

THE NONLINEAR MEMBRANE MODEL AS VARIATIONAL LIMIT OF NONLINEAR THREE-DIMENSIONAL ELASTICITY

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ABSTRACT. — We consider a cylindrical three-dimensional nonlinearly hyperelastic body and we let its thickness go to zero. We show, under appropriate hypotheses on the applied loads, that the deformations that minimize the total energy weakly converge in a Sobolev space toward deformations that minimize a nonlinear membrane energy. The nonlinear membrane energy is obtained by computing the Γ -limit of the sequence of three-dimensional energies.

RÉSUMÉ. — On considère un corps tridimensionnel cylindrique formé d'un matériau hyperélastique non linéaire dont on fait tendre l'épaisseur vers zéro. On montre, sous des hypothèses appropriées sur l'ordre de grandeur des forces appliquées, que les déformations qui minimisent l'énergie totale convergent vers des déformations qui minimisent une énergie de membrane non linéaire. La convergence a lieu au sens de la topologie faible d'un espace de Sobolev. L'énergie de membrane non linéaire est obtenue en calculant la Γ -limite de la suite des énergies tridimensionnelles.

1. Introduction

The purpose of this article is to derive nonlinear membrane models from genuine three-dimensional nonlinear elasticity by means of a rigorous convergence result. More specifically, we consider a sequence of three-dimensional cylindrical bodies which are made of the same hyperelastic material and are submitted to given loadings and boundary conditions of place on the lateral surface. We show that, in a rescaled sense, the deformations that minimize the total energy, or almost minimize it as the case may be, weakly converge in a Sobolev space towards deformations that minimize a nonlinear membrane energy as the thickness goes to zero. The expression of the nonlinear membrane energy is obtained by Γ -convergence arguments.

The main impetus for our work was provided by an article of Acerbi, Buttazzo and Percivale [1] which contains a result, the first one to the best of our knowledge to give firm grounds to lower dimensional models in nonlinear elasticity through a convergence analysis. We must also mention, in a quite different direction, the work of Mielke [1], who rigorously derives nonlinear rod models from nonlinear three-dimensional elasticity via center manifold arguments. There are numerous works on the formal derivation of lower dimensional models for rods, plates, shells and so on, but convergence results were only available in the case of linear elasticity, cf. Destuynder [1] and Ciarlet and Kesavan [1], up

to the article of Acerbi, Buttazzo and Percivale. These authors deal with strings, *i.e.*, one dimensional models, and use the tools of Γ -convergence theory to obtain their convergence result. For the two-dimensional models we consider here, nonlinear membranes, little seemed to be known concerning an asymptotic derivation, apart from the work of Fox, Raoult and Simo [1], [2]. In this work, Fox, Raoult and Simo obtain among other results a nonlinear membrane model for the Saint Venant-Kirchhoff material by formal asymptotic expansions in powers of the thickness. It turns out, and it is very surprising, that such natural formal expansions do not always yield the right result, as the application of our convergence theorem to the Saint Venant-Kirchhoff material shows.

The article is as follows. In Section 2, we state the three-dimensional problem and describe the rescaling that makes it amenable to the Γ -convergence analysis we perform in Section 3. In Acerbi, Buttazzo and Percivale [1], the limit model is one-dimensional so that convexity arguments can be used. Here, the limit model is two-dimensional, hence it involves the more complicated notion of quasiconvexity, *see* Morrey [1] and also Pipkin [1] for elastic membranes. In Section 4, the Γ -convergence result is translated in terms of convergence of the rescaled energy minimizers and we discuss the nonlinear membrane model thus obtained. Section 5 is devoted to an analysis of the consequences of material frame-indifference for nonlinear membranes. We show that the nonlinear membrane model is frame-indifferent, a property shared by geometrically exact model, *cf.* Antman [1], Naghdi [1], but often enough not satisfied by limit models obtained via an asymptotic procedure. A striking consequence of frame-indifference is that, if the reference configuration is a natural state and an absolute minimizer of the stored energy function of the three-dimensional bodies, then the corresponding nonlinear membranes offer no resistance to crumpling. This is an empirical fact, witnessed by anyone who ever played with a deflated balloon. Material symmetries are also considered. Finally, we close the article by giving an explicit formula for the nonlinear membrane energy corresponding to a Saint Venant-Kirchhoff material.

The results of this article were announced in Le Dret and Raoult [1], [2].

2. The three-dimensional and rescaled problems

For all $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^3; (x_1, x_2) \in \omega, |x_3| < \varepsilon\}$, where ω is an open, bounded subset of \mathbb{R}^2 with Lipschitz boundary. Let M_3 be the space of real 3×3 matrices endowed with the usual Euclidean norm $\|F\| = \sqrt{\text{tr}(F^T F)}$. For all $z_i \in \mathbb{R}^3$, $i = 1, 2, 3$, we note $(z_1|z_2|z_3)$ the matrix whose i -th column is z_i . Let $W : M_3 \rightarrow \mathbb{R}$ be a continuous function that satisfies the following growth and coercivity hypotheses:

$$(1) \quad \begin{cases} \exists C > 0, \quad \exists p \in]1, +\infty[, \quad \forall F \in M_3, & |W(F)| \leq C(1 + \|F\|^p), \\ \exists \alpha > 0, \quad \exists \beta \geq 0, \quad \forall F \in M_3, & W(F) \geq \alpha \|F\|^p - \beta. \end{cases}$$

We assume that Ω_ε is the reference configuration of a hyperelastic homogeneous three-dimensional body whose stored energy function is W . We do not consider here the more physical case of a function W such that $W(F) = +\infty$ if $\det F \leq 0$ and $W(F) \rightarrow +\infty$ when $\det F \rightarrow 0^+$. This case will be treated in a further work.

We assume for simplicity that the bodies are submitted to the action of dead loading body force densities $f^\varepsilon \in L^q(\Omega_\varepsilon; \mathbb{R}^3)$ and surface traction densities $g^\varepsilon \in L^r(S_\varepsilon; \mathbb{R}^3)$ on $S_\varepsilon = \omega \times \{\pm\varepsilon\}$, the top and bottom surfaces of Ω_ε . For the sake of definiteness, we assume that $q = r$ and $1/p + 1/q = 1$, but other choices are indeed possible at no extra cost. Let $\Gamma_\varepsilon = \partial\omega \times]-\varepsilon, \varepsilon[$ be the lateral surface of Ω_ε . We assume that the deformations of the bodies satisfy a boundary condition of place on Γ_ε . The equilibrium problem may be formulated as a minimization problem:

$$(2) \quad \text{find } \phi^\varepsilon \in \Phi_\varepsilon \text{ such that } I_\varepsilon(\phi^\varepsilon) = \inf_{\psi \in \Phi_\varepsilon} I_\varepsilon(\psi),$$

where the total energy I_ε is

$$(3) \quad I_\varepsilon(\psi) = \int_{\Omega_\varepsilon} W(\nabla \psi) dx - \int_{\Omega_\varepsilon} f^\varepsilon \cdot \psi dx - \int_{S_\varepsilon} g^\varepsilon \cdot \psi d\sigma$$

and the set of admissible deformations is

$$(4) \quad \Phi_\varepsilon = \{\psi \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3); \psi(x) = x \text{ on } \Gamma_\varepsilon\}.$$

See Wang and Truesdell [1], Marsden and Hughes [1] or Ciarlet [1], among others, for general references on three-dimensional nonlinear elasticity. A key-ingredient in existence proofs using the direct method in the calculus of variations is the sequential weak lower semi-continuity of the energy functional I_ε on $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$. Under the assumptions (1), it is known that the energy functional I_ε in problem (2) is sequentially weakly lower semi-continuous on $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ if and only if the function W is quasiconvex, i.e.

$$(5) \quad \forall F \in M_3, \quad \forall \varphi \in W_0^{1,\infty}(D; \mathbb{R}^3), \quad \int_D W(F + \nabla \varphi(x)) dx \geq (\text{meas } D) W(F),$$

where D is any bounded domain of \mathbb{R}^3 , see Morrey [1], Acerbi and Fusco [1], Dacorogna [1]. Problem (2) was solved in the case $W(F) = +\infty$ if $\det F \leq 0$ and $W(F) \rightarrow +\infty$ when $\det F \rightarrow 0^+$ by Ball [1], under an assumption of polyconvexity of W , a notion more restrictive than quasiconvexity, plus appropriate growth and coercivity assumptions. For our purposes here, it is not desirable to assume at the onset that W is quasiconvex or polyconvex. There are two reasons for this. First of all, the zero thickness limit model we obtain always involves a quasiconvexification, which has to be effected whether W is quasiconvex or not. Secondly, we do not want to rule out important examples, such as the Saint Venant-Kirchhoff stored energy function which is neither polyconvex nor quasiconvex, see Raoult [1]. Consequently, we *do not* assume that W is quasiconvex and problem (2) may well not possess any solutions. Naturally, if it does have solutions which are thus actual equilibrium deformations of the bodies, our results apply to these deformations.

Let us thus be given a diagonal minimizing sequence ϕ^ε for the sequence of energies I_ε over the sets Φ_ε . More specifically, we assume that

$$(6) \quad \phi^\varepsilon \in \Phi_\varepsilon, \quad I_\varepsilon(\phi^\varepsilon) \leq \inf_{\psi \in \Phi_\varepsilon} I_\varepsilon(\psi) + \varepsilon h(\varepsilon),$$

where h is a positive function such that $h(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. Such a sequence always exists and if the minimization problems have solution, ϕ^ε may be chosen to be such a solution.

In order to obtain a membrane model in the limit, it is of crucial importance to specify the order of magnitude of the applied loads. In effect, it is always possible to stretch all thin cylinders Ω_ε into the same block, say Ω_1 , by applying sufficiently large forces. For such forces, the limit behavior is obviously not that of a membrane. It turns out that the right order of magnitude is given by $\|f^\varepsilon\|_{L^q(\Omega_\varepsilon; \mathbb{R}^3)} \leq C\varepsilon^{1/q}$ and $\|g^\varepsilon\|_{L^q(S_\varepsilon; \mathbb{R}^3)} \leq C\varepsilon$ where the constant C does not depend on ε . For example, the weight of the material, $F^\varepsilon(x) = (0, 0, -\rho g)^T$, is allowed. Note that it was known for a long time that a nonlinear plate could not sustain its own weight, see for example Ciarlet and Destuynder [1]. This is no longer surprising as was already noted by Fox, Raoult and Simo [1], [2] and in view of our results, since a thin nonlinearly elastic body submitted to its own weight does not behave like a plate, but indeed like a membrane. The scalings of displacements that are used to derive plate equations are inappropriate for such loadings. These scalings imply for example that the flexural displacements should be of the order of ε , whereas we obtain here that these displacements are roughly speaking of the order of 1, so that nonlinear plate models are unable to handle this situation.

In order to rescale the problem, we let $\Omega = \Omega_1$, $\Gamma = \Gamma_1$ and $S = S_1$ and define a rescaling operator Θ_ε by $(\Theta_\varepsilon \psi)(x_1, x_2, x_3) = \psi(x_1, x_2, \varepsilon x_3)$. Let $\phi(\varepsilon) = \Theta_\varepsilon \phi^\varepsilon$ and $\phi_0(\varepsilon)(x) = (x_1, x_2, \varepsilon x_3)$. Note that all components are treated in the same way: we only transport ϕ^ε on the fixed domain Ω . This is the same rescaling as that used in Fox, Raoult and Simo [1]. The rescaled displacement $u(\varepsilon) = \phi(\varepsilon) - \phi_0(\varepsilon)$ belongs to $V = W_{\Gamma}^{1,p}(\Omega; \mathbb{R}^3)$. We accordingly rescale the energies by setting $I(\varepsilon)(\psi) = \varepsilon^{-1} I_\varepsilon(\Theta_\varepsilon^{-1} \psi)$, i.e.,

$$(7) \quad I(\varepsilon)(\psi) = \int_{\Omega} W \left(\left(\partial_1 \psi \mid \partial_2 \psi \mid \frac{\partial_3 \psi}{\varepsilon} \right) \right) dx - \int_{\Omega} f(\varepsilon) \cdot \psi dx - \int_S \varepsilon^{-1} g(\varepsilon) \cdot \psi d\sigma,$$

or in terms of the rescaled displacements

$$(8) \quad J(\varepsilon)(v) = \int_{\Omega} W \left(\left(e_1 + \partial_1 v \mid e_2 + \partial_2 v \mid e_3 + \frac{\partial_3 v}{\varepsilon} \right) \right) dx \\ - \int_{\Omega} f(\varepsilon) \cdot (\phi_0(\varepsilon) + v) dx - \int_S \varepsilon^{-1} g(\varepsilon) \cdot (\phi_0(\varepsilon) + v) d\sigma,$$

where $f(\varepsilon) = \Theta_\varepsilon f^\varepsilon$ and $g(\varepsilon) = \Theta_\varepsilon g^\varepsilon$. It is immediate that

$$(9) \quad J(\varepsilon)(u(\varepsilon)) \leq \inf_{v \in V} J(\varepsilon)(v) + h(\varepsilon).$$

For simplicity, we assume that $f(\varepsilon) = f$ and $\varepsilon^{-1} g(\varepsilon) = g$ are independent of ε .

3. Computation of the Γ -limit of the rescaled energies

We use Γ -convergence theory to determine the asymptotic behavior of the rescaled displacements $u(\varepsilon)$ when $\varepsilon \rightarrow 0$. In the sequel, the thickness parameter ε will take its values in a sequence $\varepsilon_n \rightarrow 0$. Since the results do not depend on the sequence in question, and for notational brevity, we will simply use the notation ε . Let us recall that a sequence of functions G_ε from a metric space X into $\overline{\mathbb{R}}$ is said to Γ -converge toward G_0 for the topology of X if the following two conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_\varepsilon \rightarrow x, & \liminf G_\varepsilon(x_\varepsilon) \geq G_0(x), \\ \exists y_\varepsilon \rightarrow x, & G_\varepsilon(y_\varepsilon) \rightarrow G_0(x). \end{cases}$$

If the sequence G_ε Γ -converges, its Γ -limit is alternatively given by

$$G_0(x) = \min \{ \liminf G_\varepsilon(x_\varepsilon); x_\varepsilon \rightarrow x \}.$$

In addition, the set of functions from X into $\overline{\mathbb{R}}$ has a sequential compactness property with respect to Γ -convergence in the sense that any sequence $G_\varepsilon: X \rightarrow \overline{\mathbb{R}}$ admits a Γ -convergent subsequence. The main interest of Γ -convergence is that if the minimizers of G_ε stay in a compact set of X for all ε , then their limit points are minimizers of G_0 , see De Giorgi and Franzoni [1], Attouch [1], Dal Maso [1].

We do not use $J(\varepsilon)$ directly, since this would imply working with the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$, which is non metrizable on unbounded sets. Instead, we extend the energies to $L^p(\Omega; \mathbb{R}^3)$ by setting

$$(10) \quad \forall v \in L^p(\Omega; \mathbb{R}^3), \quad \tilde{J}(\varepsilon)(v) = \begin{cases} J(\varepsilon)(v) & \text{if } v \in V, \\ +\infty & \text{otherwise.} \end{cases}$$

This is a classical trick used in the applications of Γ -convergence. It has the additional virtue of incorporating the boundary conditions in the energy functional.

Let us now proceed to compute the Γ -limit of the sequence $\tilde{J}(\varepsilon)$ for the strong topology of $L^p(\Omega; \mathbb{R}^3)$. Let $M_{3,2}$ be the space of 3×2 real matrices endowed with the usual Euclidean norm $\|\overline{F}\| = \sqrt{\text{tr}(\overline{F}^T \overline{F})}$. We note $(z_1|z_2)$ the matrix of $M_{3,2}$ whose α -th column is $z_\alpha \in \mathbb{R}^3$. For all $\overline{F} = (z_1|z_2) \in M_{3,2}$ and $z \in \mathbb{R}^3$, we also note $(\overline{F}|z)$ the matrix whose first two columns are z_1 and z_2 and whose third column is z .

As in Acerbi, Buttazzo and Percivale [4] for elastic strings, we define $W_0: M_{3,2} \rightarrow \mathbb{R}$ by

$$(11) \quad W_0(\overline{F}) = \inf_{z \in \mathbb{R}^3} W((\overline{F}|z)).$$

Due to the coercivity assumption (1), it is clear that this function is well defined. Besides, since W is continuous, the infimum is attained.

PROPOSITION 1. — *The function W_0 is continuous and satisfies the growth and coercivity estimates:*

$$(12) \quad \begin{cases} \exists C' > 0, \quad \forall \overline{F} \in M_{3,2}, & |W_0(\overline{F})| \leq C'(1 + \|\overline{F}\|^p), \\ \forall \overline{F} \in M_{3,2}, & W_0(\overline{F}) \geq \alpha \|\overline{F}\|^p - \beta. \end{cases}$$

Proof. — Since W_0 is an infimum of continuous functions, it is upper semi-continuous. Let $\bar{F} \in M_{3,2}$ and consider a sequence $\bar{F}^n \in M_{3,2}$ such that $\bar{F}^n \rightarrow \bar{F}$ as $n \rightarrow +\infty$. Because of the coercivity assumption (1), there exists a compact set K such that for all \bar{F}^n the infimum in definition (11) is attained at a point $z^n \in K$. Consider a subsequence, still denoted n , such that $W_0(\bar{F}^n)$ converges when $n \rightarrow +\infty$. We extract a further subsequence such that $z^n \rightarrow z \in K$. By continuity of W , $W_0(\bar{F}^n) = W((\bar{F}^n|_{z^n})) \rightarrow W((\bar{F}|_z)) \geq W_0(\bar{F})$. As this is true for all subsequences such that $W_0(\bar{F}^n)$ converges, it follows that $\liminf W_0(\bar{F}^n) \geq W_0(\bar{F})$, hence W_0 is lower semi-continuous.

For all $\bar{F} \in M_{3,2}$, let z_0 be a point where the infimum in definition (11) is attained. Thus, $W_0(\bar{F}) = W((\bar{F}|_{z_0})) \geq \alpha \|(\bar{F}|_{z_0})\|^p - \beta \geq \alpha \|\bar{F}\|^p - \beta$. Hence W_0 is coercive. Therefore, W_0 is nonnegative outside of a compact set K' . Since $|W_0|$ is continuous, it is bounded on K' and for $\bar{F} \notin K'$, $|W_0(\bar{F})| = W_0(\bar{F}) \leq W((\bar{F}|_0)) \leq C(1 + \|(\bar{F}|_0)\|^p) = C(1 + \|\bar{F}\|^p)$, which proves the growth estimate. \square

Let $QW_0 = \sup \{Z : M_{3,2} \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq W_0\}$ be the quasiconvex envelope of W_0 , see Dacorogna [1] for the definition and properties of quasiconvex functions and quasiconvex envelopes. Let us introduce the space

$$(13) \quad V_M = \{v \in V; \partial_3 v = 0\},$$

which we call the space of membrane displacements. It is canonically isomorphic to $W_0^{1,p}(\omega; \mathbb{R}^3)$ and we let \bar{v} denote the element of $W_0^{1,p}(\omega; \mathbb{R}^3)$ that is associated with $v \in V_M$ through this isomorphism. The expression of the Γ -limit of the sequence $\tilde{J}(\varepsilon)$ is given in the following theorem.

THEOREM 2. — *The sequence $\tilde{J}(\varepsilon)$ Γ -converges for the strong topology of $L^p(\Omega; \mathbb{R}^3)$ when $\varepsilon \rightarrow 0$. Let $\tilde{J}(0)$ be its Γ -limit. For all $v \in L^p(\Omega; \mathbb{R}^3)$, $\tilde{J}(0)(v)$ is given by*

$$(14) \quad \tilde{J}(0)(v) = \begin{cases} 2 \int_{\omega} QW_0((e_1 + \partial_1 \bar{v}|e_2 + \partial_2 \bar{v})) dx_1 dx_2 \\ - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2 \\ + \infty \end{cases} \quad \begin{array}{l} \text{if } v \in V_M, \\ \text{otherwise.} \end{array}$$

where $\mathcal{F}(x_1, x_2) = \int_{-1}^1 f(x_1, x_2, x_3) dx_3 + g(x_1, x_2, 1) + g(x_1, x_2, -1)$.

For clarity, we break the proof of Theorem 2 into a series of lemmas.

We begin by extracting a Γ -convergent subsequence and call $\tilde{J}(0)$ its Γ -limit. The uniqueness of $\tilde{J}(0)$ will make the extraction of this subsequence superfluous *a posteriori*.

LEMMA 3. — *Let $v(\varepsilon) \in L^p(\Omega; \mathbb{R}^3)$ be a sequence such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \leq C < +\infty$ where C does not depend on ε . Then $v(\varepsilon)$ is uniformly bounded in V and its limit points for the weak topology of V belong to V_M .*

Proof. — Let $v(\varepsilon) \in L^p(\Omega; \mathbb{R}^3)$ be such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \leq C < +\infty$. Then, the definition (10) of the function $\tilde{J}(\varepsilon)$ implies first of all that $v(\varepsilon) \in V$ for all $\varepsilon > 0$. Let us

call $\psi(\varepsilon) = v(\varepsilon) + \phi_0(\varepsilon)$ the deformation that is associated with the displacement $v(\varepsilon)$. The coercivity of the function W and the assumed uniform bound for the energies imply that

$$(15) \quad \alpha \int_{\Omega} \|(\partial_1 \psi(\varepsilon) | \partial_2 \psi(\varepsilon) | \varepsilon^{-1} \partial_3 \psi(\varepsilon))\|^p dx \leq C' (1 + \|\psi(\varepsilon)\|_{W^{1,p}(\Omega; \mathbb{R}^3)})$$

where C' does not depend on ε . It is clear that for all $\varepsilon \leq 1$, $\|(z_1 | z_2 | \varepsilon^{-1} z_3)\| \geq \|(z_1 | z_2 | z_3)\|$. Therefore (15) implies that

$$(16) \quad \alpha \|\nabla \psi(\varepsilon)\|_{L^p(\Omega; \mathbb{R}^3)}^p \leq C' (1 + \|\psi(\varepsilon)\|_{W^{1,p}(\Omega; \mathbb{R}^3)}),$$

which, together with the boundary condition of place $\psi(\varepsilon) = \phi_0(\varepsilon)$ on Γ , yields the desired uniform bound for $\psi(\varepsilon)$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ by Poincaré's inequality. Since $\phi_0(\varepsilon)$ is obviously uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^3)$, the same holds true for $v(\varepsilon)$.

On the other hand, since $\|(z_1 | z_2 | \varepsilon^{-1} z_3)\| \geq \varepsilon^{-1} \|z_3\|$, upon using the bound just established above in inequality (15), we obtain that $\|\partial_3 \psi(\varepsilon)\|_{L^p(\Omega; \mathbb{R}^3)} \leq C'' \varepsilon$, so that $\partial_3 \psi(\varepsilon) \rightarrow 0$ strongly in $L^p(\Omega; \mathbb{R}^3)$. If we let ψ denote any limit point of the sequence $\psi(\varepsilon)$ for the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$, it follows at once that $\partial_3 \psi = 0$. If v denotes the corresponding limit point of the sequence $v(\varepsilon)$, since $v = \psi - \phi_0(0)$ and $\partial_3 \psi_0(0) = 0$, we obtain that v belongs to V_M . \square

COROLLARY 4. — *If $v \in L^p(\Omega; \mathbb{R}^3)$ but $v \notin V_M$, then $\tilde{J}(0)(\tilde{v}) = +\infty$.*

Proof. — Indeed, if $\tilde{J}(0)(v) < +\infty$, there exists a sequence $v(\varepsilon)$ that converges strongly to v in $L^p(\Omega; \mathbb{R}^3)$ and such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \rightarrow \tilde{J}(0)(v)$. Therefore, by Lemma 3, $v \in V_M$. \square

We thus only have to compute the value of the Γ -limit for displacements in V_M . We first give a technical lemma.

LEMMA 5. — *Let $X \hookrightarrow Y$ be two Banach spaces such that X is reflexive and compactly embedded in Y . Consider a functional $G : X \rightarrow \mathbb{R}$ such that for all $v \in X$, $G(v) \geq g(\|v\|_X)$ where g is such that $g(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Let $\tilde{G} : Y \rightarrow \mathbb{R}$ be defined by $\tilde{G}(v) = G(v)$ if $v \in X$, $\tilde{G}(v) = +\infty$ otherwise. Let $\Gamma\text{-}G$ denote the sequential lower semi-continuous envelope of G for the weak topology of X and $\Gamma\text{-}\tilde{G}$ denote the lower semi-continuous envelope of \tilde{G} for the strong topology of Y . Then $\Gamma\text{-}\tilde{G} = \tilde{\Gamma\text{-}G}$.*

Proof. — Both lower semi-continuous envelopes admit a representation formula

$$(17) \quad \begin{cases} \Gamma\text{-}G(v) = \min \{\liminf G(v^\varepsilon) : v^\varepsilon \rightharpoonup v \text{ in } X\}, \\ \Gamma\text{-}\tilde{G}(v) = \min \{\liminf \tilde{G}(v^\varepsilon) : v^\varepsilon \rightarrow v \text{ in } Y\}. \end{cases}$$

Note that these lower semi-continuous envelopes are nothing but the Γ -limits of the constant sequences G and \tilde{G} .

Consider first $v \in X$. Then $\Gamma\text{-}\tilde{G}(v) < +\infty$ and there exists a sequence $v^\varepsilon \rightarrow v$ strongly in Y such that $\tilde{G}(v^\varepsilon) = G(v^\varepsilon) \rightarrow \Gamma\text{-}\tilde{G}(v)$. By the coercivity of G and since X is reflexive, $v^\varepsilon \rightharpoonup v$ weakly in X as well and the sequential weak lower semi-continuity

of Γ - G implies at one that $\Gamma\tilde{G}(v) \geq \Gamma-G(v)$. Conversely, if $v^\varepsilon \rightharpoonup v$ weakly in X is such that $G(v^\varepsilon) \rightarrow \Gamma-G(v)$, then by the compact embedding $v^\varepsilon \rightarrow v$ strongly in Y and $\Gamma-G(v) \geq \Gamma\tilde{G}(v)$. Therefore, $\Gamma-G(v) \geq \Gamma\tilde{G}(v)$ for all $v \in X$.

Let now $v \in Y \setminus X$. Then, $\Gamma\tilde{G}(v) = +\infty$. Indeed, assume for contradiction that $\Gamma\tilde{G}(v) < +\infty$. Thus, there exist a sequence $v^\varepsilon \rightarrow v$ in Y such that $G(v^\varepsilon)$ is uniformly bounded from above. The coercivity of G then implies that v^ε is bounded in X , hence contains a weakly convergent subsequence. The continuous embedding implies that the limit of such a subsequence is v , so that $v \in X$. This contradicts the hypothesis. \square

We now are in a position to give a bound from below for the Γ -limit functional.

PROPOSITION 6. — For all $v \in V_M$, we have that

$$(18) \quad \tilde{J}(0)(v) \geq 2 \int_{\omega} QW_0((e_1 + \partial_1 \bar{v}|e_2 + \partial_2 \bar{v})) dx_1 dx_2 - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

Proof. — Consider any $v \in V_M$. Since $J(\varepsilon)(v)$ is obviously bounded from above independently of ε , it follows that $\tilde{J}(0)(v) < +\infty$. By the definition of Γ -convergence, there exists a sequence $v(\varepsilon) \in V$ such that $v(\varepsilon) \rightarrow v$ strongly in $L^p(\Omega; \mathbb{R}^3)$ and $\tilde{J}(\varepsilon)(v(\varepsilon)) \rightarrow \tilde{J}(0)(v)$. Then, by Lemma 3, $v(\varepsilon) \rightharpoonup v$ weakly in V .

First of all, it is clear that, when $\varepsilon \rightarrow 0$,

$$(19) \quad \int_{\Omega} f \cdot (\phi_0(\varepsilon) + v(\varepsilon)) dx + \int_S g \cdot (\phi_0(\varepsilon) + v(\varepsilon)) d\sigma \rightarrow \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

For the elastic energy, we have that (with $\psi(\varepsilon) = v(\varepsilon) + \phi_0(\varepsilon)$ as usual)

$$(20) \quad \int_{\Omega} W \left(\left(\partial_1 \psi(\varepsilon) | \partial_2 \psi(\varepsilon) | \frac{\partial_3 \psi(\varepsilon)}{\varepsilon} \right) \right) dx \geq \int_{\Omega} W_0((\partial_1 \psi(\varepsilon) | \partial_2 \psi(\varepsilon))) dx \\ \geq \int_{\Omega} QW_0((\partial_1 \psi(\varepsilon) | \partial_2 \psi(\varepsilon))) dx.$$

Let $G : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}$ be defined by

$$(21) \quad G(\psi) = \int_{\Omega} QW_0((\partial_1 \psi | \partial_2 \psi)) dx.$$

Let us define a function $Z : M_3 \rightarrow \mathbb{R}$ by $Z((z_1|z_2|z_3)) = QW_0((z_1|z_2))$. Since QW_0 is quasiconvex, Z is also quasiconvex. Indeed, let d be the unit square in \mathbb{R}^2 and $D = d \times]0, 1[$. Consider any function $\varphi \in \mathcal{D}(D; \mathbb{R}^3)$. For all $y \in]0, 1[$, the function φ_y defined by $\varphi_y(x_1, x_2) = \varphi(x_1, x_2, y)$ belongs to $\mathcal{D}(d; \mathbb{R}^3)$. Hence, for all $F \in M_3$,

$$\int_D Z(F + \nabla \varphi) dx = \int_D QW_0((z_1 + \partial_1 \varphi | z_2 + \partial_2 \varphi)) dx \\ = \int_0^1 \left(\int_d QW_0((z_1 + \partial_1 \varphi_y | z_2 + \partial_2 \varphi_y)) dx_1 dx_2 \right) dy \\ \geq \int_0^1 QW_0((z_1|z_2)) dy = Z(F).$$

This implies that Z is quasiconvex, cf. Dacorogna [1].

We now remark that Z is quasiconvex, bounded below by $-\beta$ and satisfies the growth condition (1) since QW_0 satisfies (12). Therefore, the function G is sequentially weakly lower semi-continuous on $W^{1,p}(\Omega; \mathbb{R}^3)$, see Acerbi and Fusco [1]. Consequently, since $\psi(\varepsilon) \rightharpoonup \psi = v + \phi_0(0)$ in $W^{1,p}(\Omega; \mathbb{R}^3)$.

$$\begin{aligned} (22) \quad & \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} W \left(\left(\partial_1 \psi(\varepsilon) | \partial_2 \psi(\varepsilon) | \frac{\partial_3 \psi(\varepsilon)}{\varepsilon} \right) \right) dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} G(\psi(\varepsilon)) \geq G(\psi) \\ & = 2 \int_{\omega} QW_0((e_1 + \partial_1 \bar{v} | e_1 + \partial_2 \bar{v})) dx, \end{aligned}$$

and the proof is complete. \square

Let us now turn to proving the reverse inequality.

PROPOSITION 7. — For all $v \in V_M$, we have that

$$(23) \quad \tilde{J}(0)(v) \leq 2 \int_{\omega} QW_0((e_1 + \partial_1 \bar{v} | e_2 + \partial_2 \bar{v})) dx_1 dx_2 - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

Proof. — Let us consider $v \in V_M$. For all $w \in W_0^{1,p}(\omega; \mathbb{R}^3)$, we define a displacement

$$(24) \quad v(\varepsilon)(x) = \bar{v}(x_1, x_2) + \varepsilon x_3 w(x).$$

Obviously, $v(\varepsilon) \rightarrow v$ strongly in $W^{1,p}(\Omega; \mathbb{R}^3)$. Let us examine the limit behavior of the sequence $\tilde{J}(\varepsilon)(v(\varepsilon))$. By the dominated convergence theorem and the growth estimate, it is clear that

$$\begin{aligned} & \int_{\Omega} W((\partial_1(\bar{\psi} + \varepsilon x_3 w) | \partial_2(\bar{\psi} + \varepsilon x_3 w) | e_3 + w)) dx \\ & \rightarrow \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx \end{aligned}$$

when $\varepsilon \rightarrow 0$. Consequently,

$$(25) \quad \tilde{J}(\varepsilon)(v(\varepsilon)) \rightarrow \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

As this is true for all $w \in W_0^{1,p}(\omega; \mathbb{R}^3)$, it follows from the definition of Γ -convergence that

$$(26) \quad \tilde{J}(0)(v) \leq \inf_{w \in W_0^{1,p}(\omega; \mathbb{R}^3)} \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

We remark that

$$(27) \quad \inf_{w \in W_0^{1,p}(\omega; \mathbb{R}^3)} \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx \\ = \inf_{w \in L^p(\omega; \mathbb{R}^3)} \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx,$$

by the density of $W_0^{1,p}(\omega; \mathbb{R}^3)$ in $L^p(\omega; \mathbb{R}^3)$ and by the dominated convergence theorem. The function $g : \omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $g(x, z) = W((\partial_1 \bar{\psi}(x) | \partial_2 \bar{\psi}(x) | e_3 + z))$ is a Carathéodory function. Hence, the measurable selection lemma, cf. Ekeland and Temam [1], shows that there exists a measurable function w_0 such that

$$W_0((\partial_1 \bar{\psi}(x) | \partial_2 \bar{\psi}(x))) = W((\partial_1 \bar{\psi}(x) | \partial_2 \bar{\psi}(x) | e_3 + w_0(x)))$$

for almost all $x \in \omega$. Due to the coercivity estimate, $w_0 \in L^p(\omega; \mathbb{R}^3)$ and thus

$$(28) \quad \inf_{w \in L^p(\omega; \mathbb{R}^3)} \int_{\Omega} W((\partial_1 \bar{\psi} | \partial_2 \bar{\psi} | e_3 + w)) dx \leq \int_{\Omega} W_0((\partial_1 \bar{\psi} | \partial_2 \bar{\psi})) dx.$$

Let $G : W_0^{1,p}(\omega; \mathbb{R}^3) \rightarrow \mathbb{R}$ be defined by

$$(29) \quad G(\bar{v}) = 2 \int_{\omega} W_0((\partial_1 \bar{\psi} | \partial_2 \bar{\psi})) dx_1 dx_2 - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

It follows from (28) that for all $v \in V_M$

$$(30) \quad \tilde{J}(0)(v) \leq G(\bar{v}).$$

Let \tilde{G} be defined on $L^p(\Omega; \mathbb{R}^3)$ by $\tilde{G}(v) = G(\bar{v})$ if $v \in V_M$, $\tilde{G}(v) = +\infty$ otherwise. Corollary 4 and (30) then imply that for all $v \in L^p(\Omega; \mathbb{R}^3)$

$$(31) \quad \tilde{J}(0)(v) \leq \tilde{G}(v).$$

Since $\tilde{J}(0)$ is lower semi-continuous on $L^p(\Omega; \mathbb{R}^3)$, it is smaller than the lower semi-continuous envelope of \tilde{G} . It is known, see Acerbi and Fusco [1], that the sequential weak lower semi-continuous envelope $\Gamma\text{-}G$ of G on $W_0^{1,p}(\omega; \mathbb{R}^3)$ is given by

$$(32) \quad \Gamma\text{-}G(\bar{v}) = 2 \int_{\omega} QW_0((\partial_1 \bar{\psi} | \partial_2 \bar{\psi})) dx_1 dx_2 - \int_{\omega} \mathcal{F} \cdot (\phi_0(0) + \bar{v}) dx_1 dx_2.$$

Therefore, Lemma 5 with $X = V_M$, $Y = L^p(\Omega; \mathbb{R}^3)$ and $g(t) = \alpha(t^p - 1)$ implies that

$$(33) \quad \tilde{J}(0)(v) \leq \Gamma\text{-}\tilde{G}(v) = \widetilde{\Gamma\text{-}G}(\bar{v}),$$

which proves the Proposition. \square

Proof of Theorem 2. – Use Corollary 4 for the $v \notin V_M$ and Propositions 6 and 7 for the case $v \in V_M$. \square

4. Convergence of the rescaled deformations and the nonlinear membrane model

We now use Theorem 2 to characterize the asymptotic behavior of diagonal minimizing sequences of rescaled deformations $\phi(\varepsilon)$ for the sequence of rescaled energies $I(\varepsilon)$, which are such that $I(\varepsilon)(\phi(\varepsilon)) \leq \inf_{\psi \in \Phi(\varepsilon)} I(\varepsilon)(\psi) + h(\varepsilon)$ where h is a positive function such that $h(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$ and the sets of admissible deformations are

$$\Phi(\varepsilon) = \{\psi \in W^{1,p}(\Omega; \mathbb{R}^3); \psi(x) = \phi_0(\varepsilon)(x) \text{ on } \Gamma\}.$$

We introduce the space of membrane deformations as $\Phi_M = \{\psi \in W^{1,p}(\Omega; \mathbb{R}^3), \partial_3 \psi = 0, \psi(x) = (x_1, x_2, 0)^T \text{ on } \Gamma\}$, which is isomorphic to $\bar{\Phi}_M = \{\bar{\psi} \in W^{1,p}(\omega; \mathbb{R}^3), \bar{\psi}(x_1, x_2) = (x_1, x_2, 0)^T \text{ on } \partial\omega\}$.

THEOREM 8. — *The sequence $\phi(\varepsilon)$ is relatively compact in $W^{1,p}(\Omega; \mathbb{R}^3)$. Its limit points ϕ belong to Φ_M and are identified with elements $\bar{\phi}$ of $\bar{\Phi}_M$, solutions of the minimization problem $I(0)(\bar{\phi}) = \inf_{\bar{\psi} \in \bar{\Phi}_M} \bar{I}(0)(\bar{\psi})$ where the membrane energy $\bar{I}(0)$ is given by*

$$(34) \quad \bar{I}(0)(\bar{\psi}) = 2 \int_{\omega} Q W_0(\nabla \bar{\psi}) dx_1 dx_2 - \int_{\omega} \mathcal{F} \cdot \bar{\psi} dx_1 dx_2.$$

Proof. — With the rescaled deformation $\phi(\varepsilon)$, we associate a rescaled displacement $u(\varepsilon) \in V$ by $u(\varepsilon) = \phi(\varepsilon) - \phi_0(\varepsilon)$. Obviously, $J(\varepsilon)(u(\varepsilon)) \leq \inf_{v \in V} J(\varepsilon)(v) + h(\varepsilon)$. In particular, $J(\varepsilon)(u(\varepsilon))$ is uniformly bounded and Lemma 3 shows that the sequence $u(\varepsilon)$ is relatively weakly compact in V and that its limit points belong to V_M . This implies in particular that $u(\varepsilon)$ is relatively compact in $L^p(\Omega; \mathbb{R}^3)$.

On the other hand, it is clear that $\tilde{J}(\varepsilon)(u(\varepsilon)) \leq \inf_{v \in L^p(\Omega; \mathbb{R}^3)} \tilde{J}(\varepsilon)(v) + h(\varepsilon)$. Let u be a limit point of $u(\varepsilon)$ for the strong topology of $L^p(\Omega; \mathbb{R}^3)$ (such a u is also a limit point of $u(\varepsilon)$ for the weak topology of V). Without loss of generality, we may assume that $u(\varepsilon) \rightarrow u$ strongly in $L^p(\Omega; \mathbb{R}^3)$. Let v be an arbitrary element of $L^p(\Omega; \mathbb{R}^3)$ and consider a sequence $v(\varepsilon) \in L^p(\Omega; \mathbb{R}^3)$ such that

$$v(\varepsilon) \rightarrow v, \quad \tilde{J}(\varepsilon)(v(\varepsilon)) \rightarrow \tilde{J}(0)(v).$$

Such a sequence exists by the very definition of Γ -convergence. Since $\tilde{J}(\varepsilon)(u(\varepsilon)) \leq \tilde{J}(\varepsilon)(v(\varepsilon)) + h(\varepsilon)$, it follows that

$$\tilde{J}(0)(u) \leq \liminf \tilde{J}(\varepsilon)(u(\varepsilon)) \leq \liminf (\tilde{J}(\varepsilon)(v(\varepsilon)) + h(\varepsilon)) = \tilde{J}(0)(v).$$

Therefore, u is a minimizer of $\tilde{J}(0)$. Rewriting this statement in terms of $\bar{\phi}$, we obtain Theorem 8. \square

Remarks. — (i) The argument used in this proof is due to De Giorgi.

(ii) It is classical in Γ -convergence theory that the energies converge as well, in the sense that $I(\varepsilon)(\phi(\varepsilon)) \rightarrow \bar{I}(0)(\bar{\phi})$.

(iii) Once we know that the rescaled deformations converge weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$, hence in $L^p(-1, 1; W^{1,p}(\omega; \mathbb{R}^3))$, it follows immediately that the averages

$\bar{\phi}^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^\varepsilon \phi^\varepsilon dx_3$ converge weakly in $W^{1,p}(\omega; \mathbb{R}^3)$ toward the same limit $\bar{\phi}$. Simply use the standard properties of the Bochner integral, see Diestel and Uhl [1].

Comments on Theorem 8. – i) If the three-dimensional elasticity problem has solutions that minimize the elastic energy, Theorem 8 applies to these minimizers. Such is the case for the polyconvex stored energy functions introduced by Ball [1], which are of the form

$$W(F) = g(F, \operatorname{cof} F, \det F)$$

where $\operatorname{cof} F$ is the cofactor matrix of F and $g : M_3 \times M_3 \times \mathbb{R} \rightarrow \mathbb{R}$ is convex. We assume here that

$$g(F, H, \delta) \geq \alpha(\|F\|^p + \|H\|^r + |\delta|^s - 1), \quad |g(F, H, \delta)| \leq C(\|F\|^p + 1),$$

with $p > 3$, $1 < r \leq p/2$ and $1 < s \leq p/3$ so that the coercivity assumption does not conflict with the growth assumption. More general exponents will be considered in a further work. Examples of realistic stored energy functions that satisfy the above hypotheses include Ogden's stored energy functions with appropriate exponents. The case of stored energy functions that are such that $W(F) \rightarrow +\infty$ as $\det F \rightarrow 0^+$ presents technical difficulties and will be addressed elsewhere.

(ii) Even if the three-dimensional elasticity problem does not have any solutions, the limit minimization problem always has solutions, either by virtue of resulting from the Γ -convergence of a sequence of equicoercive functionals, or by virtue of being a quasiconvex coercive problem with growth conditions. In particular, our analysis applies to the Saint-Venant-Kirchhoff material that was used by Fox, Raoult and Simo [1], [2] to derive a nonlinear membrane model by formal asymptotic expansions in powers of the thickness. It is known that the Saint-Venant-Kirchhoff stored energy function is not quasiconvex, see Raoult [1], so that the corresponding minimization problem may not always have solutions. We will go into more detail concerning the Saint-Venant-Kirchhoff material further on.

(iii) In the case of a three-dimensional stored energy function which is not quasiconvex, it is a legitimate question to ask whether the quasiconvexification of W_0 can be avoided by using the relaxed three-dimensional energy in place of the original energy. Indeed, the Γ -limits of both sequences coincide, cf. Dal Maso [1]. In other words, we always have $Q((QW)_0) = QW_0$. However, we will show below that, even if Z is quasiconvex, in general Z_0 is not quasiconvex, see Corollary 11 and the remarks following it. In particular $Q((QW)_0) < (QW)_0$ and there is no escaping the quasiconvexification of W_0 , except in special cases.

(iv) It should be noted that the weak limits ϕ of weakly convergent subsequences of $\phi(\varepsilon)$ do not depend on x_3 , in which sense the problem becomes two-dimensional in the limit. If regular enough, the associated function $\bar{\phi}$ describes a deformation of ω into a surface of \mathbb{R}^3 . The elastic energy of such a deformation, $2 \int_\omega QW_0(\nabla \bar{\phi}) dx_1 dx_2$, only depends on its first derivatives. There are thus no bending effects associated with curvature, in which sense the resulting model is a nonlinear membrane model.

(v) If the function QW_0 is smooth enough, the Euler-Lagrange equations for the membrane problem assume the form

$$(35) \quad -\partial_\beta \left(\frac{\partial}{\partial \bar{F}} QW_0(\nabla \bar{\phi}) \right)_{i\beta} = \frac{1}{2} \mathcal{F}_i \text{ in } \omega, \quad \bar{\phi}(x_1, x_2) = (x_1, x_2, 0)^T \text{ on } \partial\omega.$$

Here, as well as in the sequel, Greek indices take their values in the set $\{1, 2\}$, Latin indices in the set $\{1, 2, 3\}$ and the summation convention is understood. System (35) is a system of three quasilinear partial differential equations in the three unknowns $\bar{\phi}_i$. If the applied load is the weight, assuming the membrane is horizontal in its reference configuration, the limit load is of the form $1/2 \mathcal{F} = (0, 0, -\rho g)^T$. The function $\bar{F} \mapsto \bar{T}_R(\bar{F}) = \frac{\partial}{\partial \bar{F}} QW_0(\bar{F})$ appears as the constitutive law for the analogue of the first Piola-Kirchoff stress tensor in the membrane. It gives the Lagrangian description of the tensile stresses in the membrane. It appears that the limit membrane problem retains the full quasilinear structure of three-dimensional elasticity, in contrast with such nonlinear plate models as the von Kármán equations that are only semilinear. It would be interesting to obtain a derivation based on Γ -convergence arguments of nonlinear plate models that incorporate bending effects, including the von Kármán equations and more invariant models. A derivation of the von Kármán model by formal asymptotic expansions is given in Ciarlet [2]. A derivation of a frame-indifferent inextensional bending model by formal asymptotic expansions is given in Fox, Raoult and Simo [1], [2].

(vi) It is a straightforward matter to extend the above analysis to the case of more general loadings as well as more general boundary conditions such as, for example, the case of a cylinder clamped on part of its lateral surface only, the other part of the lateral surface being submitted to given tractions. In this case, we would obtain the same nonlinear membrane energy and additional force terms coming from the edge tractions.

5. Properties of the nonlinear membrane energy

In the Γ -convergence analysis, we have ignored the fact that the stored energy function W of the three-dimensional bodies has to satisfy material frame-indifference, since this was irrelevant for the convergence proof. In this section, we will investigate what are the consequences of material frame-indifference for the nonlinear membrane energy QW_0 as well as the consequences of material symmetry assumptions.

First of all, recall that the principle of material frame-indifference states that to be legitimate from the standpoint of continuum mechanics, a stored energy function W has to satisfy

$$(36) \quad \forall F \in M_3, \quad \forall R \in SO(3), \quad W(RF) = W(F).$$

This is a requirement of objectivity that essentially means that the energy should not depend on the orthonormal cartesian frame used by the observer, *see e.g.* Ciarlet [1], Wang and Truesdell [1] or Marsden and Hughes [1]. Note that (36) is usually restricted to matrices

$F \in M_3$ such that $\det F > 0$ that are the only ones that make sense as three-dimensional deformation gradients. As we do not impose this restriction here for technical reasons, we extend the principle to the whole of M_3 .

THEOREM 9. — *Let the stored energy function W satisfy the principle of material frame-indifference (36). Then, the nonlinear membrane energy QW_0 is frame-indifferent as well, in the sense that*

$$(37) \quad \forall \bar{F} \in M_{3,2}, \quad \forall R \in SO(3), \quad QW_0(R\bar{F}) = QW_0(\bar{F}).$$

Moreover, there exists a function $\mathcal{W}_0 : \text{Sym}_2^{\geq} \rightarrow \mathbb{R}$, where Sym_2^{\geq} is the set of 2×2 positive semi-definite symmetric matrices, such that

$$(38) \quad \forall \bar{F} \in M_{3,2}, \quad QW_0(\bar{F}) = \mathcal{W}_0(\bar{F}^T \bar{F}).$$

Proof. — We start by proving the analogue of (37) for W_0 . Let us be given $\bar{F} \in M_{3,2}$ and $R \in SO(3)$. By definition,

$$(39) \quad \begin{aligned} W_0(R\bar{F}) &= \inf_{z \in \mathbb{R}^3} W((R\bar{F}|z)) \\ &= \inf_{y \in \mathbb{R}^3} W(R(\bar{F}|y)) \\ &= \inf_{y \in \mathbb{R}^3} W((\bar{F}|y)) = W_0(\bar{F}), \end{aligned}$$

which proves the claim for W_0 . Let us now use Dacorogna's representation formula for the quasiconvex envelope of W_0 , cf. Dacorogna [1], [2], which states that

$$(40) \quad QW_0(\bar{F}) = \inf \left\{ \frac{1}{\text{meas } D} \int_D W_0(\bar{F} + \nabla \varphi(x)) dx; \varphi \in W_0^{1,\infty}(D; \mathbb{R}^3) \right\},$$

where D is a bounded open subset of \mathbb{R}^2 (this infimum does not depend on the choice of D). Since for all $R \in SO(3)$ and all $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^3)$, $R\varphi \in W_0^{1,\infty}(D; \mathbb{R}^3)$ as well and $\nabla(R\varphi)(x) = R\nabla\varphi(x)$, the same argument as above gives (37).

Proving the representation formula (38) is now an essentially algebraic matter. In fact, it holds true for any function that satisfies (37). Let thus $Z : M_{3,2} \rightarrow \mathbb{R}$ be such that $Z(R\bar{F}) = Z(\bar{F})$ for all $\bar{F} \in M_{3,2}$ and $R \in SO(3)$. We introduce a kind of polar factorization for 3×2 matrices as follows. Let

$$(41) \quad \bar{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{U} = \sqrt{\bar{F}^T \bar{F}}.$$

By construction, $\bar{U} \in \text{Sym}_2^{\geq}$. We claim that for all $\bar{F} \in M_{3,2}$, there exists a rotation $R \in SO(3)$ such that

$$(42) \quad \bar{F} = R\bar{J}\bar{U}.$$

Assume that (42) is proved, then the representation formula follows at once since

$$(43) \quad Z(\bar{F}) = Z(\bar{J}\bar{U})$$

and it suffices to define $Z : \text{Sym}_2^+ \rightarrow \mathbb{R}$ by

$$(44) \quad Z(\bar{B}) = Z(\bar{J}\sqrt{\bar{B}}),$$

so that

$$(45) \quad Z(\bar{F}) = Z(\bar{F}^T \bar{F}).$$

Let us thus turn to proving claim (42). We begin with the case when \bar{U} is invertible. It is easily checked that this is equivalent to $\text{rank } \bar{F} = 2$. In this case, we consider the matrix $\bar{F}\bar{U}^{-1}$. We have that

$$(46) \quad (\bar{F}\bar{U}^{-1})^T \bar{F}\bar{U}^{-1} = \bar{U}^{-1} \bar{U}^2 \bar{U}^{-1} = I_2,$$

where I_2 is the identity matrix in M_2 . Therefore, the column vectors of $\bar{F}\bar{U}^{-1} = (z_1|z_2)$ are orthonormal. Hence, $R = (z_1|z_2|z_1 \wedge z_2)$ is a rotation and

$$(47) \quad \bar{F}\bar{U}^{-1} = R\bar{J}$$

and this proves the claim if \bar{U} is invertible. If \bar{U} is not invertible, i.e. $\text{rank } \bar{F} < 2$, we consider a sequence $\bar{F}^n \rightarrow \bar{F}$ with $\text{rank } \bar{F}^n = 2$ (the set of rank 2-matrices is open and dense in $M_{3,2}$) and let R^n denote the associated rotations. As $SO(3)$ is compact, there exists a subsequence such that $R^n \rightarrow R \in SO(3)$ and

$$R\bar{J}\bar{U} = \lim (R^n \bar{J}\bar{U}^n) = \lim \bar{F}^n = \bar{F},$$

since the mapping $\bar{F} \mapsto \bar{U}$ is continuous and the proof is complete. \square

Remark. – (i) The meaning of frame-indifference for a membrane is exactly the same as for three-dimensional continuum mechanics. In this sense, the nonlinear membrane model is a two-dimensional model that satisfies automatically this fundamental requirement of objectivity. This is to be contrasted with classical nonlinear plate models which have the drawback of not obeying this invariance.

(ii) If the deformation $\bar{\phi}$ is smooth enough, say an immersion, the tensor $\nabla \bar{\phi}^T \nabla \bar{\phi}$ is nothing but the metric of the deformed surface expressed in the chart $\bar{\phi}$. The membrane energy thus only depends on this metric, which is consistent with the intuition that the stress state in an elastic membrane depends only on the stretching that the deformed surface undergoes.

(iii) If we differentiate formula (38) with respect to \bar{F} , we see that there exists a 2×2 tensor-valued function N such that

$$(48) \quad \forall \bar{F} \in M_{3,2}, \quad \bar{T}_R(\bar{F}) = \bar{F} N(\bar{F}^T \bar{F}).$$

Indeed, $N(\bar{C}) = 2 \frac{\partial}{\partial C} W_0(\bar{C})$ is a symmetric stress tensor which is analogous to the second Piola-Kirchhoff stress tensor of three-dimensional elasticity. To obtain an Eulerian description of the membrane, we assume that $\bar{\phi}$ is a C^2 -immersion of ω in \mathbb{R}^3 . This implies that $\bar{\phi}(\omega)$ is a C^2 -surface in \mathbb{R}^3 and that $\bar{\phi}$ is a chart for this surface. In particular, the matrix $(\nabla \bar{\phi})^T \nabla \bar{\phi}$ is everywhere nonsingular and it is easy to check that $N = ((\nabla \bar{\phi})^T \nabla \bar{\phi})^{-1} (\nabla \bar{\phi})^T \bar{T}_R (\nabla \bar{\phi})$. Expressed in terms of N , equation (35) becomes

$$(49) \quad -\partial_\beta (\partial_\alpha \bar{\phi} N_{\alpha\beta}) = \mathcal{F}/2 \text{ in } \omega.$$

Let $a_\alpha = \partial_\alpha \bar{\phi}$ be the basis tangent vectors to the deformed surface, $a_3 = a_1 \wedge a_2 / \|a_1 \wedge a_2\|$ the normal vector, (a^1, a^2, a^3) the dual basis, $b_{\alpha\beta} = -a_3 \cdot \partial_\beta a_\alpha = a_\alpha \cdot \partial_\beta a_3$ the second fundamental form, $\Gamma_{\alpha\beta}^\gamma = a^\gamma \cdot \partial_\beta a_\alpha$ the Christoffel symbols and $a = \|a_1 \wedge a_2\|$ the area element of the surface. It only requires a routine albeit somewhat lengthy calculation to check that the tensor $n = a^{-1/2} N$ satisfies

$$(50) \quad \begin{cases} -n_{\alpha\beta|\beta} = p_\alpha, \\ n_{\alpha\beta} b_{\alpha\beta} = p_3, \end{cases} \quad \text{on } \bar{\phi}(\omega),$$

where the covariant derivative is defined by

$$n_{\alpha\beta|\gamma} = \partial_\gamma n_{\alpha\beta} + \Gamma_{\mu\gamma}^\alpha n_{\mu\beta} + \Gamma_{\mu\gamma}^\beta n_{\alpha\mu}$$

and the loads are given by

$$p_\alpha = \frac{a^{-1/2}}{2} \mathcal{F} \cdot a^\alpha, \quad p_3 = \frac{a^{-1/2}}{2} \mathcal{F} \cdot a_3.$$

Equations (50) are the usual equilibrium equations for a membrane given for example in Green and Zerna [1], although in our case the constitutive law for the Eulerian stress tensor n fully nonlinearly elastic:

$$n(\bar{F}) = \det(\bar{F}^T \bar{F})^{-1/2} (\bar{F}^T \bar{F})^{-1} \bar{F}^T \frac{\partial}{\partial \bar{F}} QW_0(\bar{F}).$$

A similar observation was made in Fox, Raoult and Simo [2] for the Saint-Venant-Kirchhoff membrane.

Note that it is not possible to define the second Piola-Kirchhoff stress N without any assumption on the constitutive law for the first Piola-Kirchhoff stress—in our case frame-indifference, which is not really an assumption but rather a requirement. This is quite different from three-dimensional elasticity where the second Piola-Kirchhoff stress tensor Σ is defined in terms of the first Piola-Kirchhoff stress tensor T_R via the formula $T_R = F \Sigma$ where, with standard notation in elasticity, $F \in M_3$ is a nonsingular matrix that stands for the deformation gradient. In effect, a necessary and sufficient condition for the existence of a factorization of the form $\bar{T}_R = \bar{F} N$ in the case of membranes is that, considering \bar{T}_R and \bar{F} as linear mappings from \mathbb{R}^2 into \mathbb{R}^3 , we should have $\text{Im } \bar{T}_R \subset \text{Im } \bar{F}$. This is not always the case: take for example the unphysical membrane stored energy function $Z(\bar{F}) = \bar{F}_{11}$. The corresponding first Piola-Kirchhoff stress tensor cannot be expressed

in terms of a second Piola-Kirchhoff stress tensor even when the rank of \bar{F} is maximal. Of course, this energy is not frame-indifferent. It should also be noted that factorization (48) implies that the Piola-Kirchhoff stress vectors always belong to the tangent plane to the deformed surface.

We may also define an analogue of the Cauchy stress tensor by setting $T = a^{-1/2} \bar{T}_R \bar{F}^T$, which is a 3×3 tensor, and write equilibrium equations in covariant form for this tensor.

(iv) The expression of material frame-indifference for a membrane in terms of the various stress tensors introduced in the comments of Theorem 8 and above can be obtained by differentiating relation (37) with respect to \bar{F} . Namely, for the Piola-Kirchhoff stress tensor \bar{T}_R , we find that

$$(51) \quad \forall R \in SO(3), \quad \forall \bar{F} \in M_{3,2}, \quad \bar{T}_R(R\bar{F}) = R\bar{T}_R(\bar{F}),$$

for the Eulerian stress tensor n that

$$(52) \quad \forall R \in SO(3), \quad \forall \bar{F} \in M_{3,2}, \quad n(R\bar{F}) = n(\bar{F}).$$

and for the Cauchy stress tensor T that

$$(53) \quad \forall R \in SO(3), \quad \forall \bar{F} \in M_{3,2}, \quad (R\bar{F}) = RT(\bar{F})R^T,$$

so that these tensors behave as they should under the left action of $SO(3)$. See Marsden and Hughes [1], Simo and Marsden [1] and Simo, Marsden and Krishnaprasad [1] for a discussion of the differential geometric nature of the various stress tensors.

(v) It should be noted that, whereas in three-dimensional elasticity it is reasonable to restrict attention to deformation gradients in M_3^+ , there is no such notion for membranes. Indeed, deformations such as $\bar{\phi}(x_1, x_2) = (x_1^2, x_2, 0)^T$, which represent a folding of the membrane on itself, are legitimate. In this deformation, the deformation gradient is rank-one along the fold ($x_1 = 0$) and the orientation is reversed across the fold. The problem of global noninterpenetration of matter for a membrane is another question altogether. It is now fairly well understood in three-dimensional elasticity, see in this direction Ball [2], Ciarlet and Nečas [1], Tang Qi [1], Giaquinta, Modica and Souček [1].

We now proceed to show that, due to frame indifference, if the three-dimensional stored energy function has a global minimum at $F = I$, the corresponding nonlinear membrane energy is constant in a subset of $M_{3,2}$ with nonempty interior. To begin with, we introduce a definition.

Definition. — For all $\bar{F} \in M_{3,2}$, we call *right singular values* of \bar{F} , and we note $v_1(\bar{F})$, $v_2(\bar{F})$, the eigenvalues of the matrix $\sqrt{(\bar{F}^T \bar{F})}$ (in increasing order).

Remarks. — By definition, the right singular values are nonnegative real numbers. In three-dimensional elasticity, the singular values of F are the three eigenvalues of $\sqrt{(F^T F)}$. These eigenvalues are also called principal stretches. The right singular values play the same role for a membrane.

THEOREM 10. — Assume that the three-dimensional stored energy function W is such that $W(I) = 0$ and $W(F) \geq 0$ for all $F \in M_3$. Then, $QW_0(\bar{F}) = 0$ for all $\bar{F} \in M_{3,2}$ such that $0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1$.

Proof. — Let $\bar{F} \in M_{3,2}$ be such that $0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1$. Let $\bar{U} = \sqrt{(\bar{F}^T \bar{F})}$ and let $\bar{R} \in SO(2)$ be such that

$$\bar{R} \bar{U} \bar{R}^T = \begin{pmatrix} v_1(\bar{F}) & 0 \\ 0 & v_2(\bar{F}) \end{pmatrix}.$$

Since $W_0 \geq 0$, it follows immediately that $QW_0(\bar{F}) \geq 0$. Indeed, the null function is trivially quasiconvex and below W_0 . Moreover, $W_0(\bar{G}) = 0$ for all \bar{G} such that $\bar{G}^T \bar{G} = I_2$, i.e., $v_1(\bar{G}) = v_2(\bar{G}) = 1$, by frame-indifference, viz. formula (38). Hence, $QW_0(\bar{G}) = 0$ for such matrices as well.

Let D be the unit disk of \mathbb{R}^2 . The idea is to construct a sequence of membrane deformations on D that have zero elastic energy and whose gradients weakly converge to a matrix \bar{F}' such that $(\bar{F}')^T \bar{F}' = \bar{F}^T \bar{F}$. We will then conclude by weak lower semi-continuity of the membrane energy.

We consider the function $\theta_1 : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\theta_1(t) = \begin{cases} (1 - v_1(\bar{F}))t & \text{if } 0 \leq t \leq \frac{1 + v_1(\bar{F})}{2}, \\ (-1 - v_1(\bar{F}))(t - 1) & \text{if } \frac{1 + v_1(\bar{F})}{2} \leq t \leq 1, \end{cases}$$

(note that since $v_1(\bar{F}) \in [0, 1]$, $\frac{1 + v_1(\bar{F})}{2} \in [0, 1]$ and the function θ_1 is well defined and continuous) and extend it by periodicity to \mathbb{R} . It is clear that $\theta_1 \in W^{1,\infty}(\mathbb{R})$. We also define a function θ_2 in the same way but using $v_2(\bar{F})$ instead $v_1(\bar{F})$. Finally, let

$$(54) \quad \bar{\phi}^n(\bar{R}^T x) = \begin{pmatrix} v_1(\bar{F})x_1 + \frac{1}{n}\theta_1(nx_1) \\ v_2(\bar{F})x_2 + \frac{1}{n}\theta_2(nx_2) \\ 0 \end{pmatrix}.$$

We have

$$(55) \quad \bar{\phi}^n(\bar{R}^T x) - \bar{\phi}^0(\bar{R}^T x) = \begin{pmatrix} v_1(\bar{F})x_1 \\ v_2(\bar{F})x_2 \\ 0 \end{pmatrix},$$

weakly-* in $W^{1,\infty}(D; \mathbb{R}^3)$, so that

$$(56) \quad \nabla \bar{\phi}^n \rightharpoonup \nabla \bar{\phi}^0 = \begin{pmatrix} v_1(\bar{F}) & 0 \\ 0 & v_2(\bar{F}) \\ 0 & 0 \end{pmatrix} \bar{R},$$

weakly-* in $L^\infty(D; M_{3,2})$. Since

$$\begin{aligned} (\nabla \bar{\phi}^0)^T \nabla \bar{\phi}^0 &= \bar{R}^T \begin{pmatrix} v_1(\bar{F}) & 0 & 0 \\ 0 & v_2(\bar{F}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} v_1(\bar{F}) & 0 \\ 0 & v_2(\bar{F}) \\ 0 & 0 \end{pmatrix} \bar{R} = \bar{R}^T (\bar{R} \bar{U}^2 \bar{R}^T) \bar{R} = \bar{U}^2, \end{aligned}$$

it follows from Theorem 9 that

$$(57) \quad QW_0(\bar{F}) = QW_0(\nabla \bar{\phi}^0)$$

(which is independent of (x_1, x_2)).

On the other hand, it is easily checked that

$$(58) \quad \nabla \bar{\phi}^n(\bar{R}^T x) = \begin{pmatrix} \varepsilon_1^n(x_1) & 0 \\ 0 & \varepsilon_2^n(x_2) \\ 0 & 0 \end{pmatrix},$$

where

$$(59) \quad \begin{cases} \varepsilon_1^n(x_1) = 1 & \text{if } nx_1 - [nx_1] \in [0, \frac{1+v_1(\bar{F})}{2}], & \varepsilon_1^n(x_1) = -1 \text{ otherwise,} \\ \varepsilon_2^n(x_2) = 1 & \text{if } nx_2 - [nx_2] \in [0, \frac{1+v_2(\bar{F})}{2}], & \varepsilon_2^n(x_2) = -1 \text{ otherwise,} \end{cases}$$

so that

$$(60) \quad (\nabla \bar{\phi}^n)^T \nabla \bar{\phi}^n = I_2$$

and by frame-indifference

$$(61) \quad QW_0(\nabla \bar{\phi}^n) = 0.$$

Since QW_0 is quasiconvex, the functional $\bar{\phi} \mapsto \int_D QW_0(\nabla \bar{\phi}(x)) dx_1 dx_2$ is sequentially weakly-* lower semi-continuous on $W^{1,\infty}(D; \mathbb{R}^3)$, so that we can write

$$(62) \quad \begin{aligned} 0 &= \liminf_{n \rightarrow \infty} \left\{ \int_D QW_0(\nabla \bar{\phi}^n(x)) dx_1 dx_2 \right\} \\ &\geq \int_D QW_0(\nabla \bar{\phi}^0(x)) dx_1 dx_2 = \pi QW_0(\bar{F}), \end{aligned}$$

which completes the proof. \square

Remarks. – (i) The assumption implies in particular that the reference configuration is a natural state for the three-dimensional body. This assumption is thus fairly natural and the resulting degeneracy of the membrane energy is not at all a pathological phenomenon.

(ii) The meaning of Theorem 10 is that crumpling a membrane does not require any energy. In effect, deformations such that $0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1$ correspond to compressions of the membrane. As the limit problem is a relaxed problem, we can interpret Theorem 10 by saying that if we try to achieve such a compression in a given membrane, the natural tendency of the membrane will be, instead of achieving it in its own tangent plane, to jump out of the tangent plane and crumple itself more and more finely as the thickness goes to zero, without modifying its metric very much so that only little energy is needed. In the weak limit, these oscillations average out and result in the required compression. This is exactly what the sequence $\bar{\phi}^n$ does, although already in the zero thickness limit: it folds the membrane on itself periodically on a finer and finer scale, without changing the metric. It is possible, although more complicated, to construct explicitly a similar sequence of three-dimensional deformations in $\Omega_{1/n}$, rescale it and obtain the same net effect.

(iii) A similar phenomenon was exhibited by Acerbi, Buttazzo and Percivale [1] in the case of elastic strings (note that they allow the energy to satisfy $W(F) \rightarrow +\infty$ as $\det F \rightarrow 0^+$). In their case, the limit deformations map a segment Σ into \mathbb{R}^3 , so that limit deformation gradients take their values in \mathbb{R}^3 . They define $W_0(z) = \inf \{W((z|z_2|z_3)); (z_2|z_3) \in M_{3,2}\}$ and prove that the limit string energy is given by the convex envelope W_0^{**} of W_0 . They show that if W is frame-indifferent and has a minimum at $F = I$, $W(I) = 0$, then $W_0^{**}(z)$ only depends on $\|z\|$ and is identically 0 for all $\|z\| \leq 1$. Such deformation gradients correspond to compressions of the string and the interpretation of this result is essentially the same as for the membrane case.

(iv) In the context of the dynamics of an elastic string, it was postulated by Gilquin and Serre [1] that the constitutive law for the tension in an elastic string should vanish in compressive states. This was for physical and numerical reasons. The result of Acerbi, Buttazzo and Percivale vindicates this postulate. Our result shows the kind of degeneracy that should be expected in the study of the dynamics of a membrane.

(v) Note that Theorem 10 depends crucially on the hypothesis that the energy is minimum at $F = I$. This is a property of the reference configuration. Indeed, assume that the reference configuration is a pre-stretched state. Then it is clear that the membrane will want to strictly lower its energy by shrinking back to a natural state.

(vi) The set of matrices $\bar{F} = (z_1|z_2)$ such that $0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1$ can alternatively be characterized by $(\|z_1\|^2 + \|z_2\|^2, \|z_1 \wedge z_2\|^2) \in S$ where $S = \{(y_1, y_2) \in (\mathbb{R}_+)^2; y_2 \geq y_1 - 1 \text{ and } 4y_2 \leq y_1^2\}$. Its interior is thus nonempty.

We now turn to showing that the quasiconvexification step in the definition of the nonlinear membrane energy cannot be avoided.

COROLLARY 11. – *Let W be frame-indifferent and such that $W(I) = 0$ and $W(F) > 0$ if $F \neq SO(3)$. Then $QW_0 < W_0$.*

Proof. – We already know that $QW_0(\bar{F}) = 0$ for all \bar{F} such that $0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1$. Consider \bar{F} such that $v_1(\bar{F}) < v_2(\bar{F}) \leq 1$. We claim that $W_0(\bar{F}) > 0$.

Indeed, let $z_0 \in \mathbb{R}^3$ be such that $W_0(\bar{F}) = W((\bar{F}|z_0))$. Since $\bar{F}^T \bar{F} \neq I_2$, it follows that $(\bar{F}|z_0) \notin SO(3)$ and therefore $W((\bar{F}|z_0)) > 0$. \square

Remarks. – Corollary 11 implies in particular that even if W is quasiconvex, W_0 cannot be expected to be quasiconvex. Similarly, if W is polyconvex, in general W_0 will not be polyconvex. Consider for example $W(F) = \sum_{i=1}^3 [v_i(F)^2 - 1]_+^2 + |\det F - 1|$ where $v_i(F)$ are the singular values of F , i.e., the eigenvalues of $\sqrt{(F^T F)}$, and $[x]_+^2 = x^2$ if $x \geq 0$ and $[x]_+^2 = 0$ if $x < 0$. Note that, by contrast, if W is convex, then W_0 is also convex so that $QW_0 = W_0$ in this case. Unfortunately, it is well known that convex stored energy functions are entirely inappropriate in nonlinear elasticity.

There is a special case in which quasiconvexity (resp. polyconvexity, rank-one convexity) is conserved, although by Theorem 10 the reference configuration of such materials cannot be a strict (modulo frame-indifference) absolute minimizer of the elastic energy.

PROPOSITION 12. – Assume that the minimum in definition (11) is attained at a point $z_0 \in \mathbb{R}^3$ that is independent of \bar{F} . In this case, if W is quasiconvex (resp. polyconvex, rank-one convex), W_0 is quasiconvex (resp. polyconvex, rank-one convex).

Proof. – Let us prove the quasiconvex case. Since W is quasiconvex and satisfies appropriate growth conditions, the functional $\phi \mapsto \int_B W(\nabla \phi(x)) dx$ is weakly lower semi-continuous on $W^{1,p}(B; \mathbb{R}^3)$, where B is the unit cube in \mathbb{R}^3 . It is shown in Ball and Murat [1], that then

$$(63) \quad \int_B W(\nabla \phi(x)) dx \geq W\left(\int_B \nabla \phi(x) dx\right)$$

for all $\phi \in W^{1,p}(B; \mathbb{R}^3)$ such that $\nabla \phi$ is B -periodic.

Let \bar{B} be the unit square in \mathbb{R}^2 so that $B = \bar{B} \times]0, 1[$. Let $\bar{F} \in M_{3,2}$ and for all $\bar{\phi} \in \mathcal{D}(\bar{B}; \mathbb{R}^3)$ define a function ϕ on B by

$$(64) \quad \phi(x_1, x_2, x_3) = \bar{F}(x_1, x_2)^T + \bar{\phi}(x_1, x_2) + x_3 z_0.$$

Then $\nabla \phi = (\bar{F} + \nabla \bar{\phi}|z_0)$ is clearly B -periodic and (63) implies that

$$(65) \quad \begin{aligned} \int_{\bar{B}} W_0(\bar{F} + \nabla \bar{\phi}(x)) dx_1 dx_2 &= \int_B W((\bar{F} + \nabla \bar{\phi}(x)|z_0)) dx \\ &\geq W\left(\int_B \nabla \phi(x) dx\right) = W((\bar{F}|z_0)) = W_0(\bar{F}). \end{aligned}$$

Hence W_0 is quasiconvex.

We omit the proof for polyconvexity and rank-one convexity, which is simply a restriction argument. \square

We now investigate the consequences on the membrane energy of material symmetry assumptions on the three-dimensional stored energy function. Recall first that an elastic material is said to be isotropic if

$$(66) \quad \forall F \in M_3, \quad \forall R \in SO(3), \quad W(FR) = W(F).$$

Although formally similar to the principle of material frame-indifference, isotropy does not share the same status. Indeed, it is not a universal property of all materials, but rather a property of the reference configuration of certain materials. Its meaning is that if we perform either a given deformation or first a rotation and then the same deformation, the elastic energy is not changed, or in other words the Cauchy stress tensor is not changed. Therefore in this sense, the material has the same material properties in all directions of space in the reference configuration. For nonlinear membranes, isotropy translates in the following fashion.

THEOREM 13. — Assume that the stored energy function W is isotropic (66). Then, the nonlinear membrane energy QW_0 is isotropic as well, in the sense that

$$(67) \quad \forall \bar{F} \in M_{3,2}, \quad \forall \bar{R} \in SO(2), \quad QW_0(\bar{F}\bar{R}) = QW_0(\bar{F}).$$

Moreover, there exists a function $w_0 : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, symmetric, such that

$$(68) \quad \forall \bar{F} \in M_{3,2}, \quad QW_0(\bar{F}) = w_0(v_1(\bar{F}), v_2(\bar{F})).$$

Proof. — The proof is similar to that of Theorem 9. Let $\bar{F} \in M_{3,2}$ and $\bar{R} \in SO(2)$. Then

$$R = \left(\begin{array}{c|c} \bar{R} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \hline 0 & 1 \end{array} \right) \in SO(3)$$

and if we let $F = (\bar{F}|z)$ then $FR = (\bar{F}\bar{R}|z)$. Therefore, $W(F) = W(FR) = W((\bar{F}\bar{R}|z))$ so that $W_0(\bar{F}) = W_0(\bar{F}\bar{R})$ as in the proof of Theorem 9. We obtain the same invariance for QW_0 by using Dacorogna's formula again. Note that it is important to take the domain D to be a disk, so that for all $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^3)$ and all $\bar{R} \in SO(2)$, we have $\varphi_{\bar{R}} \in W_0^{1,\infty}(D; \mathbb{R}^3)$ where $\varphi_{\bar{R}}(x) = \varphi(\bar{R}x)$.

By frame-indifference, we already now that

$$QW_0(\bar{F}) = W_0(\bar{F}^T \bar{F}).$$

Therefore, by (67), for all $\bar{C} \in \text{Sym}_2^{\geq}$ and all $\bar{R} \in SO(2)$,

$$W_0(\bar{R}^T \bar{C} \bar{R}) = W_0(\bar{C})$$

and the existence of w_0 follows at once from a well known result on isotropic scalar functions on Sym_2^{\geq} . \square

Remarks. – (i) The meaning of isotropy for a membrane is exactly the same as for three-dimensional elasticity, except that since the membrane is two-dimensional, the rotations to be performed in its reference configuration must belong to $SO(2)$.

(ii) In terms of the stress tensors, isotropy reads:

$$(69) \quad \forall \bar{R} \in SO(2), \quad \forall \bar{F} \in M_{3,2}, \quad \bar{T}_R(\bar{F}\bar{R}) = \bar{T}_R(\bar{F})\bar{R},$$

$$(70) \quad \forall \bar{R} \in SO(2), \quad \forall \bar{F} \in M_{3,2}, \quad n(\bar{F}\bar{R}) = \bar{R}^T n(\bar{F})\bar{R}$$

and

$$(71) \quad \forall \bar{R} \in SO(2), \quad \forall \bar{F} \in M_{3,2}, \quad T(\bar{F}\bar{R}) = T(\bar{F}).$$

(iii) A symmetric function of the right singular values of \bar{F} is also a function of the pair $(\|z_1\|^2 + \|z_2\|^2, \|z_1 \wedge z_2\|^2)$, which we might thus call right principal invariants of $\bar{C} = \bar{F}^T \bar{F}$ by analogy with three-dimensional elasticity. Indeed, $v_1(\bar{F})^2 + v_2(\bar{F})^2 = \|z_1\|^2 + \|z_2\|^2 = \text{tr } \bar{C}$ and $v_1(\bar{F})^2 v_2(\bar{F})^2 = \|z_1 \wedge z_2\|^2 = \det \bar{C}$. Theorem 13 is thus a two-dimensional analogue of the Rivlin-Ericksen theorem.

(iv) Note that $\sqrt{\|\partial_1 \bar{\phi}\|^2 + \|\partial_2 \bar{\phi}\|^2}$ and $\|\partial_1 \bar{\phi} \wedge \partial_2 \bar{\phi}\|$ are respectively the square root of the trace and the square root of the determinant of the deformed metric. The second function is also the surface element on the deformed surface. Both functions $\sqrt{\|z_1\|^2 + \|z_2\|^2} = \|\bar{F}\|$ and $\|z_1 \wedge z_2\| = \|\text{adj}_2(\bar{F})\|$ are polyconvex functions of \bar{F} , so that any function $\bar{F} \mapsto g(\sqrt{\|z_1\|^2 + \|z_2\|^2}, \|z_1 \wedge z_2\|)$ with $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ convex and nondecreasing in each variable defines a frame-indifferent, isotropic and polyconvex nonlinear membrane energy (if the first variable is missing, this is reminiscent of a minimal surface energy). In terms of the right singular values, such a function may be written as $\bar{F} \mapsto h(v_1(\bar{F}), v_2(\bar{F}), v_1(\bar{F})v_2(\bar{F}))$ with $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ convex, symmetric in the first two variables and nondecreasing in each variable (see Ball [1] and Le Dret [1] for a discussion of the polyconvexity of such functions in the $n \times n$ square matrix case and Lemma 15 below for the convexity in the 3×2 case). If we want to enforce additionally the degeneracy of nonlinear membrane energies, it suffices to ask that $h(v_1, v_2, w) = 0$ whenever $0 \leq v_1, v_2, w \leq 1$.

For the case of material symmetries other than isotropy, we have a less complete result whose proof is identical to that of Theorem 13. Let $\mathfrak{G} = \{R \in SO(3); \forall F \in M_3, W(FR) = W(F)\}$ be the material symmetry group of the three-dimensional material and define $\bar{\mathfrak{G}} = \{\bar{R} \in M_2; \forall \bar{F} \in M_{3,2}, QW_0(\bar{F}\bar{R}) = QW_0(\bar{F})\}$ to be the material symmetry set for the membrane (it is not *a priori* clear that $\bar{\mathfrak{G}} \subset SO(2)$ nor that $\bar{\mathfrak{G}}$ is a group). Then we have:

PROPOSITION 14. – If $\bar{R} \in SO(2)$ is such that

$$R = \left(\begin{array}{c|c} \bar{R} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \hline 0 & 1 \end{array} \right) \in \mathfrak{G},$$

then $\bar{R} \in \bar{\mathfrak{G}}$.

We conclude this article by examining the case of the Saint-Venant-Kirchhoff material in more detail. Recall that the Saint-Venant-Kirchhoff stored energy function is given by

$$W(F) = \frac{\mu}{4} \operatorname{tr}(F^T F - I)^2 + \frac{\lambda}{8} (\operatorname{tr}(F^T F - I))^2$$

where μ and λ are the Lamé moduli, which we assume to be such that $\mu > 0$ and $\lambda \geq 0$. This energy satisfies our basic hypotheses with $p = 4$. As it is not quasiconvex, see Raoult [1], we may only conclude that approximate minimizers for the sequence of three-dimensional problems converge towards minimizers of the corresponding membrane energy. The quasiconvex envelope of the Saint-Venant-Kirchhoff stored energy function has been explicitly computed in Le Dret and Raoult [3], [4], by the same kind of arguments as those used below for the membrane energy. Let us first state a lemma, which is a 3×2 version of a result on isotropic convex functions of square matrices of Thompson and Freede [1], see also Ball [1]. The simplified proof given in Le Dret [1] also works here.

LEMMA 15. — *Let $\Phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a symmetric, convex function that is nondecreasing in each variable. Then the function $Z(\bar{F}) = \Phi(v_1(\bar{F}), v_2(\bar{F}))$ is convex on $M_{3,2}$.*

Proof. — As in the square matrix case, the proof is based on the fact that given any triple (s_1, s_2, s_3) such that $0 \leq s_1 \leq s_2 \leq s_3$, the function $F \mapsto \sum_{i=1}^3 s_i v_i(F)$, where $v_i(F)$ are the singular values of F , is convex on M_3 . This in turn results from a trace inequality of von Neumann, which says that given two $n \times n$ matrices A and B ,

$$\operatorname{tr} AB \leq \sum_{i=1}^n v_i(A) v_i(B)$$

where the singular values $v_i(A)$ and $v_i(B)$ are arranged in increasing order, see Mirsky [1].

Since the singular values of the matrix $F = (\bar{F}|0)$ are $0, v_1(\bar{F})$ and $v_2(\bar{F})$, we deduce from the above result that for all $0 \leq s_2 \leq s_3$, the function $\bar{F} \mapsto \sum_{i=1}^2 s_{i+1} v_i(\bar{F})$ is convex on $M_{3,2}$. We conclude as in Le Dret [1] by expressing the function Z as a supremum of such functions obtained from the subdifferentials of Φ . \square

The membrane energy for the Saint-Venant-Kirchhoff material is computed in following fashion.

PROPOSITION 16. — *For the Saint-Venant-Kirchhoff stored energy function, we have*

$$(72) \quad \begin{aligned} W_0(\bar{F}) = & \frac{\mu}{4} \operatorname{tr}(\bar{F}^T \bar{F} - I_2)^2 + \frac{\lambda\mu}{4(\lambda + 2\mu)} h(\bar{F})^2 \\ & + \frac{1}{8(\lambda + 2\mu)} ([\lambda h(\bar{F}) - (\lambda + 2\mu)]_+)^2 \end{aligned}$$

where $h(\bar{F}) = \text{tr}(\bar{F}^T \bar{F} - I_2)$ and the membrane energy is

$$(73) \quad QW_0(\bar{F}) = \frac{E}{8} ([v_2(\bar{F})^2 - 1]_+)^2 \\ + \frac{E}{8(1-\nu^2)} ([v_1(\bar{F})^2 + \nu v_2(\bar{F})^2 - (1+\nu)]_+)^2 \\ + \frac{E}{8(1-\nu^2)(1-2\nu)} ([\nu(v_1(\bar{F})^2 + v_2(\bar{F})^2) - (1+\nu)]_+)^2,$$

where $v_1(\bar{F}) \leq v_2(\bar{F})$ are the right singular values of \bar{F} and $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is the Young modulus and $\nu = \frac{\lambda}{2(\mu + \lambda)}$ is Poisson's ratio.

Proof. — Let us express $W(F)$ in terms of the column vectors of F :

$$(74) \quad W(F) = \frac{\mu}{4} \left(\sum_{i,j=1}^3 (z_i \cdot z_j - \delta_{ij})^2 \right) + \frac{\lambda}{8} \left(\sum_{i=1}^3 (\|z_i\|^2 - 1) \right)^2 \\ = \frac{\mu}{4} \left(\sum_{\alpha,\beta=1}^2 (z_\alpha \cdot z_\beta - \delta_{\alpha\beta})^2 \right) + \frac{\lambda}{8} \left(\sum_{\alpha=1}^2 (\|z_\alpha\|^2 - 1) \right)^2 \\ + \frac{\mu}{2} \left(\sum_{\alpha=1}^2 (z_\alpha \cdot z_3)^2 \right) + \frac{2\mu + \lambda}{8} (\|z_3\|^2 - 1)^2 \\ + \frac{\lambda}{4} (\|z_3\|^2 - 1) \left(\sum_{\alpha=1}^2 (\|z_\alpha\|^2 - 1) \right).$$

In view of (74), it is clear that in order to minimize $W((\bar{F}|z_3))$ with respect to z_3 we can always choose z_3 to be orthogonal to z_1 and z_2 . Hence, we are left with minimizing the function

$$(75) \quad f(t) = \frac{2\mu + \lambda}{8} (t^2 - 1)^2 + \frac{\lambda}{4} (t^2 - 1) \left(\sum_{\alpha=1}^2 (\|z_\alpha\|^2 - 1) \right)$$

over the set $t \geq 0$, depending on the values of the parameters $\|z_1\|$ and $\|z_2\|$. As this is a quadratic function in $y = t^2 - 1$, if we let $h = \sum_{\alpha=1}^2 (\|z_\alpha\|^2 - 1)$, it thus suffices to minimize

$$(76) \quad g(y) = \frac{2\mu + \lambda}{8} y^2 + \frac{\lambda}{4} hy$$

over the set $y \geq -1$. Therefore

$$(77) \quad \min g = \begin{cases} -\frac{\lambda^2}{8(2\mu + \lambda)} h^2 & \text{if } h \leq \frac{2\mu + \lambda}{\lambda}, \\ \frac{2\mu + \lambda}{8} - \frac{\lambda}{4} h & \text{if } h > \frac{2\mu + \lambda}{\lambda}, \end{cases}$$

from which we obtain formula (72).

To compute the quasiconvex envelope of W_0 we first notice that for all $\bar{F} \in M_{3,2}$, there exists $\bar{G} = (z_1|z_2)$ with $z_1 \cdot z_2 = 0$ such that $v_\alpha(\bar{F}) = v_\alpha(\bar{G})$. By Theorem 13, $QW_0(\bar{F}) = QW_0(\bar{G})$, it therefore suffices to consider matrices \bar{F} whose column-vectors are orthogonal, which simplifies subsequent computations. Note that for such matrices $\{v_1(\bar{F}), v_2(\bar{F})\} = \{\|z_1\|, \|z_2\|\}$.

The proof is based on the following observation. Since QW_0 is quasiconvex, it is rank-one convex. Hence, for z_2 fixed, the function $k_{z_2} : (z_2)^\perp \rightarrow \mathbb{R}$ defined by $k_{z_2}(z_1) = QW_0((z_1|z_2))$ is convex. Since $k_{z_2}(z_1) \leq l_{z_2}(z_1) = W_0((z_1|z_2))$, it follows that $k_{z_2}(z_1) \leq l_{z_2}^{**}(z_1)$, where $l_{z_2}^{**}$ is the convex envelope of l_{z_2} .

Let us consider the case $\|z_2\| \geq 1$. In this case, a direct computation shows that

$$(78) \quad l_{z_2}^{**}(z_1) = \begin{cases} \frac{E}{8} (\|z_2\|^2 - 1)^2 & \text{if } \|z_1\|^2 + \nu \|z_2\|^2 \leq 1 + \nu, \\ l_{z_2}(z_1) & \text{if } \|z_1\|^2 + \nu \|z_2\|^2 \geq 1 + \nu. \end{cases}$$

By the previous remark, this function gives an upper bound for $QW_0(\bar{F})$ on the set of matrices such that $z_1 \cdot z_2 = 0$ and $\|z_2\| \geq 1$. Similarly,

$$(79) \quad l_{z_1}^{**}(z_2) = \begin{cases} \frac{E}{8} (\|z_1\|^2 - 1)^2 & \text{if } \|z_2\|^2 + \nu \|z_1\|^2 \leq 1 + \nu, \\ l_{z_1}(z_2) & \text{if } \|z_2\|^2 + \nu \|z_1\|^2 \geq 1 + \nu, \end{cases}$$

(with obvious notation) gives an upper bound for $QW_0(\bar{F})$ on the set of matrices such that $z_1 \cdot z_2 = 0$ and $\|z_1\| \geq 1$. We now define a function Z_0 on the set of matrices such that $z_1 \cdot z_2 = 0$ by

$$(80) \quad Z_0(\bar{F}) = \begin{cases} 0 & \text{if } 0 \leq \|z_1\|, \|z_2\| \leq 1, \\ l_{z_2}^{**}(z_1) & \text{if } 0 \leq \|z_1\| \leq 1 \leq \|z_2\| \text{ and } \|z_1\|^2 + \nu \|z_2\|^2 \leq 1 + \nu, \\ l_{z_1}^{**}(z_2) & \text{if } 0 \leq \|z_2\| \leq 1 \leq \|z_1\| \text{ and } \|z_2\|^2 + \nu \|z_1\|^2 \leq 1 + \nu, \\ W_0(\bar{F}) & \text{if } \|z_1\|^2 + \nu \|z_2\|^2 \geq 1 + \nu \text{ and } \|z_2\|^2 + \nu \|z_1\|^2 \geq 1 + \nu, \end{cases}$$

Theorem 10 applies to the Saint-Venant-Kirchhoff stored energy function. Therefore, $QW_0(\bar{F}) = 0$ for $0 \leq \|z_1\|, \|z_2\| \leq 1$. Consequently, $QW_0(\bar{F}) \leq Z_0(\bar{F})$ for all \bar{F} such that $z_1 \cdot z_2 = 0$. On the other hand, it also follows from (78) and (79) that $Z_0(\bar{F}) \leq W_0(\bar{F})$ for all \bar{F} such that $z_1 \cdot z_2 = 0$.

We now extend the function Z_0 to $M_{3,2}$ in an isotropic and frame-indifferent fashion by letting for all $\bar{F} \in M_{3,2}$:

$$(81) \quad Z_0(\bar{F}) = \begin{cases} 0 & \text{if } 0 \leq v_1(\bar{F}) \leq v_2(\bar{F}) \leq 1, \\ \frac{E}{8} (v_2(\bar{F})^2 - 1)^2 & \text{if } 0 \leq v_1(\bar{F}) \leq 1 \leq v_2(\bar{F}) \\ & \text{and } v_1(\bar{F})^2 + \nu v_2(\bar{F})^2 \leq 1 + \nu, \\ W_0(\bar{F}) & \text{otherwise.} \end{cases}$$

It is clear that for all \bar{F} such that $z_1 \cdot z_2 = 0$ definitions (80) and (81) coincide. Consequently, by frame-indifference and isotropy, $QW_0 \leq Z_0 \leq W_0$ on $M_{3,2}$. Let us introduce a function Φ by

$$(82) \quad \Phi(v_1, v_2) = \frac{E}{8} ([v_2^2 - 1]_+)^2 + \frac{E}{8(1 - \nu^2)} ([v_1^2 - 1 + \nu(v_2^2 - 1)]_+)^2 \\ + \frac{E}{8(1 - \nu^2)(1 - 2\nu)} ([\nu(v_1^2 + v_2^2) - (1 + \nu)]_+)^2,$$

for $v_1 \leq v_2$. A straightforward inspection of all possible cases shows that the expression of Z_0 in terms of the right singular values of \bar{F} (81) is precisely the right-hand side of formula (73), in other words $Z_0(\bar{F}) = \Phi(v_1(\bar{F}), v_2(\bar{F}))$.

We now remark that the extension of Φ by symmetry for $v_1 \geq v_2$ is convex and non-decreasing in each variable, so that Z_0 is convex on $M_{3,2}$ by Lemma 15. Therefore, $QW_0 = Z_0$. \square

Remarks. – (i) It is possible to derive the expression of QW_0 completely algebraically without appealing to Theorem 10. This is slightly more complicated.

(ii) The function W_0 agrees with the expression found by Fox, Raoult and Simo [1], [2] by formal asymptotic expansions as long as $\lambda h(\bar{F}) - (\lambda + 2\mu) \leq 0$. The fact that this function is not quasiconvex already implied that it had to be relaxed in order to give rise to a well-posed problem. However, the failure of formal asymptotic expansions to capture the additional term $\frac{1}{8(\lambda + 2\mu)} ([\lambda h(\bar{F}) - (\lambda + 2\mu)]_+)^2$ seems hard to explain. It should be noted that Fox, Raoult and Simo where led to restrict their attention to deformations such that $\lambda h(\bar{F}) - (\lambda + 2\mu) \leq 0$ in order to be able to pursue their asymptotic expansions beyond the zero order term and obtain frame-indifferent, bending governed plate models.

(iii) It is useful to draw a picture of the various regions of the (v_1, v_2) -plane according to the values of QW_0 (see Fig. 1). The curve is an arc of the ellipse of equation $v_1^2 + \nu v_2^2 = 1 + \nu$. In the union of the interior of this ellipse and the half unit square, the function W_0 is relaxed as indicated. Outside of this set, the quasiconvex envelope QW_0 coincides with the function W_0 itself. The circle is the curve on which the term $\frac{1}{8(\lambda + 2\mu)} ([\lambda h(\bar{F}) - (\lambda + 2\mu)]_+)^2$ becomes strictly positive. It is tangent to the ellipse. It is only in the hachured region delimited by this circle and the ellipse that the energy found

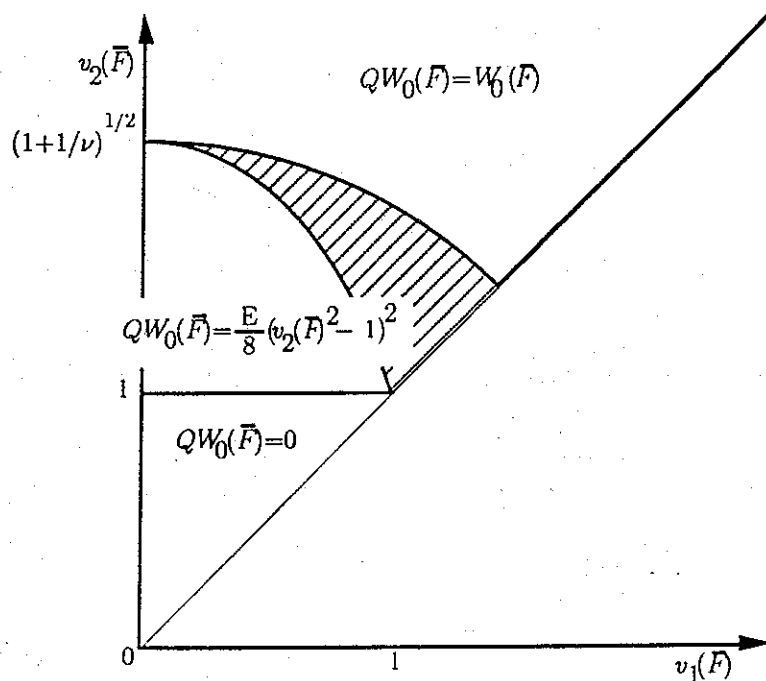


Fig. 1

by formal asymptotic expansions is the correct energy for the Saint-Venant-Kirchhoff membrane.

(iv) For $\lambda = 0$, the expression of the quasiconvex envelope QW_0 is much simpler:

$$QW_0(\bar{F}) = \frac{\mu}{4} (([v_1(\bar{F})^2 - 1]_+)^2 + ([v_2(\bar{F})^2 - 1]_+)^2).$$

(v) This is a case when $(QW_0) = QW_0$. The minimum in (QW_0) is attained at one point and furthermore QW happens to be convex, cf. Proposition 12 and the remarks following Corollary 11. See Le Dret and Raoult [3], [4] for the explicit computation of QW if W is the Saint-Venant-Kirchhoff stored energy function.

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