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Convergence in \mathcal{D}' and in L^1 under strict convexity

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Dedicated to Enrico Magenes with esteem and affection

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let (u_n) be a sequence in $L^1(\Omega; \mathbb{R}^M)$ which converges "weakly" to some limit $u \in L^1(\Omega; \mathbb{R}^M)$. Let $j : \mathbb{R}^M \rightarrow \mathbb{R}$ be a convex function such that

$$(1) \quad \limsup_{n \rightarrow \infty} \int_{\Omega} j(u_n) \leq \int_{\Omega} j(u).$$

Many authors have studied the question whether (u_n) converges *strongly* in L^1 if, in addition, j is assumed to be strictly convex (see [8], [2], [3], [4], [6] and the references therein). In these works it is often assumed that (u_n) converges to u *weakly* in L^1 , that is, for the weak $\sigma(L^1, L^\infty)$ topology. Unfortunately, the convergence in $\sigma(L^1, L^\infty)$ is a *very restrictive assumption* and it is desirable to replace it by the *much weaker and more natural assumption* that (u_n) converges to u in the sense of distributions

$$(2) \quad u_n \rightarrow u \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^M).$$

Throughout this paper we shall assume, for convenience, that $j : \mathbb{R}^M \rightarrow \mathbb{R}$ is a continuous convex function such that

$$(3) \quad |j(t)| \leq C(|t| + 1) \quad \forall t \in \mathbb{R}^M$$

for some constant C .

Our main result is the following

THEOREM 1. Let (u_n) be a sequence in $L^1(\Omega; \mathbb{R}^M)$ and let $u \in L^1(\Omega; \mathbb{R}^M)$ be such that (1) and (2) hold. Assume that

$$(4) \quad j \text{ is strictly convex.}$$

(i) Then

$$(5) \quad u_n \rightarrow u \text{ strongly in } L^1_{loc}(\Omega; \mathbb{R}^M).$$

(ii) If, in addition, we suppose that

$$(6) \quad \lim_{|t| \rightarrow \infty} j(t) = +\infty$$

then

$$(7) \quad u_n \rightarrow u \text{ strongly in } L^1(\Omega; \mathbb{R}^M).$$

Remark 1. If we assume that $u_n \rightharpoonup u$ weakly in $\sigma(L^1, L^\infty)$, then (1) and (4) imply (7) without having to assume (6) (see [8], Theorem 2). However, if we assume only (1), (2) and (4) without (6) then conclusion (7) may fail as the following example shows :

Example 1. Let j be any (smooth) strictly convex function on \mathbb{R} satisfying

$$(8) \quad j(t) > 0 \quad \forall t \in \mathbb{R}$$

and

$$(9) \quad \lim_{t \rightarrow +\infty} j(t) = 0.$$

Let $\Omega = (0, 1)$ and let

$$u_n(x) = \begin{cases} 0 & \text{if } 0 < x < 1 - \frac{1}{n} \\ n^2 & \text{if } 1 - \frac{1}{n} < x < 1 \end{cases}$$

so that

$$u_n \rightarrow 0 \quad \text{in } \mathcal{D}'(\Omega).$$

We have

$$\int_{\Omega} j(u_n) = (1 - \frac{1}{n})j(0) + \frac{1}{n}j(n^2)$$

and thus (1) holds. But (7) fails and we even have $\|u_n\|_{L^1} \rightarrow \infty$.

An easy consequence of Theorem 1 is the following :

COROLLARY 1. Let (u_n) be a sequence in $W^{1,1}(\Omega; \mathbb{R})$ and let $u \in W^{1,1}(\Omega; \mathbb{R})$ be such that

$$(10) \quad u_n \rightarrow u \quad \text{in } L^1_{loc}(\Omega).$$

Assume that $j : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies (3) and (4) and that

$$(11) \quad \limsup_{n \rightarrow \infty} \int_{\Omega} j(\nabla u_n) \leq \int_{\Omega} j(\nabla u).$$

(i) Then we have

$$(12) \quad \nabla u_n \rightarrow \nabla u \quad \text{strongly in } L^1_{loc}(\Omega; \mathbb{R}^N).$$

(ii) If, in addition, (6) holds then we have

$$(13) \quad \nabla u_n \rightarrow \nabla u \quad \text{strongly in } L^1(\Omega; \mathbb{R}^N).$$

Assertion (ii) in Corollary (1) corresponds essentially to the conclusion of Theorem 8.6 in [1].

The proof of Theorem 1 is divided into 6 steps.

Step 1. Assume j is a convex function satisfying (3) and that (2) holds, then

$$(14) \quad \liminf_{n \rightarrow \infty} \int_{\Omega} j(u_n) \zeta \geq \int_{\Omega} j(u) \zeta \quad \forall \zeta \in C^\infty(\Omega) \text{ with } 0 \leq \zeta \leq 1.$$

Proof. Let j^* be the conjugate convex function of j . Then

$$(15) \quad \int_{\Omega} j(u_n) \zeta \geq \int_{\Omega} u_n \varphi \zeta - \int_{\Omega} j^*(\varphi) \zeta$$

for every $\varphi \in \mathcal{D}(\Omega; \mathbb{R}^M)$. Passing to the limit in (15) we see that

$$(16) \quad \liminf_{n \rightarrow \infty} \int_{\Omega} j(u_n) \zeta \geq \int_{\Omega} u \varphi \zeta - \int_{\Omega} j^*(\varphi) \zeta.$$

Next we observe (as in [5], Proposition 1) that

$$\sup_{\varphi \in \mathcal{D}(\Omega; \mathbb{R}^M)} \left\{ \int_{\Omega} u \varphi \zeta - \int_{\Omega} j^*(\varphi) \zeta \right\} = \int_{\Omega} j(u) \zeta.$$

Remark 2. The spirit of Step 1 has been essentially known for a long time (see e.g. [7]).

Step 2. Assume (1), (2) and (4). Then, there is a subsequence (u_{n_k}) such that

$$(17) \quad u_{n_k} \rightarrow u \quad \text{a.e.}$$

Proof. Set

$$(18) \quad f_n = \frac{1}{2}j(u) + \frac{1}{2}j(u_n) - j\left(\frac{u+u_n}{2}\right) \geq 0.$$

By (1) and Step 1 (applied to $\frac{u+u_n}{2}$ and $\zeta \equiv 1$) we have

$$(19) \quad \limsup_{n \rightarrow \infty} \int_{\Omega} f_n \leq \int_{\Omega} j(u) - \liminf_{n \rightarrow \infty} \int_{\Omega} j\left(\frac{u+u_n}{2}\right) \leq 0.$$

Hence $f_n \rightarrow 0$ in $L^1(\Omega)$ and thus there is a subsequence n_k such that

$$(20) \quad f_{n_k} \rightarrow 0 \quad \text{a.e.}$$

We conclude easily with the help of the following standard

LEMMA 1. Assume j is strictly convex on \mathbb{R}^M . Let $a \in \mathbb{R}^M$ and let (b_n) be a sequence in \mathbb{R}^M such that

$$\frac{1}{2}j(a) + \frac{1}{2}j(b_n) - j\left(\frac{a+b_n}{2}\right) \rightarrow 0.$$

Then $b_n \rightarrow a$.

Step 3. Assume (1), (2) and (4). Then

$$(21) \quad j(u_{n_k}) \rightarrow j(u) \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. Since j is convex there exist some $\underline{z} \in \mathbb{R}^M$ and a constant C such that

$$(22) \quad j(t) \geq \underline{z}t - C \quad \forall t \in \mathbb{R}^M.$$

Set

$$\tilde{j}(t) = j(t) - \underline{z}t + C$$

so that \tilde{j} is convex and $\tilde{j}(t) \geq 0 \forall t$. Set

$$g_k(x) = \tilde{j}(u_{n_k}(x)) - [\tilde{j}(u_{n_k}(x)) - \tilde{j}(u(x))],$$

so that

$$(23) \quad |g_k(x)| \leq \tilde{j}(u(x))$$

(since $\tilde{j} \geq 0$).

On the other hand, by Step 2, we know that

$$(24) \quad g_k(x) \rightarrow \tilde{j}(u(x)) \quad \text{a.e.}$$

We deduce from (23) and (24), by dominated convergence, that

$$g_k \rightarrow \tilde{j}(u) \quad \text{in } L^1(\Omega).$$

But

$$g_k - \tilde{j}(u) = -2(\tilde{j}(u_{n_k}) - \tilde{j}(u))^-$$

and thus we conclude that

$$(25) \quad \int_{\Omega} (\tilde{j}(u_{n_k}) - \tilde{j}(u))^- \rightarrow 0.$$

Finally, we observe that

$$(26) \quad |\tilde{j}(u_{n_k}) - \tilde{j}(u)| = \tilde{j}(u_{n_k}) - \tilde{j}(u) + 2(\tilde{j}(u_{n_k}) - \tilde{j}(u))^-.$$

Let $\zeta \in \mathcal{D}(\Omega)$ with $0 \leq \zeta \leq 1$ and write

$$(27) \quad \begin{aligned} \int_{\Omega} \tilde{j}(u_{n_k}) \zeta &= \int_{\Omega} [j(u_{n_k}) \zeta - \xi u_{n_k} \zeta + C \zeta] \\ &= \int_{\Omega} j(u_{n_k}) - \int_{\Omega} j(u_{n_k})(1 - \zeta) - \int_{\Omega} \xi u_{n_k} \zeta + C \int_{\Omega} \zeta. \end{aligned}$$

Passing to the limit in (27) with the help of (1) and Step 1 (applied with $1 - \zeta$ in place of ζ) we are led to

$$(28) \quad \limsup_{k \rightarrow \infty} \int_{\Omega} \tilde{j}(u_{n_k}) \zeta \leq \int_{\Omega} \tilde{j}(u) \zeta.$$

Combining (25), (26) and (28) we obtain

$$(29) \quad \limsup_{k \rightarrow \infty} \int_{\Omega} |\tilde{j}(u_{n_k}) - \tilde{j}(u)| \zeta \leq 0.$$

In particular, $\tilde{j}(u_{n_k}) \rightarrow \tilde{j}(u)$ in $\mathcal{D}'(\Omega)$ and consequently $j(u_{n_k}) = \tilde{j}(u_{n_k}) + \xi u_{n_k} - C$ converges in $\mathcal{D}'(\Omega)$ to $j(u)$.

Step 4. We shall need the following :

LEMMA 2. Let (ψ_n) be a sequence in $L^1(\Omega; \mathbb{R})$ and let $\psi \in L^1(\Omega; \mathbb{R})$ such that

$$(30) \quad \psi_n \geq 0 \quad \text{a.e., } \forall n,$$

$$(31) \quad \psi_n \rightarrow \psi \quad \text{a.e.}$$

and

$$(32) \quad \psi_n \rightarrow \psi \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$(33) \quad \psi_n \rightarrow \psi \quad \text{in } L^1_{loc}(\Omega).$$

Proof. Note that, by (30),

$$\psi - \psi_n \leq \psi$$

and thus

$$(\psi - \psi_n)^+ \leq \psi.$$

By dominated convergence we deduce that

$$(34) \quad (\psi - \psi_n)^+ \rightarrow 0 \quad \text{in } L^1(\Omega).$$

But

$$(\psi - \psi_n)^- = (\psi - \psi_n)^+ - (\psi - \psi_n)$$

and thus

$$(35) \quad \int_{\Omega} (\psi - \psi_n)^- \zeta = \int_{\Omega} (\psi - \psi_n)^+ \zeta - \int_{\Omega} (\psi - \psi_n) \zeta \rightarrow 0$$

for every $\zeta \in \mathcal{D}(\Omega)$, by (32) and (34). The conclusion (33) follows from (34) and (35).

Step 5. We shall need the following :

6

LEMMA 3. Let K be a closed convex set in \mathbb{R}^M , $K \neq \mathbb{R}^M$, and K strictly convex. Let (v_n) be a sequence in $L^1(\Omega; \mathbb{R}^M)$ and let $v \in L^1(\Omega; \mathbb{R}^M)$. Assume

$$v_n(x) \in K \quad \text{a.e., } \forall n,$$

$$v_n \rightarrow v \quad \text{a.e.}$$

and

$$(38) \quad v_n \rightarrow v \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^M).$$

Then

$$(39) \quad v_n \rightharpoonup v \quad \text{in } L^1_{loc}(\Omega; \mathbb{R}^M).$$

Proof. Let I_K^* denote the conjugate function of the indicator function I_K of K , i.e.

$$I_K^*(y) = \sup_{x \in K} yx, \quad \text{for } y \in \mathbb{R}^M.$$

Note that $I_K^*(0) = 0$, $I_K^*(y) \in [0, \infty] \forall y$ and $I_K^*(\lambda y) = \lambda I_K^*(y) \forall \lambda > 0, \forall y$. Hence

$$D(I_K^*) = \{y \in \mathbb{R}^M; I_K^*(y) < \infty\}$$

is a convex cone with vertex at 0. We claim that

$$(40) \quad D(I_K^*) \quad \text{has non empty interior.}$$

For otherwise $D(I_K^*)$ would be contained in some hyperplane, say $y_M = 0$. Then

$$I_K(x) = \sup_{y \in \mathbb{R}^M} \{xy - I_K^*(y)\} = \sup_{[y_M=0]} \{xy - I_K^*(y)\}$$

and consequently

$$I_K(x + te_M) = I_K(x) \quad \forall t \in \mathbb{R}, \forall x$$

where e_M denote the unit vector normal to the hyperplane $y_M = 0$. This means that K is a cylinder of the form

$$K = Q \times \mathbb{R}$$

where $Q = K \cap [y_M = 0]$. This is impossible since K is assumed to be strictly convex and $K \neq \mathbb{R}^M$. Hence we have proved the claim (40).

7

Next, let $\xi_1, \xi_2, \dots, \xi_M$ be a collection of unit vectors in $D(I_K^*)$ which are linearly independent (such a collection exists by (40)). Set

$$C_i = I_K^*(\xi_i) < \infty.$$

For each fixed $i = 1, 2, \dots, M$, consider the function

$$\psi_n^i(x) = C_i - v_n(x)\xi_i.$$

It is easy to see, using (36)-(38), that ψ_n^i satisfies (30)-(32) and therefore, by Lemma 2,

$$\psi_n^i \xrightarrow{n \rightarrow \infty} \psi_i \quad \text{in } L_{\text{loc}}^1, \quad \forall i.$$

Since the directions ξ_i are linearly independent we conclude that

$$v_n \rightarrow v \quad \text{in } L_{\text{loc}}^1(\Omega; \mathbb{R}^M).$$

Step 6. Proof of Theorem 1.

Part (i). Let $K = \text{epi } j = \{[t, \lambda] \in \mathbb{R}^M \times \mathbb{R}; \lambda \geq j(t)\}$, so that K is a closed convex set in \mathbb{R}^{M+1} , $K \neq \mathbb{R}^{M+1}$, and K is strictly convex (because j is strictly convex). Set

$$v_n(x) = [u_n(x), j(u_n(x))].$$

Clearly, $v_n(x) \in K$ a.e., $\forall n$. By Step 2 we know that

$$v_{n_k} \rightarrow v = [u, j(u)] \quad \text{a.e.}$$

By assumption (2) and by Step 3 we know that

$$v_{n_k} \rightarrow v \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^{M+1}).$$

Applying Lemma 3 (with $(M+1)$ instead of M) we conclude that

$$v_{n_k} \rightarrow v \quad \text{in } L_{\text{loc}}^1(\Omega; \mathbb{R}^{M+1})$$

and in particular

$$u_{n_k} \rightarrow u \quad \text{in } L_{\text{loc}}^1(\Omega; \mathbb{R}^M).$$

The uniqueness of the limit implies, as usual, that

$$u_n \rightarrow u \quad \text{in } L_{\text{loc}}^1(\Omega; \mathbb{R}^M).$$

8

Part (ii). The additional assumption (6) implies that

$$j(t) \geq \alpha|t| - C \quad \forall t$$

for some constants $\alpha > 0$ and C . Adding a constant to j we may always assume that

$$(41) \quad j(t) \geq \alpha|t| \geq 0 \quad \forall t.$$

Applying Step 1 with $\zeta \equiv 1$ and combining this with assumption (1) we see that

$$(42) \quad \int_{\Omega} j(u_n) \rightarrow \int_{\Omega} j(u).$$

We write once more (as in the proof of Lemma 2)

$$(j(u) - j(u_n))^+ \leq j(u),$$

so that, by Step 2 and dominated convergence

$$(43) \quad \int_{\Omega} (j(u) - j(u_{n_k}))^+ \rightarrow 0.$$

Finally we recall that

$$(j(u) - j(u_n))^- = (j(u) - j(u_n))^+ - (j(u) - j(u_n))$$

and consequently (using (42) and (43)) we conclude that

$$(44) \quad \int_{\Omega} (j(u) - j(u_{n_k}))^- \rightarrow 0.$$

From (43) and (44) we deduce that

$$j(u_{n_k}) \rightarrow j(u) \quad \text{in } L^1$$

Passing to a further subsequence we may always assume that

$$(45) \quad |j(u_{n_k})| \leq f \quad \forall k, \text{ a.e.}$$

for some fixed function $f \in L^1(\Omega)$. Combining (41) and (45) we conclude that

$$|u_{n_k}| \leq \frac{1}{\alpha} f \quad \forall k, \text{ a.e.}$$

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From Step 2 and dominated convergence we infer that

$$u_{n_k} \rightarrow u \quad \text{in } L^1(\Omega; \mathbb{R}^M).$$

Again, the uniqueness of the limit implies the convergence of the full sequence.

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