} 0 & \text{if } \; u = 0, \\ -\varepsilon^{1/2} & \text{if } \; u = 1/\varepsilon \text{ and } \varepsilon \text{ is rational}, \\ \varepsilon^2 & \text{otherwise in } \mathbf{R}. \end{cases}$$

Of course each functional $\mathscr{F}_\varepsilon$ has a unique minimizer, which is 0 if $\varepsilon$ is irrational and is $1/\varepsilon$ if $\varepsilon$ is rational. The only limit point of minimizers is 0, and we have compactness only for sequences $\varepsilon_j \downarrow 0$ such that $\varepsilon_j$ is definitively irrational. In particular, (1.4) is false for a rational sequence $\varepsilon_j \downarrow 0$, for in this case $m_{\varepsilon_j} = -\varepsilon_j^{1/2}$, while $\mathscr{F}^{(0)} \equiv 0$ and $\mathscr{F}^{(1)} \equiv 0$.

The process above can be iterated in the following way. Suppose the asymptotic development of the first order (1.1) holds, recall that $m_1 = \inf_{\mathscr{U}^0} \mathscr{F}^{(1)}$, set

$$\mathscr{U}^1 = \{u \in \mathscr{U}^0: \mathscr{F}^{(1)}(u) = m_1\},$$

and suppose $m_1 < +\infty$ and $\mathscr{U}^1 \neq \varnothing$. Then consider the family of functionals

$$\mathscr{F}^{(2)}_\varepsilon = \frac{\mathscr{F}^{(1)}_\varepsilon - m_1}{\varepsilon}.$$

If we have

$$\mathscr{F}^{(2)} = \Gamma - \lim_{\varepsilon \to 0} \mathscr{F}^{(1)}_{\varepsilon_j} \qquad \text{in } \mathscr{U}^1,$$

then we say that the asymptotic development of the second order

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon\mathscr{F}^{(1)} + \varepsilon^2\mathscr{F}^{(2)} + o(\varepsilon^2)$$

holds.

Obviously, in this situation the analogs of Theorems 1.2 and 1.3 hold, so that the limit points of the minimizers of the functionals $\mathscr{F}_\varepsilon$ must minimize the functional $\mathscr{F}^2$ on the set $\mathscr{U}^1$.

The definition of the asymptotic development of order $k$ is obtained simply by recursion in the above process: if the asymptotic development of order $k-1$ holds, we set $m_{k-1} = \inf\{\mathscr{F}^{(k-1)}\}$, $\mathscr{U}^{k-1} = \{u \in \mathscr{U}^{k-2}\colon\ \mathscr{F}^{(k-1)}(u) = m_{k-1}\}$ and, assuming $m_{k-1} < +\infty$ and $\mathscr{U}^{k-1} \neq \varnothing$,

$$\mathscr{F}_\varepsilon^{(k)} = \frac{\mathscr{F}_\varepsilon^{k-1} - m_{k-1}}{\varepsilon}.$$

If

$$\mathscr{F}^{(k)} = \Gamma - \lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon^{(k)} \qquad \text{in } \mathscr{U}^{k-1},$$

we write

$$\mathscr{F}_\varepsilon = \mathscr{F}^{(0)} + \varepsilon\mathscr{F}^{(1)} + \cdots + \varepsilon^k\mathscr{F}^{(k)} + o(\varepsilon^k),$$

the development of order $k$. Of course, a limit point of minimizers of the family $\mathscr{F}_\varepsilon$ has to be a minimizer of $\mathscr{F}^{(k)}$ in $\mathscr{U}^{(k-1)}$ and

$$m_{\varepsilon_j} = m_0 + \varepsilon_j m_1 + \cdots + \varepsilon_j^k m_k + o(\varepsilon_j^k).$$

We conclude this section by giving a few very simple examples of asymptotic developments of families of functionals, showing how the above theoretical setting can give useful information in some situations. A more complex example is treated in Section 2.

We also give an example which show that our method may need improvements for more general classes of functionals.

**Example 1.2.** Consider the family of functionals in Example 1.1. We have shown that $\mathscr{F}^0(u) \equiv 0$ is the $\Gamma$-limit of order 0, and so $m_0 = 0$ and $\mathscr{U}^0 = \mathbf{R}$. On the other hand, $\mathscr{F}^{(1)}(u) = |u|$, and so $m_1 = 0$ and $\mathscr{U}^1 = \{0\}$.

This way we already have the complete information about the limit points of minimizers by arresting our development to the first-order term.

If we consider a sequence of the form

$$\mathscr{F}_\varepsilon(u) = \varepsilon^k |u|,$$

we have that all the terms of the asymptotic developments of order $<k$ are zero functional, while the asymptotic development of order $k$ is $|u| + o(\varepsilon^k)$, and so we need to compute $k$ $\Gamma$-limits in order to have the complete information on the limit points of minimizers.

Of course, in this particular case this phenomenon is due to a bad choice of the parameter $\varepsilon$: if we consider $\varepsilon^k$ as a new parameter we reach this information in one step only. The following example shows that it is not always so.

**Example 1.3.** Let $X = [-1, 1] \subset \mathbf{R}$, and consider the following family of functionals:

$$\mathscr{F}_\varepsilon(u) = \begin{cases} 0 & \text{if } \ u = 0, \\ \varepsilon^k & \text{if } \ |u| \in \left(\dfrac{1}{2^k}, \dfrac{1}{2^{k-1}}\right]. \end{cases}$$

For the development of order zero we have the same situation as before: $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F}^{(0)} \equiv 0$, and so $\mathscr{U}^0 = [-1, 1]$, $m_0 = 0$ and we have no information at all. We have

$$\mathscr{F}_\varepsilon^{(1)}(u) = \begin{cases} 0 & \text{if } \ u = 0, \\ \varepsilon^{k-1} & \text{if } \ |u| \in \left(\dfrac{1}{2^k}, \dfrac{1}{2^{k-1}}\right]. \end{cases}$$

On the set $\mathscr{U}^0$ the family $\mathscr{F}_\varepsilon^{(1)}$ Γ-converges to the following functional:

$$\mathscr{F}^{(1)}(u) = \begin{cases} 0 & \text{if } \ u \in [-\frac{1}{2}, \frac{1}{2}], \\ 1 & \text{otherwise in } \mathscr{U}^0. \end{cases}$$

Hence we have $\mathscr{U}^1 = [-\frac{1}{2}, \frac{1}{2}]$ and $m_1 = 0$. By induction on the above process we get $\mathscr{U}^k = [-1/2^k, 1/2^k]$, $m_k = 0$, and

$$\mathscr{F}^{(k)}(u) = \begin{cases} 0 & \text{if } \ |x| \le 1/2^k, \\ 1 & \text{otherwise in } \mathscr{U}^{k-1}. \end{cases}$$

As $\bigcap_{k=1}^{\infty} \mathscr{U}^k = \{0\}$, only the complete asymptotic development of $\mathscr{F}_\varepsilon$ gives us the desired information about the limit points of minimizers.

**Example 1.4.** In some cases our process does not work: this, we believe, is essentially due to the necessity of the right choice of *scaling* in our family. For instance, if we modify the family of Examples 1.1 and 1.2 as follows:

$$\mathscr{F}_\varepsilon(u) = e^{-1/\varepsilon}|u|,$$

we have that all the terms of the asymptotic development are the functional constantly equal to zero, and so with this ill-omened scaling we cannot hope to get any information at all.

A similar situation is the following:

$$\mathscr{F}_\varepsilon(u) = |u| + \frac{1}{\varepsilon}.$$

In this case the Γ-limit is constantly equal to $+\infty$, and so we cannot develop our family further.

Of course a simple rescaling obtained by multiplying the functionals by $\varepsilon$ eliminates the problem, but in general determination of the right scaling is far from easy.

**Remark.** A situation as that of Example 1.5 may arise if we have a countable sequence of functionals $\{\mathscr{F}_j\}$ instead of a continuous family.

In this case, once we have defined the functional $\mathscr{F}^{(0)}$ as the $\Gamma$-limit of the family, we may define $\mathscr{F}^{(1)}$ as the $\Gamma$-limit of the sequence

$$\mathscr{F}_j^{(1)} = \frac{\mathscr{F}_j - m_0}{\omega_j},$$

where $\omega_j$ is a suitable vanishing sequence. In this case we do not have a *natural* scaling given by the parameter $\varepsilon$, and we must carefully choose a sort of "order of zero" for the sequence $\mathscr{F}_j$.

## 2. A Concrete Example of Asymptotic Development in a Problem Related to Phase Transitions

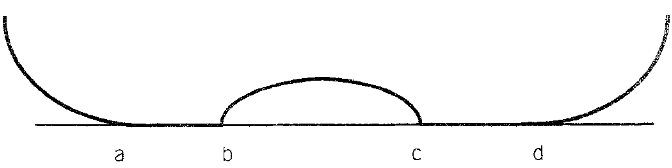
In this section we study a family $\mathscr{F}_\varepsilon$ of functionals very closely related to those already considered by Modica and Mortola [MM], and afterwards studied in many papers, also in view of their physical meaning in the theory of phase transitions, see, for example, [M1], [M2], [S], [FT], [B], [ABV], and [BB].

In the papers quoted above, the information about the asymptotic behavior of the minimizers of the family of functionals was obtained simply by computing the $\Gamma$-limit of a suitable rescaling of the family itself: in the setting we established in the previous section this corresponds to the first-order asymptotic development of the family. In this section we show that, in dimension one, the second-order development of $\mathscr{F}_\varepsilon$ can also be computed and that this gives further information about the asymptotic behavior of the minimizers. We show, furthermore, with an example, that in higher dimensions the seemingly most natural conjecture for the second $\Gamma$-limit $\mathscr{F}^{(2)}$ is indeed false. For the moment we are not able to conjecture the correct form of $\mathscr{F}^{(2)}$ in dimension greater than or equal to 2.

Throughout this section we have $X = L^1(\Omega)$, with $\Omega \subset \mathbf{R}^n$ a bounded open set with $C^2$-regular boundary. In the second part of the section we restrict ourselves to $n = 1$.

Suppose we have a function $\varphi: \mathbf{R} \to [0, +\infty)$ with the following properties:

(i) $\varphi \in C^0(\mathbf{R})$.
(ii) $Z = \{u: \varphi(u) = 0\} = [a, b] \cup [c, d]$ with $a < b < c < d$.
(iii) There is a $K > 0$ such that $\varphi(u)$ decreases for $u < -K$, but increases for $u > K$.



Let $g\colon \partial\Omega \to [a, d]$ be a lipschitz-continuous function, and define our family of functionals as follows:

$$\mathscr{F}_\varepsilon(u) = \begin{cases} \varepsilon^2 \displaystyle\int_\Omega |Du|^2\, dx + \int_\Omega \varphi(u)\, dx & \text{if } \ u \in H^1(\Omega), u|_{\partial\Omega} = g, \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Up to rescaling our family of functionals is the same as in the paper by Modica and Mortola, but in our case the function $\varphi$ vanishes in two intervals instead of two points. We shall see that this difference makes it possible and interesting to compute a step more in the asymptotic development. Here we also have a boundary condition which is not present in [MM], but this makes only a technical difference.

We remark that each functional $\mathscr{F}_\varepsilon$ has at least one minimizer in $L^1(\Omega)$ and that, under suitable hypothesis on the growth of $\varphi$ at infinity (a superlinear growth is enough), each family of minimizers $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^1(\Omega)$ (see [M1], [FT], and [B]). Hence we are lawfully allowed to speak of "limit points of minimizers."

**Theorem 2.1.** *We have* $\Gamma - \lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon = \mathscr{F}^{(0)}$ *in* $L^1(\Omega)$, *where*

$$\mathscr{F}^{(0)}(u) = \int_\Omega \varphi(u(x))\, dx.$$

*Proof.* We fix a sequence $\varepsilon_j \downarrow 0$ and a function $u \in L^1(\Omega)$.

*First step.* We must show first that, for each sequence $\{u_j\} \subset L^1(\Omega)$ with $u_j \to u$ in $L^1(\Omega)$, the following inequality holds:

$$\liminf_{j\to+\infty} \mathscr{F}_{\varepsilon_j}(u_j) \ge \mathscr{F}^{(0)}(u).$$

This follows easily from Fatou's lemma.

*Second step.* We must construct a sequence $\{u_j\} \subset L^1(\Omega)$ with $u_j \to u$ in $L^1(\Omega)$ and such that

$$\lim_{j\to+\infty} \mathscr{F}_{\varepsilon_j}(u_j) = \mathscr{F}^{(0)}(u).$$

To do this, we first choose a sequence $\{v_j\} \subset W^{1,\infty}(\Omega)$ such that $v_j$ converge to $u$ in $L^1(\Omega)$ and almost everywhere in $\Omega$. We may suppose that $v_{j|\partial\Omega} = g$ (this can be obtained by modifying the original sequence $v_j$ on small neighborhoods of $\partial\Omega$).

If $u$ is bounded, $\{v_j\}$ can also be taken uniformly bounded in $L^\infty$, and we obviously have

$$\lim_{j\to+\infty} \int_\Omega \varphi(v_j)\, dx = \int_\Omega \varphi(u)\, dx.$$

If $u$ is not bounded, we can obtain the same result by approximating the truncations

$$w_m(x) = \max\{-m, \min\{u(x), m\}\}$$

of $u$, and by taking a suitable diagonal sequence. This is possible by property (iii) of $\varphi$.

If the following inequality holds

$$\int_{\Omega} |Dv_j|^2 \leq \frac{1}{\varepsilon_j} \tag{2.1}$$

our claim is proved.

This is obtained simply by "slowing down" the convergence of the sequence $\{v_j\}$: the sequence $\{u_j\}$ is obtained from $\{v_j\}$ simply by repeating each term $v_k$ many times in order to have the estimate (2.1). □

The $\Gamma$-limit (or the asymptotic development of order zero, which is the same) does not tell us very much about the asymptotic behavior of the minimizers of the family $\mathscr{F}_\varepsilon$ as $\varepsilon \to 0$. In fact the only thing we can say at this point is that a limit point of minimizers has to be a minimizer of $\mathscr{F}^{(0)}$, that is an element of the very large space

$$\mathscr{U}^0 = \{u \in L^1(\Omega)\colon u(x) \in Z \text{ for almost all } x \in \Omega\}.$$

Hence we look for higher-order developments, according to the definitions established in Section 1.

By using the notation of the previous section we have

$$m_0 = \inf_{u \in L^1(\Omega)} \mathscr{F}^{(0)}(u) = 0,$$

$$\mathscr{F}_\varepsilon^{(1)}(u) = \frac{\mathscr{F}_\varepsilon(u) - m_0}{\varepsilon}$$

$$= \begin{cases} \varepsilon \displaystyle\int_{\Omega} |Du|^2\, dx + \frac{1}{\varepsilon} \int_{\Omega} \varphi(u)\, dx & \text{if } \quad u \in H^1(\Omega), \quad u = g \text{ on } \partial\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

We remark that the functionals $\mathscr{F}_\varepsilon^{(1)}$, apart from the boundary condition, are the same ones considered in [MM].

We use the following notation:

$$\Phi(t) = \int_b^t \varphi^{1/2}(s)\, ds, \qquad c_0 = \int_b^c \varphi^{1/2}(s)\, ds = \Phi(c).$$

If $u \in \mathscr{U}^0$ we set

$$A_u = \{x \in \Omega\colon u(x) \in [c, d]\}.$$

Notice that in this case

$$\Phi \circ u = c_0 \chi_{A_u},$$

$$\int_{\Omega} |D(\Phi(u))| = c_0 P_\Omega(A_u),$$

where $P_\Omega(A_u)$ is the perimeter of the set $A_u$ (see [G]).

We consider the functional $\mathscr{F}^{(1)}$ defined on $\mathscr{U}^0$ as follows:

$$\mathscr{F}^{(1)}(u) = \begin{cases} 2c_0 P_\Omega(A_u) + 2\int_{\partial\Omega} |\widetilde{\Phi(u)} - \Phi(g)|\, d\mathscr{H}^{n-1} & \text{if} \quad P_\Omega(A_u) < +\infty, \\ +\infty & \text{otherwise in } \mathscr{U}^{(0)}, \end{cases}$$

where $\widetilde{\Phi(u)}$ denotes the trace of $\Phi(u)$ on $\partial\Omega$, which is well defined whenever the perimeter of $A_u$ is finite.

**Theorem 2.2.** *We have*

$$\Gamma - \lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon^{(1)} = \mathscr{F}^{(1)} \qquad \textit{in } \mathscr{U}^0.$$

*Proof.* Fix $\varepsilon_j \downarrow 0$ and $u \in \mathscr{U}^0$.

*First step.* We show that, for each sequence $u_j \to u$ in $L^1(\Omega)$, *we have*

$$\liminf_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(1)}(u_j) \geq \mathscr{F}^{(1)}(u).$$

Without loss of generality, we can restrict ourselves to the case $\{u_j\} \subset H^1(\Omega)$ with $u_{j|\partial\Omega} = g$, and we can also assume that the limit

$$\lim_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(1)}(u_j)$$

exists and is finite. In particular we may assume that $\mathscr{F}_{\varepsilon_j}^{(1)}(u_j) \leq C$.

From the inequality

$$\int_\Omega \varepsilon_j |Du_j|^2 + \int_\Omega \frac{1}{\varepsilon_j}\varphi(u_j) \geq 2\int_\Omega |Du_j|\varphi^{1/2}(u_j) = 2\int_\Omega |D(\Phi(u_j(x)))|$$

and the lower semicontinuity of the total variation we get that $\Phi \circ u \in BV(\Omega)$. In particular, the trace of $(\Phi \circ u)$ on $\partial\Omega$ is defined. Applying again the same inequality we obtain

$$\begin{aligned} \liminf_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(1)}(u_j) &\geq 2 \liminf_{j\to+\infty} \int_\Omega |D(\Phi \circ u_j)|\, dx \\ &\geq 2\left\{\int_\Omega |D(\Phi\circ u)| + \int_{\partial\Omega} |\widetilde{\Phi(u)} - \Phi(g)|\, d\mathscr{H}^{n-1}\right\} = \mathscr{F}^{(1)}(u). \end{aligned}$$

The last inequality is a consequence of the lower semicontinuity in $L^1(\Omega)$ of the functional

$$\int_\Omega |Dv| + \int_{\partial\Omega} |v - h|\, d\mathscr{H}^{n-1},$$

defined for $v \in BV(\Omega)$ and with $h \in L^1(\partial\Omega)$ fixed (see [G]).

*Second step.* We must exhibit a sequence $\{u_j\}$ converging to $u$ in $L^1(\Omega)$ such that

$$\lim_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(1)}(u_j) = \mathscr{F}^{(1)}(u).$$

The other case being obvious, we can assume that $\mathscr{F}^{(1)}(u) < +\infty$.

We can reduce the proof to a situation in which the set $\partial^* A_u$ is regular and intersects $\partial\Omega$ transversally: to do that we can use approximation results as in [BB], [B], and [OS].

As in the proof of Theorem 2.1 we can get a sequence $\{f_j\} \subset W^{1,\infty}(\Omega)$ such that $f_{j|\partial\Omega} = g, f_j \to u$ in $L^1(\Omega)$ and a.e., and such that the following estimate holds:

$$\|Df_j\|_{L^2} \leq C\varepsilon_j^{1/4}.$$

Consider the following partition of $\Omega$:

$$A_j = \{x \in A_u\colon d(x, A_u^C) > K\varepsilon_j \text{ and } d(x, \partial\Omega) > K\varepsilon_j\},$$

$$V_j = \{x \in A_u\colon d(x, A_u^C) \leq K\varepsilon_j\} \cup \{x \in \Omega\colon d(x, \partial\Omega) \leq K\varepsilon_j\},$$

$$B_j = \Omega \backslash (A_j \cup V_j),$$

where $K$ is a suitable positive constant. We now define

$$u_j(x) = \max\{\min\{f_j(x), b\}, a\} \qquad \text{if} \quad x \in B_j,$$

$$u_j(x) = \max\{\min\{f_j(x), d\}, c\} \qquad \text{if} \quad x \in A_j.$$

Finally, we may extend the function $u_j$ with functions constructed as in [B], [BB], and [OS]: these functions are responsible, at the limit, for the perimeter and the boundary integral which appear in $\mathscr{F}^{(1)}$. On the other hand, the integrals on $A_j$ and $B_j$ vanish.

A simplified discussion of the ideas used in the papers above can be found in a remark on p. 186, of [ABV], where the proof of case $n = 1$ is outlined. □

With this last Γ-limit we have gained further information on the asymptotic behavior of the minimizers of our family of functionals, in particular, we can now say that, for a limit point $u$ of minimizers, the set $A_u$ has minimal boundary. In other words, by employing the usual notation we have

$$\mathscr{U}^1 = \{u \in \mathscr{U}^0\colon A_u \text{ minimizes } H(A)\},$$

where

$$H(A) := 2c_0 P_\Omega(A) + 2\int_{\partial\Omega} |c_0\chi_A - \Phi(g)|\, d\mathscr{H}^{n-1}$$

is defined on the subsets of $\Omega$ with finite perimeter and

$$m_1 = \inf \mathscr{F}^{(1)} = \min_A H(A).$$

As we have said before, we are able to compute a further step in the asymptotic development of $\mathscr{F}_\varepsilon$ when $\Omega$ is an open interval of $\mathbf{R}$, for example, $\Omega = (0, 1)$.

Put

$$g(x) = \begin{cases} u_0 & \text{if} \quad x = 0, \\ u_1 & \text{if} \quad x = 1, \end{cases}$$

with $u_0 \in (a, b)$ and $u_1 \in (c, d)$. In this case we have

$$m_1 = \inf_{\mathscr{U}^0} \mathscr{F}^{(1)} = 2c_0,$$

$$\mathscr{U}^1 = \{u \in \mathscr{U}^0 \colon A_u \text{ is of the type } (\bar{x}, 1) \text{ with } \bar{x} \in [0, 1]\}.$$

We define

$$\mathscr{F}_\varepsilon^{(2)}(u) = \frac{\mathscr{F}_\varepsilon^{(1)}(u) - m_1}{\varepsilon}$$

and

$$\mathscr{F}^{(2)}(u) = \begin{cases} \displaystyle\int_{\Omega\setminus\{\bar{x}\}} (u')^2\, dx & \text{if } \quad u \in \mathscr{W}, \\ +\infty & \text{otherwise in } \mathscr{U}^1, \end{cases}$$

where

$$\mathscr{W} = \{u \in \mathscr{U}^1 \text{ with } \bar{x} \neq 0 \text{ and } \bar{x} \neq 1,\ u \in H^1(A_u) \cap H^1(\Omega\setminus A_u),$$
$$u(0) = u_0,\ u(1) = u_1,\ u^-(\bar{x}) = b,\ u^+(\bar{x}) = c\}.$$

Here $u^-(\bar{x})$ and $u^+(\bar{x})$ denote the traces of $u$ from the left and from the right at the point $\bar{x}$.

**Theorem 2.3.** *If* $\Omega = (0, 1) \subset \mathbf{R}$, *we have*

$$\Gamma - \lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon^{(2)} = \mathscr{F}^{(2)} \qquad \textit{in } \mathscr{U}^1.$$

*Proof.* As usual, fix $\varepsilon_j \downarrow 0$ and $u \in \mathscr{U}^1$.

*First step.* We show that for each sequence $u_j \to u$ in $L^1(\Omega)$ we have the inequality

$$\liminf_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(2)}(u_j) \geq \mathscr{F}^{(2)}(u).$$

With no loss of generality, we may suppose that the limit

$$\lim_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(2)}(u_j)$$

exists and is finite. In particular we assume $\mathscr{F}_{\varepsilon_j}^{(2)}(u_j) < C < +\infty$.

The set $A_u$ is of the form $(\bar{x}, 1]$, with $\bar{x} \in [0, 1]$. Suppose for the moment that $\bar{x} \in (0, 1)$, later we shall see that if this is not the case we must have

$$\liminf_{j\to+\infty} \mathscr{F}_{\varepsilon_j}^{(2)}(u_j) = +\infty.$$

Next put

$$x_1 = \operatorname{ess\,sup}\{x \in [0, 1]: u(x) < b\},$$
$$x_2 = \operatorname{ess\,inf}\{x \in [0, 1]: u(x) > c\}.$$

Of course, $x_1 \le \bar{x} \le x_2$.

As a consequence of our position, for each $\delta > 0$ there exists a $\sigma > 0$ and a subset of positive measure of $(x_2, x_2 + \delta)$ such that $u(x) > c + \sigma$. As $u_j$ can be supposed to converge to $u$ almost everywhere in $\Omega$, we can find a point $x_2^* \in (x_2, x_2 + \delta)$ such that $u_j(x_2^*) > c$ for $j$ large enough.

For the same reason, for every $\delta > 0$ there is a point $x_1^* \in (x_1 - \delta, x_1)$ such that $u_j(x_1^*) < b$ for $j$ large enough.

By denoting $I_\delta^*$ the interval $(x_1^*, x_2^*)$ we have

$$C \ge \mathscr{F}_{\varepsilon_j}^{(2)}(u_j) = \frac{1}{\varepsilon_j}\left[\varepsilon_j \int_{I_\delta^*} (u_j')^2 + \frac{1}{\varepsilon_j}\int_{I_\delta^*} \varphi(u_j) - m_1\right] + \int_{\Omega\setminus I_\delta^*} (u_j')^2 + \frac{1}{\varepsilon_j^2}\int_{\Omega\setminus I_\delta^*} \varphi(u_j). \tag{2.2}$$

The expression between the square brackets is positive: in fact, the sum of the two integrals is greater than or equal to

$$2\int_{I_\delta^*} |D(\Phi \circ u_j)|,$$

and this last expression is greater than $m_1$ because $u_j(x_1^*) < b$ and $u_j(x_2^*) > c$.

Because the last integral of (2.2) is also positive, we get the estimate

$$\int_{\Omega\setminus I_\delta^*} (u_j')^2\, dx \le C,$$

hence as $\delta$ is arbitrary and the constant $C$ does not depend on $\delta$ we get that $u \in H^1(\Omega\setminus(\{x_1\} \cup \{\bar{x}\} \cup \{x_2\}))$ and that the $L^2$-norm of the derivative on this open set is dominated by the constant $C$ (recall that the function $u$ is indeed constant in the intervals $(x_1, \bar{x})$ and $(\bar{x}, x_2)$).

To conclude the proof in the case $\bar{x} \in (0, 1)$ we only need to show that $u^+(x_2) = c$ and $u^-(x_1) = b$. In fact, this implies that $u \in \mathscr{W}$, while (2.2) gives the desired estimate on the minimum limit of the sequence $\mathscr{F}_{\varepsilon_j}^{(2)}(u_j)$.

Suppose by contradiction that $u^+(x_2) = c + h$ with $h > 0$. By using the regularity of $u$ proved above we have that there exists a constant $\beta > 0$ such that $u(x) > c + \frac{2}{3}h$ in $(x_2, x_2 + \beta)$ and $u(x) \le c$ in $(x_2 - \beta, x_2)$. As the $u_j$ converge to $u$ almost everywhere, for each fixed $k \in \mathbf{N}$ we can find an index $j_k$ and two points $z_k', z_k'' \in (x_2 - 1/k, x_2 + 1/k)$ with $z_k' < z_k''$ such that $j_k \to +\infty$, $c \le u_{j_k}(z_k') < c + \frac{1}{3}h$, and $u_{j_k}(z_k'') > c + \frac{2}{3}h$. In this situation we have the estimate

$$\mathscr{F}_{\varepsilon_j}^{(2)}(u_{j_k}) \ge \frac{1}{\varepsilon_{j_k}}\left[2\int_0^{z_{j_k}'} |D(\Phi \circ u_{j_k})|\, dx\right] + \int_{z_{j_k}'}^{z_{j_k}''} (u_{j_k}')^2\, dx.$$

The part between the square brackets is positive because $u_{j_k}(z'_k) \geq c$, while the last integral is greater than or equal to

$$(z''_k - z'_k)\left(\frac{h/3}{z'_k - z''_k}\right)^2 = \frac{h^2 k}{18}.$$

As the last expression goes to $+\infty$ as $k \to +\infty$ we get a contradiction with the assumption that

$$\liminf_{j \to +\infty} \mathscr{F}^{(2)}_{\varepsilon_j}(u_j) < +\infty.$$

With a similar proof we can show that if $\bar{x} = 0$ or $\bar{x} = 1$ we necessarily have

$$\liminf_{j \to +\infty} \mathscr{F}^{(2)}_{\varepsilon_j}(u_j) = +\infty,$$

and the first step of the proof is complete.

*Second step.* We exhibit a sequence $\{u_j\}$ such that $u_j \to u$ in $L^1(\Omega)$ and

$$\lim_{j \to +\infty} \mathscr{F}^{(2)}_{\varepsilon_j}(u_j) = \mathscr{F}^{(2)}(u).$$

We can assume $\mathscr{F}^{(2)}(u) < +\infty$, because otherwise the construction is trivial, and so we have $u \in H^1(\Omega \backslash \{\bar{x}\})$ and $u(0) = u_0$, $u(1) = u_1$, $u^-(\bar{x}) = b$, $u^+(\bar{x}) = c$.

We define

$$u_j(x) = \begin{cases} u(x) & \text{if } 0 \leq x \leq \bar{x}, \\ \eta_j(x - \bar{x}) & \text{if } \bar{x} \leq x \leq \bar{x} + \xi_j, \\ u\left(\dfrac{1 - \bar{x}}{1 - \bar{x} - \xi_j}(x - \bar{x} - \xi_j) + \bar{x}\right) & \text{if } \bar{x} + \xi_j \leq x \leq 1. \end{cases}$$

The functions $\eta_j$ and the sequence $\xi_j$ are defined in the following lemma.

**Lemma 2.4.** *It is possible to find increasing solutions of the following sequence of differential problems*:

$$(\mathrm{P}_j) \quad \begin{cases} \varepsilon_j \eta'_j = \varphi^{1/2}(\eta_j) + \varepsilon_j \delta_j, \\ \eta_j(0) = b, \quad \eta_j(\xi_j) = c, \end{cases}$$

*with* $\xi_j \to 0$. *The sequence* $\delta_j$ *above is defined by*

$$\delta_j^2 = \operatorname{meas}(\{s \in [b, c]: \varphi^{1/2}(s) \leq \varepsilon_j^{1/2}\}).$$

*Obviously*, $\delta_j \to 0$.

*Proof.* Consider a solution of the Cauchy problem corresponding to $(\mathrm{P}_j)$ without the condition $\eta_j(\xi_j) = c$. The solution of this problem is globally defined and is strictly increasing. We call $\xi_j$ the time this solution takes to reach the value $c$. More precisely we put $\xi_j = (\inf\{t \in \mathbf{R}: \eta_j(t) = c\})$.

We have to show that the sequence $\xi_j$ goes to zero as $j \to +\infty$. Put $E_j = \{s \in [b, c]: \varphi^{1/2}(s) > \varepsilon_j^{1/2}\}$. We have

$$\begin{aligned}
\xi_j &= \int_b^c \frac{\varepsilon_j \, ds}{\varphi^{1/2}(s) + \varepsilon_j \delta_j} \\
&= \int_{E_j} \frac{\varepsilon_j \, ds}{\varphi^{1/2}(s) + \varepsilon_j \delta_j} + \int_{[b,c]\setminus E_j} \frac{\varepsilon_j \, ds}{\varphi^{1/2}(s) + \varepsilon_j \delta_j} \\
&\le \varepsilon_j^{1/2}(c - b) + \frac{\operatorname{meas}([b - c]\setminus E_j)}{\delta_j} \\
&= \varepsilon_j^{1/2}(c - b) + \delta_j.
\end{aligned}$$

As the last expression vanishes as $j \to +\infty$ the lemma is proved. □

We are now ready to conclude the second step of the proof of Theorem 2.3, by estimating the maximum limit of $\mathscr{F}^{(2)}_{\varepsilon_j}(u_j)$ (of course the sequence $u_j$ converges to $u$ in $L^1)\Omega)$).

On $(0, 1)\setminus[\bar{x}, \bar{x} + \xi_j]$ we get

$$\begin{aligned}
&\limsup_{j \to +\infty} \int_{(0,1)\setminus[\bar{x}, \bar{x}+\xi_j]} \left[ (u_j')^2 + \frac{1}{\varepsilon_j^2} \varphi(u_j) \right] dx \\
&\quad \le \limsup_{j \to +\infty} \int_0^{\bar{x}} (u_j')^2 \, dx + \limsup_{j \to +\infty} \int_{\bar{x}+\xi_j}^1 (u_j')^2 \, dx = \int_{(0,1)\setminus\{\bar{x}\}} (u')^2 \, dx.
\end{aligned}$$

Finally, on $(\bar{x}, \bar{x} + \xi_j)$ we have the estimate

$$\begin{aligned}
&\limsup_{j \to +\infty} \left[ \int_{\bar{x}}^{\bar{x}+\xi_j} \left[ (u_j')^2 + \frac{1}{\varepsilon_j^2} \varphi(u_j) \right] dx - \frac{2}{\varepsilon_j} \int_b^c \varphi^{1/2}(s) \, ds \right] \\
&\quad = \limsup_{j \to +\infty} \int_0^{\xi_j} \left[ (\eta_j')^2 + \frac{1}{\varepsilon_j^2} \varphi(\eta_j) - \frac{2}{\varepsilon_j} \eta_j' \varphi^{1/2}(\eta_j) \right] dx \\
&\quad = \limsup_{j \to +\infty} \int_0^{\xi_j} \left[ \eta_j' - \frac{\varphi^{1/2}(\eta_j)}{\varepsilon_j} \right]^2 dx \\
&\quad = \limsup_{j \to +\infty} \int_0^{\xi_j} \delta_j^2 \, dx = 0.
\end{aligned}$$

By adding these estimates we get the claim, and the proof is complete. □

In the more general situation $\Omega \subset \mathbf{R}^n$ with $n > 1$, the simplest generalization of the limit functional would be

$$\mathscr{F}^{(2)}(u) = \begin{cases} \displaystyle\int_{\Omega\setminus\partial\bar{A}} |\nabla u|^2 \, dx & \text{if} \quad u \in \mathscr{W}, \\ +\infty & \text{otherwise in } \mathscr{U}^1, \end{cases}$$

where

$$\mathscr{W} = \{u \in \mathscr{U}^1 \cap BV(\Omega)\colon (Du - Du_{|\partial^* A_u}) \in L^2(\Omega);\ \widetilde{\Phi(u)} = \Phi(g)\ \mathscr{H}^{n-1}\text{-a.e. whenever } g \in [a, b[ \cup ]c, d];\ u^-(x) = b \text{ and } u^+(x) = c\ \mathscr{H}^{n-1}\text{-a.e. on } \partial A_u\}.$$

Here $u^-$ and $u^+$ denote the traces of $u$ on the two sides of $\partial^* u$. We remark that in the case where $\partial A_u$ is closed we simply have

$$\mathscr{W} = \{u \in \mathscr{U}^1 \cap H^1(\Omega \backslash \partial A_u)\colon \widetilde{\Phi(u)} = \Phi(g)\ \mathscr{H}^{n-1}\text{-a.e. whenever } g \in [a, b[ \cup ]c, d];\ u^-(x) = b \text{ and } u^+(x) = c\ \mathscr{H}^{n-1}\text{-a.e. on } \partial A_u\}.$$

The following example shows that this functional $\mathscr{F}^{(2)}$ cannot be the Γ-limit of the sequence $\mathscr{F}_\varepsilon^{(2)}$.

**Example 2.1.** Let $\Omega = \{x \in \mathbf{R}^2\colon |x| < 1\}$ and $g(y) \equiv \alpha$ on $\partial\Omega$, $\alpha \in (b, c)$. On the integrand $\varphi$ we make the further assumption that $\int_\alpha^c \varphi^{-1/2}(s)\, ds < +\infty$ (this is essentially an assumption of the order of zero of $\varphi$ at $c$).

If we choose $\alpha \in (b, c)$ in such a way that $\Phi(c) - \Phi(\alpha) < \Phi(\alpha)$, we have

$$\mathscr{U}^{(1)} = \{u \in L^1(\Omega)\colon u(x) \in [c, d] \text{ a.e. in } \Omega\}.$$

$$m_1 = 4\pi(\Phi(c) - \Phi(\alpha)).$$

In particular, the function $u_\infty \equiv c$ is in $\mathscr{U}^{(1)}$. What we do now is build a sequence $\{u_j\} \subset H^1(\Omega)$ with $u_j \equiv g$ on $\partial\Omega$ and $u_j \to u_\infty$ in $L^1(\Omega)$ such that $\mathscr{F}_{\varepsilon_j}^{(2)}(u_j) = -\beta$, with $\beta > 0$. This shows that the positive functional $\mathscr{F}^{(2)}$ cannot be the Γ-limit of the sequence $\mathscr{F}_\varepsilon^{(2)}$.

Let us consider the solution of the following Cauchy problem:

$$\begin{cases} v'(t) = \varphi^{(1/2)}(v(t)), \\ v(0) = \alpha. \end{cases} \tag{2.3}$$

$v(t)$ is a strictly increasing function on the interval $[0, K]$ with $K = \int_\alpha^c \varphi^{-1/2}(s)\, ds$, and $v(K) = c$. If $h(x)$ denotes the distance of $x \in \Omega$ from $\partial\Omega$, we define

$$u_j(x) = \begin{cases} v\left(\dfrac{h(x)}{\varepsilon_j}\right) & \text{if } \quad h(x) < K\varepsilon_j, \\ c & \text{otherwise in } \Omega. \end{cases}$$

By using tubular neighborhood coordinates, observing that

$$\Phi(c) - \Phi(\alpha) = \int_0^{K\varepsilon_j} \varphi^{1/2}\left(v\left(\frac{t}{\varepsilon_j}\right)\right) \frac{d}{dt}\, v\left(\frac{t}{\varepsilon_j}\right) dt,$$

and recalling (2.3) we get

$$\begin{aligned}
\mathscr{F}^{(2)}_{\varepsilon_j}(u_j) &= \int_\Omega \left[ |Du_j|^2 + \frac{\varphi(u_j)}{\varepsilon_j^2} \right] dx - \frac{4\pi}{\varepsilon_j} (\Phi(c) - \Phi(\alpha)) \\
&= 2\pi \int_0^{K\varepsilon_j} \left\{ \left[ \left| \frac{d}{dt} v\left(\frac{t}{\varepsilon_j}\right) \right|^2 + \frac{1}{\varepsilon_j^2} \varphi\left(v\left(\frac{t}{\varepsilon_j}\right)\right) \right] (1-t) \right. \\
&\qquad \left. - \frac{2}{\varepsilon_j} \varphi^{1/2}\left(v\left(\frac{t}{\varepsilon_j}\right)\right) \frac{d}{dt} v\left(\frac{t}{\varepsilon_j}\right) \right\} dt \\
&= 2\pi \int_0^{K\varepsilon_j} \left[ \frac{d}{dt} v\left(\frac{t}{\varepsilon_j}\right) - \frac{1}{\varepsilon_j} \varphi^{1/2}\left(v\left(\frac{t}{\varepsilon_j}\right)\right) \right]^2 dt \\
&\quad - 4\pi \int_0^{K\varepsilon_j} t \frac{1}{\varepsilon_j} \varphi^{1/2}\left(v\left(\frac{t}{\varepsilon_j}\right)\right) \frac{d}{dt} v\left(\frac{t}{\varepsilon_j}\right) dt \\
&= -4\pi \int_0^K \tau \varphi^{1/2}(v(\tau)) \frac{d}{d\tau} v(\tau)\, d\tau = -4\pi \int_0^K \tau \varphi(v(\tau))\, d\tau = -\beta.
\end{aligned}$$

It is heuristically convincing that sequences of this kind can be constructed whenever $\partial\Omega$ has a curved portion, or whenever $u$ jumps on a curved surface, and the negative term we get seems to depend on some manner on an integral of the curvatures of the surface itself. In particular, a similar phenomenon may occur around the intersections between interior jumps and the boundary of $\Omega$, where we may think there is some concentrated curvature of $\partial A_u$. Anyhow, we have no generally reliable conjecture about the correct form of the second $\Gamma$-limit in a dimension higher than one.

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