EXISTENCE AND LINEARIZATION FOR THE
SOUZA-AURICCHIO MODEL AT FINITE STRAINS

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Abstract. We address the analysis of the Souza-Auricchio model for shape-memory alloys in the finite-strain setting. The model is formulated in variational terms and the existence of quasistatic evolutions is obtained within the classical frame of energetic solvability. The finite-strain model is proved to converge to its small-strain counterpart for small deformations via a variational convergence argument.

1. Introduction. Shape-memory alloys recover large strains during mechanical or thermomechanical cycling. This results from an abrupt and diffusionless solid-solid phase transformation between different crystallographic configurations, namely the austenite (mostly cubic, predominant at high temperature and low stresses) and the martensites (lower symmetry variants, favored at low temperature or high stresses) [27]. The resulting thermomechanical behavior is at the basis of a variety of innovative applications ranging from sensors and actuators, to Aerospace, Biomedical, and Seismic Engineering [21], among others, and has triggered an intense research activity in the last decades.

The complex behavior of shape-memory materials is nowadays addressed by a menagerie of models [28, 44, 61] focusing on different scales (atomistic, microscopic with micro-structures [9], mesoscopic with volume fractions, macroscopic [13, 64, 63]), emphasizing different principles (minimization of stored energy vs. maximization of dissipation, phenomenology vs. rational crystallography and Thermodynamics [62]), considering different structures (single crystals vs. polycrystalline aggregates, possibly including intragranular interaction and viscosity [58]), and having ambition for different ranges of applicability, including electromagnetic coupling [66, 67]. A minimal selection of macroscopic thermomechanically coupled models can be found in [3, 32, 37, 43, 59, 60, 70].

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Among the many available options, we focus here on a specific phenomenological, internal-variable-type model for polycrystalline materials. The model was originally proposed by Souza, Mamiya, & Zouain [69] and then combined with finite elements by Auricchio & Petrini [5, 6, 7]. The Souza-Auricchio model is remarkably simple (in terms of number of material parameters) and features a sound variational structure that in turn makes it amenable to a quite satisfactory mathematical treatment. The reader is referred to the recent survey [33] for a detailed discussion on the capabilities of the model as well as an account of its many extensions to encompass thermal, ferromagnetic [65], and plastic effects.

The original Souza-Auricchio model is formulated in the small-deformation regime. Still, martensitic-transformation strains can be indeed as large as 8% and the small-strain assumption turns out to be often questionable in real processes. Moving from these considerations, the original, small-strain Souza-Auricchio model has been extended to finite strains by Evangelista, Marfia, & Sacco [22, 23]. At finite strains the model retains its variational structure, although in a geometrically nonlinear setting, and the energy and the flow rule are checked to be pointwise converging to the original Souza-Auricchio model as deformations are small [22, 23]. A first analytical investigation of the model is in [30] where the constitutive relation, corresponding indeed to a strongly nonlinear ODE for tensors is proved to admit suitably weak solutions. Specifically, incremental time-discretizations of the constitutive relation are proved to converge to energetic solutions [51].

The aim of this note is twofold. Firstly, we extend the finite-dimensional analysis of [30] by combining the constitutive material relation with the quasistatic equilibrium system and moving to the full three-dimensional problem. This is again accomplished within the frame of energetic solvability and, in particular, extends the results by Mielke & Petrov [50] to the finite-strain case. To this aim, exactly as in the small-strain regime [4], a specific nonlocal term in the internal variable is introduced in order to prevent fine-scale microstructuring. Moreover, the temperature of the medium is assumed to be known and space-homogeneous. This would ideally correspond to the case of a thin body in at least one direction and mechanical cycles with suitably low frequency. In this situation, one can assume that the heat produced in the specimen is immediately transferred to the surrounding environment acting indeed as a time-dependent heat bath. Note that the Souza-Auricchio model with unknown temperature has been addressed in one space dimension only [38, 39].

The second focus of the paper is the rigorous proof of the convergence of the finite strain model to its small-strain counterpart for small deformations. This in particular delivers a sort of cross-justification of the finite-strain model by connecting it with the by-now well validated small-deformation version. The small-deformation limit is accomplished by means of a variational convergence proof along the general theory of [52]. This in particular provides an extension of the original pointwise limiting argument in [22, 23] and entails the actual convergence of trajectories. Note that such an evolutionary convergence argument has already been applied to inelastic evolution in the context of finite plasticity with hardening [53, 35], of perfect plasticity in one dimension [31], and combined with dimension reduction [18, 19].

The paper is organized as follows. We recall the basic formulation of the Souza-Auricchio model, both in small and finite strains, in Section 2. In particular, the two constitutive models and the corresponding quasistatic evolutions are detailed.
The existence of an energetic solution to the quasistatic evolution problem at finite strains is commented in Section 3. Eventually, the small-deformation limit is rigorously ascertained in Section 4.

2. The mechanical model. This section is devoted to a brief recollection of the constitutive model and its coupling with quasistatic equilibrium, both in the small- and in the finite-strain situation.

2.1. Notation. All tensorial quantities considered in this paper are relative to the three-dimensional euclidean space \( \mathbb{R}^3 \). Matrices \( \mathbf{A} \in \mathbb{R}^{3\times3} \) will be indicated with boldface letters, while their elements are denoted by \( A_{ij} \). In particular, \( \mathbf{I} \in \mathbb{R}^{3\times3} \) with \( I_{ij} = \delta_{ij} \) (Kronecker) denotes the identity matrix. For any \( \mathbf{A} \in \mathbb{R}^{3\times3} \), \( \mathbf{A}^\top \) denotes the transpose of \( \mathbf{A} \) and \( \text{tr}(\mathbf{A}) \) its trace. The deviatoric part of \( \mathbf{A} \) is defined by \( \text{dev}(\mathbf{A}) := \mathbf{A} - (\text{tr}(\mathbf{A}))\mathbf{I}/3 \). For \( \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times3} \) we denote contraction by \( \mathbf{A} : \mathbf{B} := \text{tr}(\mathbf{A}^\top \mathbf{B}) = A_{ij}B_{ij} \), with summation over repeated indeces. This defines a scalar product between matrices which induces the Frobenius norm \( |\mathbf{A}|^2 := \mathbf{A} : \mathbf{A} = A_{ij}A_{ij} \). The following matrices sets will be mentioned hereafter

\[
\begin{align*}
\mathbb{R}_{\text{sym}}^{3\times3} &= \{ \mathbf{A} \in \mathbb{R}^{3\times3} \mid \mathbf{A}^\top = \mathbf{A} \}, \\
\mathbb{R}_{\text{sym}+}^{3\times3} &= \{ \mathbf{A} \in \mathbb{R}_{\text{sym}}^{3\times3} \mid \mathbf{A} \geq 0 \}, \\
\mathbb{R}_{\text{dev}}^{3\times3} &= \{ \mathbf{A} \in \mathbb{R}_{\text{sym}}^{3\times3} \mid \text{tr}\mathbf{A} = 0 \}, \\
\text{GL}(3) &= \{ \mathbf{A} \in \mathbb{R}^{3\times3} \mid \det\mathbf{A} \neq 0 \}, \\
\text{GL}_+(3) &= \{ \mathbf{A} \in \text{GL}(3) \mid \mathbf{A} > 0 \}, \\
\text{GL}_{\text{sym}+}(3) &= \text{GL}(3) \cap \mathbb{R}_{\text{sym}+}^{3\times3}, \\
\text{SL}(3) &= \{ \mathbf{A} \in \mathbb{R}^{3\times3} \mid \det\mathbf{A} = 1 \}, \\
\text{SO}(3) &= \{ \mathbf{A} \in \text{SL}(3) \mid \mathbf{A}^{-1} = \mathbf{A}^\top \}, \\
\text{SL}(3)_{\text{sym}+} &= \text{SL}(3) \cap \mathbb{R}_{\text{sym}+}^{3\times3}.
\end{align*}
\]

The tensor \( \text{cof}\mathbf{A} \) is the cofactor matrix of \( \mathbf{A} \). For \( \mathbf{A} \) invertible we have that \( \text{cof}\mathbf{A} = (\det\mathbf{A})\mathbf{A}^{-\top} \). For convenience we describe the collection of the minors of \( \mathbf{A} \) as the triplet \( \mathcal{M}(\mathbf{A}) = (\mathcal{M}_1(\mathbf{A}), \mathcal{M}_2(\mathbf{A}), \mathcal{M}_3(\mathbf{A})) \), where

\[
\mathcal{M}_1(\mathbf{A}) := \mathbf{A}, \quad \mathcal{M}_2(\mathbf{A}) = \text{cof}\mathbf{A}, \quad \mathcal{M}_3(\mathbf{A}) = \det\mathbf{A}.
\]

For any \( \mathbf{A} \in \mathbb{R}_{\text{sym}+}^{3\times3} \) and any real function \( f : (0, +\infty) \to \mathbb{R} \), we define the tensor-valued function \( \mathbf{A} \mapsto f(\mathbf{A}) \in \mathbb{R}_{\text{sym}+}^{3\times3} \) via diagonalization and without changing notation: if \( Q^\top \mathbf{A} Q = \mathbf{A} \), with \( Q \in \text{SO}(3) \) and \( \mathbf{A} \) diagonal with \( A_{ij} = \lambda_i \delta_{ij} \) (no summation), then

\[
f(\mathbf{A}) := Q^\top f(\mathbf{A}) Q, \quad f(\mathbf{A})_{ij} := f(\lambda_i)\delta_{ij}.
\]

In particular, \( \mathbf{A}^s \) for any \( s \in \mathbb{R} \), \( \log(\mathbf{A}) \), and \( \exp(\mathbf{A}) \) are uniquely defined on \( \mathbb{R}_{\text{sym}+}^{3\times3} \).

For all 3-tensors \( \mathbf{D} \in \mathbb{R}^{3\times3\times3} \) we define \( |\mathbf{D}|^2 := D_{ijk}D_{ijk} \), the partial transposition \( (\mathbf{D}^\top)_{ijk} := D_{ijk} \), and the product with the 2-tensor \( \mathbf{A} \) as \( (\mathbf{D}\mathbf{A})_{ijk} := D_{ijk}A_{ik} \) and \( (\mathbf{A}\mathbf{D})_{ijk} := A_{ik}D_{ijk} \). Note that, along with these definitions, \( |\mathbf{DA}|, |\mathbf{AD}| \leq |\mathbf{A}| |\mathbf{D}| \).

Given any symmetric, positive-definite 4-tensor \( \mathbf{C} \in \mathbb{R}^{3\times3\times3\times3} \), the product \( \mathbf{C}\mathbf{A} \) is defined as \( (\mathbf{C}\mathbf{A})_{ijk} := C_{ij\ell k}A_{\ell k} \). Moreover, we denote by \( |\mathbf{A}|_C^2 := \mathbf{A}^\top\mathbf{C}\mathbf{A} = C_{ij\ell k}A_{ij}A_{\ell k} \) the corresponding induced (squared) norm on \( \mathbb{R}_{\text{sym}}^{3\times3} \).

Let \( \varphi : E \to (\infty, \infty] \) on the general normed space \( E \) be either smooth or proper, convex, and lower semicontinuous. We denote the subdifferential of \( \varphi \) [14]
as $\partial \varphi : \{\varphi < \infty\} \to 2^{E^*}$ with
\[
\partial \varphi(x) = \{v \in E^* | \varphi(y) - \varphi(x) \geq \langle v, y - x \rangle\},
\]
where $E^*$ is the topological dual of $E$.

A final caveat: in the following we use the same symbol $c$ in order to indicate a
genéric constant, possibly depending on data and varying from line to line.

2.2. Small strains: constitutive model. We consider a body occupying the
reference configuration $\Omega \subset \mathbb{R}^3$, which is an open, connected, and bounded subset of
$\mathbb{R}^3$ with Lipschitz boundary $\partial \Omega$. We decompose the boundary $\partial \Omega$ as $\partial \Omega = \Gamma_D \cup \Gamma_{tr}$
where $\Gamma_D$ and $\Gamma_{tr}$ are disjoint and $\Gamma_D$ has positive surface measure. In the following,
a Dirichlet boundary condition is imposed on $\Gamma_D$ whereas traction is exerted on $\Gamma_{tr}$.

We denote by $y : \Omega \to \mathbb{R}^3$ the deformation of the body and by $u = y - \text{id} : \Omega \to \mathbb{R}^3$
the corresponding displacement field. The linearized strain $\epsilon(u) := (\nabla u + (\nabla u)^\top)/2$
is additively decomposed in an elastic part $\epsilon_{el}$ and a transformation part $z$ as
\[
\epsilon(u) = \epsilon_{el} + z.
\]
The elastic part of the deformation is linearly related to the stress $\sigma \in \mathbb{R}^{3 \times 3}_\text{sym}$ via $\epsilon_{el} = \mathbb{C}^{-1} \sigma$, where $\mathbb{C}$ stands for the isotropic elasticity tensor. The transformation
strain $z \in \mathbb{R}^{3 \times 3}_{\text{dev}}$ is assumed to be deviatoric, since the martensitic transformations
are approximately volume preserving, and can be related to the martensitic content
at a given material point. This can be made explicit in the case of a single crystal [41],
where to each martensitic variant is associated a (fixed) transformation strain $z_k$, $k = 1, \ldots, N$,
determined by crystallography and showing the same elastic response.
In a macroscopic description, one assumes the material to be a mixture of the
$N$ martensitic variants, described by a mass-fractions vector $p := (p_1, \ldots, p_N) \in [0, 1]^N$, with $|p| := \sum_{i=1}^N p_i \leq 1$ ($|p| = 0$ means that only austenite is present).
Accordingly, the transformation strain is assumed to be a function of the mass-
fractions vector $p$ through $z(p) = \sum_{i=1}^N p_i z_i$. It follows that the transformation
strain $z$ belongs to the $N$ simplex in $\mathbb{R}^{3 \times 3}_{\text{dev}}$. In the polycrystalline case, it is hence
reasonable to assume an isotropic constraint of the form $|z| \leq \epsilon_L$, where the scalar $\epsilon_L$
expresses the maximum strain realizable via martensitic transformation.

The equilibrium property of the material are encoded in the free energy functional. The Souza-Auricchio model is characterized by the Helmholtz free energy density
\[
\psi_0(\epsilon, z, \theta) := \psi_0^\text{el}(\epsilon, z) + \psi_0^\text{in}(z, \theta),
\]
where
\[
\psi_0^\text{el}(\epsilon, z) := \frac{1}{2} (\epsilon - z) : \mathbb{C} (\epsilon - z),
\]
\[
\psi_0^\text{in}(z, \theta) := \frac{1}{2} z : \mathbb{H} z + b(\theta)|z| + I_{\epsilon_L}(|z|)
\]
denote respectively the elastic and the inelastic component of the energy. Here and
in the following the subscript $0$ is used to refer to the small-strain situation. The indicator function $I_{\epsilon_L}$ is defined by
\[
I_{\epsilon_L}(r) = \begin{cases}
0 & \text{if } |r| \leq \epsilon_L \\
+\infty & \text{else},
\end{cases}
\]
and encodes the constraint $|z| \leq \epsilon_L$. The function $\theta \mapsto \beta(\theta)$ is the austenite-
martensite transformation stress at temperature $\theta$. It is usually assumed to be
vanishing below some critical temperature \( \theta_M \) and to be linearly growing for high temperatures. The original choice \( \beta(\theta) := \beta_0(\theta - \theta_M)_+ \), where \( x_+ := \max\{x, 0\} \) is the classical positive part and \( \beta_0 \) is a positive material parameter \([5, 69]\), leads to some thermodynamical inconsistency and should be replaced by a suitable regularized version \([38]\). In what follows we however only ask \( \beta \) to be Lipschitz continuous. Note nonetheless that the temperature evolution \( t \mapsto \theta(t) \) is henceforth assumed to be given.

The classical constitutive relations for the conjugated forces \( \sigma \) and \( X \) are obtained from the variation of the free energy as

\[
\sigma = \partial_\varepsilon \psi_0 = \mathbb{C}(\varepsilon - z), \quad X = \partial_z \psi_0 = -\sigma + \mathbb{H}z + \beta(\theta)\partial |z| + \partial I_{\text{vel}}(|z|).
\]

The constitutive evolution law at the material point is prescribed through a balance between dissipative and conservative forces as

\[
\partial_\dot{\varepsilon} R_0(\dot{\varepsilon}) + \partial_z \psi_0(\varepsilon, z, \theta) \ni 0.
\]

The dissipation potential \( R_0 \) is of Von Mises type, namely \( R_0(\dot{\varepsilon}) = \rho |\dot{\varepsilon}| \), where \( \rho > 0 \) denotes some given yield stress. Since \( R_0 \) is 1-homogeneous, the constitutive law (2) is rate independent. Given the explicit form of the free energy \( \psi_0 \), the material constitutive relation (2) can be spelled out as

\[
\rho \partial |\dot{\varepsilon}| + \mathbb{H}z + \beta(\theta)\partial |z| + \partial I_{\text{vel}}(|z|) \ni \sigma,
\]

to be solved in \( \mathbb{R}^{3 \times 3}_{\text{dev}} \), at all material points.

### 2.3. Small strains: quasistatic evolution

By assuming the temperature evolution \( t \mapsto \theta(t) \) to be given, the quasistatic evolution of the medium results from the combination of the constitutive model with the equilibrium system. The state of the system is hence described by the pair \( q = (u, z) \). In order to obtain the existence of quasistatic evolutions, we are forced to include in the energy a penalization of phases interfaces in the form \([33]\)

\[
\mathcal{V}_r(z) = \frac{\mu}{r} \int_\Omega |\nabla z(x)|^r \, dx, \quad r \geq 1
\]

which sets the problem within the frame of gradient inelastic theories \([55, 24, 25]\). This introduces the length scale \( \kappa^{-r} \) for interfaces. Note that \( r = 1 \) is admissible. The total energy functional, which includes the internal energy and external power sources, is given by

\[
\mathcal{E}_0(t, q) := \int_\Omega \psi_0(\varepsilon(u), z, \theta(t)) \, dx + \mathcal{V}_r(z) - \langle \ell(t), u \rangle \quad \text{(3)}
\]

where

\[
\langle \ell(t), u \rangle := \int_\Omega b(x, t) \cdot u(x) \, dx + \int_{\Gamma_\text{tr}} g(x, t) \cdot u(x) \, d\Gamma
\]

and \( b : (0, T) \times \Omega \to \mathbb{R}^3 \) and \( g : (0, T) \times \Gamma_\text{tr} \to \mathbb{R}^3 \) are given body force and boundary traction, respectively. The dissipation potential reads

\[
\mathcal{R}_0(\dot{q}) = \int_\Omega \rho |\dot{z}| \, dx
\]

and the quasistatic evolution of the medium can be formulated variationally as

\[
\partial_q \mathcal{R}_0(\dot{q}) + \delta_q \mathcal{E}_0(t, q) \ni 0,
\]
where \( \delta_q \mathcal{E} := \partial_q \mathcal{E} - \nabla \cdot \partial q \mathcal{E} \) stands for a variational derivative. In particular, for \( r > 1 \) we have the system

\[
\mathbb{C}(\varepsilon(u) - z) = \sigma, \tag{4}
\]

\[
\nabla \cdot \sigma + b = 0, \tag{5}
\]

\[
\rho \partial |z| + H z + \beta(\theta) \partial |z| + \partial I_{\varepsilon_L}(|z|) - \kappa \nabla \cdot (|z|^{-2} \nabla z) \ni \sigma \tag{6}
\]
on \((0, T) \times \Omega\), complemented with the boundary conditions

\[
u = u_D \text{ on } \Gamma_D, \quad \sigma^\nu = g \text{ on } \Gamma_{\text{tr}}, \quad (\nu \cdot \nabla) z = 0 \text{ on } \partial \Omega,
\]

where the displacement \( u_D \) is given and \( \hat{\nu} = \hat{\nu}(x) \) denotes the unit (external) normal vector at the boundary point \( x \in \partial \Omega \), as well as an initial condition for \( z \).

2.4. Finite strains: constitutive model. A finite-strain version of the original small strains Souza-Auricchio model was proposed in [22, 23] and further studied from the analytical viewpoint in [30].

Let \( F := \nabla y = I + \nabla u \) (in components \( F_{ij} = \partial y_i/\partial x_j \)) be the deformation gradient, assumed to be defined almost everywhere on \( \Omega \) with values in \( GL(3) \).

Following the pattern of the classical finite-strain plasticity [40, 42], one multiplica-
tively decomposes [23]

\[
F = F_e F_{\text{tr}}, \tag{7}
\]

where \( F_e \) denotes the elastic part of \( F \) and \( F_{\text{tr}} \) the transformation part. The volume preservation constraint amounts to assume \( \det F_{\text{tr}} = 1 \), so that \( F_{\text{tr}} \in SL(3) \).

Note that in [60] a further multiplicative splitting of the tensor \( F_{\text{tr}} \) is introduced, separating indeed the reversible contributions due to phase transitions from plastic contributions (a generalization of the same idea is introduced in [8] in the small-
strain context).

For each deformation gradient we introduce the corresponding (right) Cauchy-
Green symmetric tensors

\[
C := F^T F \in GL_{\text{sym}}(3), \quad C_e := F_e^T F_e \in GL_{\text{sym}}(3), \quad C_{\text{tr}} := F_{\text{tr}}^T F_{\text{tr}} \in SL(3)_{\text{sym}}.
\]

The Green-St. Venant tensor corresponding to the transformation strain

\[
E_{\text{tr}} := \frac{1}{2}(C_{\text{tr}} - I),
\]

plays a central role, since it replaces the infinitesimal transformation strain \( z \). Indeed, the free energy density \( \psi \) for the finite strain model is assumed to be

\[
\psi = \tilde{\psi}^{\text{el}}(C_e) + \psi^{\text{in}}(E_{\text{tr}}, \theta),
\]

where the inelastic free energy density is obtained from the infinitesimal version \( \psi^{\text{in}}_0 = \beta(\theta)|z| + \frac{1}{2}|z|^2 + I_{\varepsilon_L}(|z|) \) by just replacing \( z \) with \( E_{\text{tr}} \), namely

\[
\psi^{\text{in}}(E_{\text{tr}}, \theta) = \beta(\theta)|E_{\text{tr}}| + \frac{1}{2}|E_{\text{tr}}|^2 + I_{\varepsilon_L}(|E_{\text{tr}}|).
\]

This position is readily justified upon noting that \( E_{\text{tr}} = z + O(|z|^2) \).

The expression of the elastic energy as a function of \( C_e \) rather than \( F_e \) reflects frame indifference, namely the invariance under change of frame in the physical space. Precisely, frame indifference reads

\[
\psi^{\text{el}}(Q F_e) = \psi^{\text{el}}(F_e) \quad \forall Q \in SO(3),
\]
which implies $\psi^e(F_e) = \psi^e(C_e^{1/2}) = \hat{\psi}^e(C_e)$, thanks to the polar decomposition theorem.

The elastic Cauchy Green tensor can be expressed by (7) in terms of the total and transformation strains as $C_e = F_{tr}^{-\top}CF_{tr}^{-1}$. We assume isotropic elastic response, that is

$$
\psi^e(F_e) = \psi^e(F_eR) \quad \forall R \in SO(3),
$$

This implies that the energy depends on $C_{tr}$, rather than on $F_{tr}$. In fact, by the polar decomposition, $C_e = R(C_{tr}^{-1/2}CC_{tr}^{-1/2}) R^\top$, and we can express the elastic energy solely in terms of the Cauchy-Green tensors $C_{tr}$ and $C_{tr}$. We can therefore define the free energy as a function $\psi(C, C_{tr})$ of the state variables $C$ and $C_{tr}$.

For the sake of definiteness, let us mention the class of Ogden materials [15, Sec. 4.9] whose elastic energy density is

$$
\hat{\psi}^e(C_e) = \sum_{i=1}^{n} a_{i} tr C_e^{\gamma_i/2} + \sum_{j=1}^{m} b_{j} tr (cof C_e)^{\delta_j/2} + \Gamma (\det C_e^{1/2}),
$$

$\gamma_i, \delta_j \geq 1, \ a_i, b_j > 0, \ \left[\begin{array}{c} \gamma_i \\ \delta_j \end{array} \right] \geq 1, \ s \mapsto \Gamma(s)$ convex on $(0, \infty)$, \ \lim_{s \to 0^+} \Gamma(s) = \infty

complying indeed with all these assumptions.

The constitutive evolution equation in the finite-strain model [23] has the form of an associative flow rule for $\dot{C}_{tr}$. This evolutive equation was brought back to the variational formulation

$$
\partial_{C_{tr}} R(C_{tr}, \dot{C}_{tr}) + \partial_{\dot{C}_{tr}} \psi(C, C_{tr}) \geq 0,
$$

in [30] by means of the following dissipation potential

$$
R(C_{tr}, \dot{C}_{tr}) = \left\{ \begin{array}{ll}
\frac{\rho}{2} | C_{tr}^{-1/2} \dot{C}_{tr}C_{tr}^{-1/2} | & \text{if } tr (C_{tr}^{-1/2} \dot{C}_{tr}C_{tr}^{-1/2}) = 0, \\
+\infty & \text{else}
\end{array} \right.
$$

A discussion on the equivalence between the flow rules in terms of $F_{tr}$ and of $C_{tr}$ can be found in [34].

The flow rule (9) can be made more explicit as follows. Let $\phi : SL(3)_{sym+} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be the yield function associated with $R$, namely, let $T \mapsto \phi(C_{tr}, T)$ be the Legendre conjugate of $\dot{C}_{tr} \mapsto R(C_{tr}, \dot{C}_{tr})$ given by [30, 34]

$$
\phi(C_{tr}, T) = |\text{dev} (C_{tr}^{1/2} TC_{tr}^{1/2})| - \frac{\rho}{2},
$$

and compute

$$
\partial_T \phi(C_{tr}, T) = C_{tr} \frac{\text{dev} (C_{tr}^{1/2} TC_{tr}^{1/2})}{|\text{dev} (C_{tr}^{1/2} TC_{tr}^{1/2})|} C_{tr}^{1/2} = C_{tr} \frac{\text{dev} (TC_{tr})}{|\text{dev} (TC_{tr})|},
$$

where we interpret $0/0 := \{ F \in \mathbb{R}^{3 \times 3} \mid |F| \leq 1 \}$. Then, the flow rule (9) can be expressed via classical Kuhn-Tucker complementary conditions as

$$
\begin{cases}
\dot{C}_{tr} = \zeta C_{tr} \frac{\text{dev} (TC_{tr})}{|\text{dev} (TC_{tr})|}, \\
\zeta \geq 0, \ \phi(C_{tr}, T) \leq 0, \ \zeta \phi(C_{tr}, T) = 0.
\end{cases}
$$
where \( T \in \mathbb{R}^{3 \times 3} \) is an element of the subdifferential of the free energy, namely

\[
T \in -2\partial_{C_{\text{tr}}} \psi(C, C_{\text{tr}}) = -2F_{\text{tr}}^{-1}C_{\text{e}}\partial_{C_{\text{e}}} \psi_{\text{e}}(C_{\text{e}})F_{\text{tr}}^{-\top} - (\beta(\theta) + \partial I_{\epsilon L}(|E_{\text{tr}}|)) \frac{E_{\text{tr}}}{|E_{\text{tr}}|} - \mathbb{H}E_{\text{tr}}.
\]

2.5. Quasistatic evolution at finite strains. We now turn to the coupling of the finite strain constitutive material relation with the quasistatic equilibrium system. In analogy with the small-strain model, we shall here consider an interfacial energy contribution of the form

\[
V_{r}(C_{\text{tr}}) = \mu_{r} \int_{\Omega} |\nabla C_{\text{tr}}|^{r} \, dx, \quad r \geq 1.
\]

which is bounded for \( C_{\text{tr}} \in W^{1,r}(\Omega, \mathbb{R}^{3 \times 3}) \) if \( r > 1 \). For \( r = 1 \), we allow \( \nabla C_{\text{tr}} \) to be measure-valued, so that \( V_{1}(C_{\text{tr}}) \) actually is proportional to the total variation of \( C_{\text{tr}} \) for any \( C_{\text{tr}} \in BV(\Omega, \mathbb{R}^{3 \times 3}) \) (see [2] and Section 3.1). The state variable now reads \( q = (y, C_{\text{tr}}) \) and the total energy functional with the external actions is

\[
E(t, q) = \int_{\Omega} \psi(\nabla y^{\top} \nabla y, C_{\text{tr}}, \theta(t)) \, dx + V_{r}(C_{\text{tr}}) - \langle \ell(t), y \rangle.
\]

The dissipation is instead given by

\[
\mathcal{R}(q, \dot{q}) = \int_{\Omega} R(C_{\text{tr}}, \dot{C}_{\text{tr}}) \, dx,
\]

so that the evolution equation is formally expressed by

\[
\partial_{t} \dot{q} + \delta_{q} E(t, q) \ni 0.
\]

By computing the variational derivatives [30, 34], we can rewrite the latter for \( r > 1 \) as

\[
2F_{\text{tr}}^{-1} \partial_{C_{\text{tr}}} \psi_{\text{e}}(C_{\text{e}}^{-1/2} \nabla y^{\top} \nabla y C_{\text{tr}}^{-1/2}) F_{\text{tr}}^{-\top} = S,
\]

\[
\nabla \cdot (\nabla y S) + b = 0,
\]

\[
\partial_{C_{\text{tr}}} R(C_{\text{tr}}, C_{\text{tr}}) + \partial_{C_{\text{tr}}} \psi(C, C_{\text{tr}}) + \kappa \nabla \cdot (|\nabla C_{\text{tr}}|^{r-2} \nabla C_{\text{tr}}) \ni 0,
\]

in \((0, T) \times \Omega\) (compare with (4)-(6)) along with the initial condition

\[
(y(0), C_{\text{tr}}(0)) = (y_{0}, C_{\text{tr}, 0}) \text{ on } \Omega
\]

and the boundary conditions

\[
y = y_{D} \text{ on } \Gamma_{D}, \quad \nabla y S \hat{\nu} = g \text{ on } \Gamma_{\text{tr}},
\]

where \( u_{D} \) is a given boundary datum and (for \( r > 1 \))

\[
(\hat{\nu} \cdot \nabla) C_{\text{tr}} = 0 \text{ on } \partial \Omega.
\]

Observe that the tensor \( S \) is nothing but the second Piola-Kirchhoff stress tensor, which is defined by

\[
S = 2\partial_{C} \psi_{\text{e}},
\]

where, in comparison with (12), the invariance under rotations of the intermediate configuration is apparent.
3. **Energetic solvability.** Let us now consider the existence of solutions to (12)-(17). In particular, given suitable external actions

\[ t \in [0, T] \mapsto (\theta(t), y_D(t), f(t), g(t)) \]

and an initial datum \((y_0, C_{tr,0})\), we aim at finding a solution \(t \mapsto (y(t), C_{tr}(t))\) to the system (12)-(17). As strong solvability seems presently unaccessible, we resort to the variational notion of energetic solvability instead [46, 47, 54]. This formulation is introduced in Subsection 3.1 and existence is proved in Subsection 3.2.

3.1. **Energetic formulation.** Let us recollect here some basic material on the concept of energetic solutions. All details and motivation can be found in the recent monograph by Mielke & Roubiček [51]. An abstract rate-independent system is specified by the triple \((Q, E, R)\), where \(Q\) is the state space, \(E(t,q)\) the energy functional, and \(R(q,\dot{q})\) the dissipation potential, and reads

\[ \partial \dot{q} R(q,\dot{q}) + \delta q E(t,q) \ni 0. \]  

(18)

The positive 1-homogeneity condition \(R(q,\lambda \dot{q}) = \lambda R(q,\dot{q})\), for \(\lambda > 0\), encodes rate-independence. The energetic formulation of the differential problem (18) consists in looking for a trajectory \(q: [0, T] \to Q\) fulfilling, for all times \(t \in [0, T]\),

\[ q(t) \in S(t), \]

(19)

\[ E(t,q(t)) + \text{Diss}_{D,[0,t]}(q) = E(0,q(0)) + \int_0^t \partial_s E(s,q(s))ds \]

(20)

where

\[ D(q_0, q_1) := \inf \left\{ \int_0^1 R(q(t),\dot{q}(t))dt \mid q \in C^1([0,1], Q), q(0) = q_0, q(1) = q_1 \right\} \]

\[ S(t) := \{ q \in Q \mid E(t,q) \leq E(t,\tilde{q}) + D(q,\tilde{q}) \forall \tilde{q} \in Q \}, \]

\[ \text{Diss}_{D,[s,t]}(q) := \sup \left\{ \sum_{n=1}^n D(q(t_{n-1}),q(t_n)) \mid n \in \mathbb{N}, s = t_0 < t_1 < \ldots < t_n = t \right\} \]

for all \(0 \leq s \leq t \leq T\). The set \(S(t)\) is called the set of stable states at time \(t\). Correspondingly, relation (19) expresses the global stability of the state \(q\) at each time. Relation (20) is the energy balance instead.

Let us now leave the abstract frame and specify our notion of energetic solutions for the quasistatic evolution at finite strains. Note that existence of energetic solutions of the constitutive model has already been obtained in [30]. We hence extend here those results to the full quasistatic-evolution case.

In order to take nonhomogeneous Dirichlet boundary conditions into account, we perform a classical map-composition splitting of the displacement field [26, 45]. The deformation map is expressed as

\[ y_D(t, y(t,x)) \]

(21)

where \(y_D\) is the prescribed Dirichlet condition on \(\Gamma_D \subset \Omega\). Note that this allows to have the variable \(y\) to be defined on the set

\[ \mathcal{Y} := \{ y \in W^{1,\eta_0}(\Omega, \mathbb{R}^3) \mid y = \text{id} \text{ on } \Gamma_D \} \]

(22)
where \( q_y > 3 \). The map \( y_\theta(\cdot, t) \) is assumed to be a diffeomorphism. More precisely, by defining \( H = \nabla y_\theta \) we assume

\[
y_\theta \in C^2((0, T) \times \mathbb{R}^3; \mathbb{R}^3), \quad H \in GL_+ (3) \text{ in } (0, T) \times \Omega
\]

and \( \sup_{[0, T] \times \mathbb{R}^3} (|H(t, x)| + |H^{-1}(t, x)|) \leq c_D, \) \hspace{1cm} \text{(23)}

for some \( c_D > 0 \). The deformation gradient is then written as \( F = \nabla (y_\theta \circ y) = H \nabla y \).

The state space \( Z := Z_r \) for the transformation strain \( C_{\text{tr}} \) is defined in coordinate with the exponent \( r \) as follows.

\[
Z_r := \{ C_{\text{tr}} \in L^\infty(\Omega, \mathbb{R}^{3 \times 3}) \cap W^{1, r}(\Omega, \mathbb{R}^{3 \times 3}) \mid C_{\text{tr}} \in \text{SL}(3)_{\text{sym+}} \text{ a.e. in } \Omega \} \text{ for } r > 1,
\]

\[
Z_1 := \{ C_{\text{tr}} \in L^\infty(\Omega, \mathbb{R}^{3 \times 3}) \cap BV(\Omega, \mathbb{R}^{3 \times 3}) \mid C_{\text{tr}} \in \text{SL}(3)_{\text{sym+}} \text{ a.e. in } \Omega \} \text{ for } r = 1,
\]

so that the nonlocal term \( V_r(C_{\text{tr}}) \) is finite for all \( C_{\text{tr}} \in Z_r \). The space \( Y \) is endowed with its weak topology whereas \( Z_r \) has the weak topology of \( W^{1, r}(\Omega; \mathbb{R}^{3 \times 3}) \) for \( r > 1 \) and the weak star topology of \( BV(\Omega; \mathbb{R}^{3 \times 3}) \) for \( r = 1 \). Eventually, the full state space is defined as \( Q := Y \times Z \).

Note that the regularizing term \( V \) in combination with the constraint from \( I_\epsilon_L \) in the energy \( E \) induces strong compactness of the energy sublevels in \( L^p \), namely

\[
\forall (t, y) \in [0, T] \times Y : \quad \{ E(t, y, \cdot) < c \} \subset \subset L^p(\Omega, \mathbb{R}^{3 \times 3}) \quad \forall p \geq 1. \quad \text{(24)}
\]

We define the dissipation \( D \) as follows. Let \( R \) be the dissipation potential defined from \( (10) \). We construct the dissipation metric \( D : \text{SL}(3)_{\text{sym+}} \times \text{SL}(3)_{\text{sym+}} \to [0, +\infty] \) associated to \( R \) as

\[
D(C_0, C_1) := \inf \left\{ \int_0^1 R(\dot{C}, \dot{C}) \, dt \, \bigg| \, C \in C^1([0, 1], \text{SL}(3)_{\text{sym+}}), C(0) = C_0, C(1) = C_1 \right\}.
\]

In particular, the function \( D \) defines a distance and can be explicitly evaluated \([49, 45, 34]\) as

\[
D(C_0, C_1) = \frac{\rho}{2} \left| \log \left( \frac{C_0^{-1/2} C_1 C_0^{-1/2}}{C_1} \right) \right| \quad \forall C_0, C_1 \in \text{SL}(3)_{\text{sym+}}. \quad \text{(25)}
\]

It can be proved from \( (25) \) that \( D \) is locally Lipschitz continuous on \( \text{SL}(3)_{\text{sym+}} \) and that it admits the lower and upper bounds \([34]\)

\[
e^{-1}(|C_0| + |C_1|)^{-1} |C_0 - C_1| \leq D(C_0, C_1) \leq c(|C_0| + |C_1|). \quad \text{(26)}
\]

The dissipation is then defined as

\[
D(C_0, C_1) := \int_\Omega D(C_0(x), C_1(x)) \, dx.
\]

We collect in the following Lemma some properties, useful both in the existence and in the linearization theory.

**Lemma 3.1** (Dissipation properties).

i) \( D \) is a distance.

ii) \( D \) is weakly continuous on bounded sets.

iii) Let \( C_n \) be uniformly bounded with \( D(C, C_n) \to 0 \). Then, \( C_n \to C \) in \( Z \).
In particular, it implies that $\psi$ polyconvexity and plays an important role in finite-deformation theories \[11, 12\].

Above. Assume polyconvexity and coercivity of the local Lipschitz continuity of $D$ tensor $(\text{Existence of quasistatic evolutions})$ Existence.

We conclude by observing via the triangle inequality that $|D(C_n, C) - D(C, \hat{C})| \leq D(C_n, C) + D(\hat{C}, \hat{C}) \to 0$.

Ad iii): owing to the lower bound (26), the convergence $D(C, C_n) \to 0$ implies $C_n \to C$ in $L^1$ and the assertion follows as bounded sets in $Z$ are relatively sequential compact. \hfill $\square$

3.2. Existence. We are now ready to state and prove our existence result.

**Theorem 3.2 (Existence of quasistatic evolutions).** Let $(Q, E, D)$ be defined as above. Assume polyconvexity and coercivity of $\psi$, namely

$$
\psi^I(F_e) = \Psi(F_e, \text{cof}F_e, \det F_e) \quad \forall F_e \in GL_+(3)
$$

for some $\Psi : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}_+ \to \mathbb{R}$ convex, \hfill (27)

$$
\psi^I(F_e) \geq c_e |F_e|^{q_y} - \frac{1}{c_e} 
$$

for some $c_e > 0$, $q_y > 3$ \hfill (28)

Moreover, assume the controllability of the the Kirchhoff stress

$$
|\partial_F \psi^I(F_e) F_e^T| \leq c_K (1 + \psi^I(F_e)) \quad \forall F_e \in GL_+(3)
$$

where $c_K > 0$. Finally, let $\theta \in C^1([0, T], \mathbb{R})$, $\ell \in C^1([0, T]; (W_1^{1,q_y}(\Omega; \mathbb{R}^3))^*)$, and $y_D$ fulfill (23). Given the initial stable state $(y_0, C_{tr,0}) \in S(0)$, there exists an energetic solution corresponding to $(Q, E, D)$ starting from $(y_0, C_{tr,0})$. More precisely, for all partitions $\{0 = t_0^k < t_1^k < \ldots < t_N^k = T\}$ with time step $\tau^k = \max(t_{i+1}^k - t_i^k) - 1$ the incremental minimization problems

$$(y_i, C_{tr,i}) = \text{Argmin} \{ E(t_i^k, y_i, C_{tr}) + D(C_{tr,i-1}, C_{tr,i}) \mid (y_i, C_{tr}) \in Q \}
$$

for $i = 1, \ldots, N^k$ \hfill (30)

admit a solution $\{(y_0, C_{tr,0}), (y_1^k, C_{tr,1}), \ldots, (y_N^k, C_{tr,N^k})\}$ and, as $\tau^k \to 0$, the corresponding piecewise backward-constant interpolants $t \mapsto (y^k(t), C_{tr}^k(t))$ on the partition admit a not relabeled subsequence such that, for all $t \in [0, T]$,

$$
(y^k(t), C_{tr}^k(t)) \to (y(t), C_{tr}(t)), \\
\text{Diss}_{\mathcal{D},[0,t]}(C_{tr}^k) \to \text{Diss}_{\mathcal{D},[0,t]}(C_{tr}), \\
E(t, y^k(t), C_{tr}^k(t)) \to E(t, y(t), C_{tr}(t)),
$$

and $\partial_t E(\cdot, y^k(\cdot), C_{tr}^k(\cdot)) \to \partial_t E(\cdot, y(t), C_{tr}(\cdot))$ in $L^1(0,T)$ where $(y, C_{tr})$ in an energetic solution corresponding to $(Q, E, D)$.

Assumption (29) expresses the controllability of the so called Kirchhoff stress tensor $\partial_F \psi^I(F_e) F_e^T$ by means of the energy. This condition is compatible with polyconvexity and plays an important role in finite-deformation theories \[11, 12\]. In particular, it implies that $\psi^I$ has polynomial growth \[12, \text{Prop. 2.7}\].
The proof of Theorem 3.2 follows the general existence theory for the rate-independent systems [51] and, in particular, [26, 45] (see also [35]). Therefore, we shall not provide here a full argument, but rather comment on some specific detail. The classical strategy of the proof consists in the construction of the time-discrete approximate solutions mentioned in the statement and in a limiting procedure, combined with the check that the limit trajectory is actually an energetic solution. Accordingly, we split the proof of Theorem 3.2 into the next two subsections.

3.3. Time-discrete approximate solutions. We start by proving that, for any partition of the time interval, the incremental minimization problems in (30) admit a solution. This follows via the Direct Method by proving lower semicontinuity and coercivity of $E(t_i, \cdot) + D(C_{tr, i-1}, \cdot)$. Lemma 3.1 provides the required lower semicontinuity of the dissipation potential. The following three Lemmas address the energy instead.

**Lemma 3.3** (Coercivity of the energy). Under the assumptions of Theorem 3.2, the energy $E$ is coercive in the following sense

$$E(t, y, C_{tr}) \geq c\|y\|_{W^{1, q}}^q + c\|C_{tr}\|_{L^\infty} + c\mathcal{V}_r(C_{tr}) - \frac{1}{c}. \quad (31)$$

**Proof.** We may assume with no loss of generality that $|C_{tr}| \leq \epsilon_L$ almost everywhere. Letting $\bar{y}(t, x) := y_D(t, y(t, x))$, from the coercivity assumption (28) we deduce that

$$\frac{1}{c_1} \psi_c(F_c) + \frac{1}{c_2} \geq |\nabla \bar{y} C_{tr}^{-1/2}|^{q_v} \geq \left(\frac{|\nabla \bar{y}|}{|C_{tr}^{1/2}|}\right)^{q_v} \geq 3^{-1/2} \left(|\nabla \bar{y}|^{q_v}/|C_{tr}|^{q_v/2}\right) \geq c |\nabla \bar{y}|^{q_v} = c |\nabla \bar{y} y|^{q_v} \geq c (|\nabla y|/|H^{-1}|)^{q_v} \geq c |\nabla y|^{q_v}.$$ 

Since $\psi_{tr}$ is nonnegative, we obtain

$$\psi(\nabla \bar{y}^\top \nabla \bar{y}, C_{tr}, \theta(t)) \geq c |\nabla \bar{y}|^{q_v}.$$ 

Moreover, $|(f, y)| \leq \|f\|_{W^{1, q_v}}\|y\|_{W^{1, q_v}}$. Hence, by virtue of the Poincaré inequality and the boundedness of $C_{tr}$, we obtain (31). \qed

In order to check the lower semicontinuity of the energy, the following Lemma 3.4 on the convergence of minors is needed. According to assumption (27), we write $\psi_{tr}(H(t)\nabla y C_{tr}^{-1/2}) = \Psi(M(H(t)\nabla y C_{tr}^{-1/2}))$. The next Lemma follows as in [48, Prop. 5.1].

**Lemma 3.4** (Convergence of minors). Let $y_k \rightharpoonup y$ in $W^{1, q_v}(\Omega; \mathbb{R}^3)$ and $P_k \rightarrow P$ in $L^p(\Omega; \text{SL}(3))$ with

$$q_y > 3, \quad \frac{1}{q_y} + \frac{2}{p} \leq 1. \quad (32)$$

Let $H_k = H(t, y_k)$ and $\nabla \bar{y}_k = H_k \nabla y_k$. Then,

$$M(\nabla \bar{y}_k P_k^{-1}) \rightarrow M(\nabla \bar{y} P^{-1}) \text{ in } L^1(\Omega; \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}).$$

**Proof.** Since $\det P_k = 1$, we have

$$M(\nabla \bar{y}_k P_k^{-1}) = \left(\nabla \bar{y}_k (\text{cof} P_k)^\top, \text{cof}(\nabla \bar{y}_k) P_k^\top, \det(\nabla \bar{y}_k)\right). \quad (33)$$
From the compact embedding $W^{1,q}(\Omega; \mathbb{R}^3) \hookrightarrow C(\Omega; \mathbb{R}^3)$ we obtain $y_k \rightharpoonup y$ in $L^p(\Omega; \mathbb{R}^3)$ for some $p \geq 1$ and, by (23), we conclude that $H_k \rightarrow H = H(t, y)$ in $L^\infty(\Omega; \mathbb{R}^3)$ as well. Therefore

$$\nabla \bar{y}_k \rightharpoonup \nabla \bar{y} \text{ on } L^q(\Omega; \mathbb{R}^3).$$

(34)

In order to establish the desired convergences we will use the basic fact

$$\frac{1}{\rho} + \frac{1}{\sigma} \leq 1, \quad A_k \overset{L^p}{\rightharpoonup} A, \quad B_k \overset{L^p}{\rightharpoonup} B \Rightarrow A_k B_k \overset{L^1}{\rightharpoonup} AB,$$

applied to the three minors in (33). The classical weak continuity of the minors of the gradient [10] for $q > 3$ and (34) yields

$$\text{cof}(\nabla \bar{y}_k) \overset{L^{q_y/2}}{\rightharpoonup} \text{cof}(\nabla \bar{y}), \quad \det(\nabla \bar{y}_k) \overset{L^{q_y/3}}{\rightharpoonup} \det(\nabla \bar{y}).$$

Moreover, we clearly have that

$$P_k^{T} \overset{L^p}{\rightharpoonup} P^{T}, \quad \text{cof} P_k \overset{L^{q_y/2}}{\rightharpoonup} \text{cof} P.$$

The assertion hence follows upon checking the following conditions on the indexes

$q_y > 3, \quad \frac{1}{q_y} + \frac{2}{p} \leq 1, \quad \frac{2}{q_y} + \frac{1}{p} \leq 1.$

The first two are (32) and the last one is a direct consequence of these.

We can now ready to check the weak lower semicontinuity of the energy.

**Lemma 3.5** (Lower semicontinuity of the energy). *Under the assumptions of Theorem 3.2 the energy $E$ is weakly lower semicontinuous.*

**Proof.** Let $(y_k, C_{tr,k}) \rightharpoonup (y, C_{tr})$ in $Y \times Z$. The term $C_{tr} \mapsto \mathcal{V}_t(C_{tr})$ is lower semicontinuous for all $r \geq 1$. By the compact embedding (24) we have $C_{tr,k} \rightarrow C_{tr}$ in $L^p$, for all $p \geq 1$. Therefore, one can extract a subsequence $(C_{tr,n_k})$ converging pointwise almost everywhere to $C_{tr}$. By the lower semicontinuity of the nonnegative function $C_{tr} \mapsto \psi_{el}(C_{tr}, \theta)$ and the Fatou lemma, one concludes that, along such subsequence

$$\int_{\Omega} \psi_{el}(E_{tr}, \theta(t)) \, dx \leq \liminf_{k} \int_{\Omega} \psi_{el}((E_{tr})_{n_k}, \theta(t)) \, dx.$$

The $L^p$-convergence of $(C_{tr,n_k})$ for any $p \geq 1$ implies the $L^p$ convergence of $(C_{tr,n_k})^{1/2}$. In fact, the constraints $|(C_{tr})_{n_k} - I| \leq \epsilon_L$ and $\det C_{tr,k} = 1$ force the eigenvalues of $(C_{tr})_{n_k}$ to belong to the interval $[(1 + \epsilon_L)^{-2}, (1 + \epsilon_L)]$, entailing the Lipschitz continuity of the square root. Hence, Lemma 3.4 yields

$$\mathcal{M}(\nabla \bar{y}_{n_k}(C_{tr,n_k})^{-1/2}) \rightharpoonup \mathcal{M}(\nabla \bar{y}C_{tr}^{-1/2}) \text{ in } L^1.$$

As $\psi^1(\nabla \bar{y}C_{tr}^{-1/2}) = \Psi(\mathcal{M}(\nabla \bar{y}C_{tr}^{-1/2}))$ with $\Psi$ convex, the lower semicontinuity of the elastic energy term follows. Eventually, the time-dependent linear term is weakly continuous.
3.4. **Convergence of time discretizations.** From the solutions to the incremental minimization problems (30) we construct the corresponding backward piecewise-constant interpolants $(\bar{y}^k, \bar{C}_t^k)$ on the partition. These can be classically proved [47] to have bounded energy and dissipation, independently of the diameter of the partition $\tau$, from the coercivity of the energy, the nondegeneracy of the dissipation, and a control of the power of external actions. This last point deserves a specific lemma, for it slightly departs from the standard theory.

**Lemma 3.6** (Power of external actions). Under the assumptions of Theorem 3.2, given all $(t, q) \in [0, T] \times \mathcal{Q}$ with $\mathcal{E}(t, q) < \infty$ and all $\tilde{t} \in [0, T]$ we have

$$|\partial_t \mathcal{E}(t, q)| \leq c(1 + \mathcal{E}(t, q)), \quad (35)$$

$$|\partial_t \mathcal{E}(t, q) - \partial_t \mathcal{E}(\tilde{t}, q)| \leq \omega(|t - \tilde{t}|)(1 + \mathcal{E}(t, q)), \quad (36)$$

where the modulus of continuity $\omega$ depend only on data.

**Proof.** The power of external actions reads

$$\partial_t \mathcal{E}(t, q) = \int_{\Omega} \partial_t \psi^{\text{el}}(H(t)\nabla_y C_{tr}^{-1/2}) + \beta'(\theta(t))\dot{\theta}(t) \int_{\Omega} |E_{tr}| dx - \langle \dot{\ell}(t), y \rangle, \quad (37)$$

where the first term represents the contribution of the time-dependent Dirichlet boundary condition (21)-(22). By computing its time derivative we obtain

$$\partial_t \psi^{\text{el}}(H(t)\nabla_y C_{tr}^{-1/2}) = \partial_{F_{c}} \psi^{\text{el}}(F_{c}) : (\dot{H} \nabla_y F_{tr}^{-1}) = \partial_{F_{c}} \psi^{\text{el}}(F_{c}) : (\dot{H} H^{-1} F_{c}) =$$

$$(\partial_{F_{c}} \psi^{\text{el}}(F_{c}) F_{c}^T) : \dot{H} H^{-1},$$

Therefore, using hypothesis (23) and (29) one has

$$|\partial_t \psi^{\text{el}}(H(t)\nabla_y C_{tr}^{-1/2})| \leq c(1 + \psi^{\text{el}}(F_{c})).$$

As for the remaining two terms in the right-hand side of (37), by observing that

$$\beta' \circ \theta, \dot{\theta} \in L^\infty(0, T), \quad \dot{\ell} \in C^0([0, T]; (W^{1, 3}_Y(\Omega, \mathbb{R}^3))^*),$$

we conclude

$$|\partial_t \mathcal{E}(t, q)| \leq c \left( \int_{\Omega} (1 + \psi^{\text{el}}(F_{c}) + |C_{tr} - I|) dx + \|y\|_{W^{1, s}} \right) \leq c(1 + \mathcal{E}(t, q)),$$

namely relation (35). The continuity in time $\partial_t \mathcal{E}(\cdot, q)$ follows directly from that of $H$ (see also [45, Thm. 5.3]), $\dot{\theta}(t)$, and $\dot{\ell}$.

Owing to the energy and dissipation bound, the interpolants $(\bar{y}^k, \bar{C}_t^k)$ turn out to admit a not relabeled subsequence which converges to $(y, C)$ weakly in $Y \times Z$. The latter can be proved to be an energetic solution by virtue of the lower semicontinuity of the energy (Lemma 3.5), the continuity of the dissipation (Lemma 3.1.ii) and the continuity of the power of external actions (36). Eventually, convergence of the energies and dissipations can also be checked [51, Thm. 2.1.6, p. 55].

4. **Small-deformation limit for quasistatic evolution.** We now turn our attention to small deformation regime. In particular, we prove that in such regime the finite-strain model reduces to the small-deformation one. This consists in a evolution $\Gamma$-convergence argument [16, 20, 51], implying in particular the convergence of the corresponding energetic solutions. In the static case, the seminal contribution in this direction is [17] where a variational justification of linearization in elasticity
is provided, see also [1] for successive refinements and [56, 57, 68] for extensions of the argument have been presented.

In the inelastic, evolutive setting the corresponding small-deformation-limit technique has been presented in [52] in an abstract setting and then applied in the frame of finite plasticity in [53, 31, 34, 35]. We follow here the argument of [35] by adapting it to the different form of the energy functional. The Souza-Auricchio model is here augmented via a gradient term of the inelastic variables, both in the finite and in the infinitesimal-strain case [4]. As a consequence, we take advantage of a strong convergence notion for the inelastic variable. This was not available in the plasticity model considered in [53], where both the finite-strain model and the small-strain limit had no plastic gradient nor in [35], where the gradient of the inelastic variable was included at finite strains only.

The present small-deformation analysis is restricted to the case \( r = 2 \) in (11) for the quadratic character of the gradient energy term plays a crucial role. Letting \( \varepsilon > 0 \) and \((y, C_{ir}) \in Q \) be given, we introduce the rescaled variables

\[
u = \frac{1}{\varepsilon}(y-\text{id}), \quad z = \frac{1}{2\varepsilon} \log C_{ir}\]

and the rescaled Green-Saint Venant strain

\[
e := \frac{1}{2\varepsilon}((I+\varepsilon\nabla u)^\top (I+\varepsilon\nabla u) - I) = \nabla u^\text{sym} + \frac{\varepsilon}{2} \nabla u^\top \nabla u.
\]

The rescaling of the energy is performed in a way to obtain the density (1) to first order, namely we define the rescaled energy density as

\[
\psi_\varepsilon(e, z, \theta) := \psi_\varepsilon^e(e, z) + \psi_\varepsilon^{\text{in}}(z, \theta),
\]

\[
\psi_\varepsilon^e(e, z) := \frac{1}{\varepsilon^2} \hat{\psi}\varepsilon^e(\exp(-\varepsilon z)(I+2\varepsilon e) \exp(-\varepsilon z)),
\]

\[
\psi_\varepsilon^{\text{in}}(z, \theta) := \frac{1}{8\varepsilon^2} \exp(2\varepsilon z) - |I|_{2\varepsilon}^2 + \frac{\beta(\theta)}{2\varepsilon} \exp(2\varepsilon z) - I + I_{el} \left( \frac{1}{2\varepsilon} \exp(2\varepsilon z) - I \right).
\]

The quadratic scaling of the elastic energy is motivated by the assumption of a particular, the elastic energy is supposed to be twice differentiable at \( I \) and we define the elasticity tensor \( C \) as

\[
C := 4\partial^2_{C_{ir}} \hat{\psi}\varepsilon^e(I) = \partial^2_{F_{ir}} \hat{\psi}\varepsilon^e(I)
\]

so that the symmetries \( C_{ij\ell k} = C_{\ell kij} = C_{ij\ell k} \) hold. In this way the rescaled energy density as a function of the rescaled variables admits the Taylor approximation

\[
\psi_\varepsilon^e(e, z) = \frac{1}{2} |e - z|_C^2 + o(1).
\]

which represents the elastic part of (1). The complete rescaled energy functional reads then

\[
E_\varepsilon(t, u, z) = \int_{\Omega} \psi_\varepsilon(e, z, \theta(t)) \, dx + \int_{\Omega} |\nabla \exp(2\varepsilon z)|^2 \, dx - \langle \ell(t), u \rangle,
\]

where its small-strain counterpart (recall (3)) is

\[
E_0(t, u, z) = \int_{\Omega} \psi_0(e, z, \theta(t)) \, dx + 2\mu \int_{\Omega} |\nabla \exp(2\varepsilon z)|^2 \, dx - \langle \ell(t), u \rangle
\]
We define the rescaled dissipation distance as

$$D_\varepsilon(z_1, z_2) = \int_\Omega D_\varepsilon(z_1, z_2) \, dx$$

where

$$D_\varepsilon(z_1, z_2) := \frac{1}{\varepsilon} D(C_{tr1}, C_{tr2}) = \frac{1}{\varepsilon} D\left(\exp(2\varepsilon z_1), \exp(2\varepsilon z_2)\right).$$

(41)

The linear scaling reflects the 1-homogeneity of the dissipation potential $R$. By exploiting the explicit form of $D$ given in (25) one can compute

$$D_\varepsilon(z_1, z_2) = \frac{\rho}{2\varepsilon} \left| \log \left( \exp(-\varepsilon z_1) \exp(2\varepsilon z_2) \exp(-\varepsilon z_1) \right) \right| = \rho \| z_1 - z_2 \| + o(1),$$

To first order, the $D_\varepsilon$ hence reduces to

$$D_0(z_1, z_2) := R_0(z_1 - z_2) = \rho \| z_1 - z_2 \|.$$

(42)

Assume now to be given $\theta \in C^1([0, T], \ell \in C^1([0, T]; (W^{1, q}_\Omega(\Omega; \mathbb{R}^3))*)$, assume $y_D = id$ (for the sake of simplicity, nonhomogeneous conditions can also be considered at the expense of notational intricacies), and initial values $z_0$, such that $C_{tr0} := \exp(2\varepsilon z_0) \in S(0)$ where $S(t)$ denotes stable states at time $t \in [0, T]$ corresponding to $(Q, E, D)$. Owing to Theorem 3.2 there exists an energetic solution $(y_\varepsilon, C_{tr\varepsilon})$ corresponding to the triple

$$\left( Q, \frac{1}{\varepsilon^2}(E + (\ell(t), \cdot)) - \frac{1}{\varepsilon} (\ell(t), \cdot), \frac{1}{\varepsilon} D \right).$$

By defining $(u_\varepsilon, z_\varepsilon)$ from $(y_\varepsilon, C_{tr\varepsilon})$ via the change of variables (38) we readily find that $(u_\varepsilon, z_\varepsilon)$ is an energetic solution corresponding to $(Q_0, E_\varepsilon, D_\varepsilon)$ where the space $Q_0$ is now chosen to be

$$Q_0 = H^1_D(\Omega; \mathbb{R}^3) \times H^1(\Omega; \mathbb{R}^{3 \times 3}_{dev})$$

by simply extending trivially the functionals. Note that, for all $\varepsilon > 0$, the trajectory $(u_\varepsilon, z_\varepsilon)$ takes values in the linear, $\varepsilon$-independent state space $Q_0$. The weak convergence in $Q_0$ will hence provide the relevant topology for the $\Gamma$-convergence argument. We shall refer to $(u_\varepsilon, z_\varepsilon)$ as finite-strain quasistatic evolutions and denote the corresponding set of stable states at time $t \in [0, T]$ by $S_\varepsilon(t)$.

We are concerned with the convergence of finite-strain quasistatic evolutions $(u_\varepsilon, z_\varepsilon)$ to a solution $(u, z)$ of the small-strain Souza-Auricchio system corresponding to $(Q_0, E_0, D_0)$ where

$$E_0(t, u, z) = \int_\Omega \psi_0(\nabla u^{sym}, z, \theta(t)) \, dx + 2\mu \int_\Omega |\nabla z|^2 \, dx - \langle \ell(t), u \rangle,$$

$$D_0(z, \tilde{z}) = \int_\Omega D_0(z, \tilde{z}) \, dx.$$  

In particular, $(u, z)$ is an energetic solution of the equilibrium system (4)-6 with $r = 2$ along with the homogeneous Neumann condition $(\nabla z) \tilde{\nu} = 0$ and an initial condition for $z$ [36].

We now state our main convergence result.
Theorem 4.1 (Small-deformation limit of the quasistatic evolution). Under the assumptions of Theorem 3.2, let \( \tilde{\psi}^{cl} \) be quadratic at the identity, namely
\[
\forall \delta > 0 \exists \delta > 0 \forall |A| \leq c_\delta : \left| \tilde{\psi}^{cl}(I + 2A) - \frac{1}{2}|A|^2 \right| \leq \delta |A|^2,
\]
and control the distance to rotations, namely
\[
\psi^{cl}(F) \geq c \text{dist}^2(F, \text{SO}(3)) \quad \forall F \in \text{GL}_+(3).
\]
Let \( \mathcal{E}_\varepsilon, \mathcal{E}_0, \mathcal{D}_\varepsilon, \) and \( \mathcal{D}_0 \) be defined by (39), (40), (41), and (42), respectively. Let \( (u_\varepsilon, z_\varepsilon) \) be finite-transformation quasistatic evolutions starting from well-prepared initial data \( (u_{0\varepsilon}, z_{0\varepsilon}) \in \mathcal{S}_\varepsilon(0) \), namely,
\[
(u_{0\varepsilon}, z_{0\varepsilon}) \to (u_0, z_0) \quad \text{in} \quad Q_0 \quad \text{and} \quad \mathcal{E}_\varepsilon(0, u_{0\varepsilon}, z_{0\varepsilon}) \to \mathcal{E}_0(0, u_0, z_0).
\]
Then, for all \( t \in [0, T] \),
\[
(u_\varepsilon(t), z_\varepsilon(t)) \to (u(t), z(t)) \quad \text{in} \quad Q_0,
\]
\[
\text{Diss}_{\mathcal{D}_\varepsilon,[0,t]}(z_\varepsilon) \to \text{Diss}_{\mathcal{D}_0,[0,t]}(z),
\]
\[
\mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \to \mathcal{E}_0(t, u(t), z(t))
\]
where \( (u, z) \) is an energetic solution of the small-displacement system (4) - (6) starting from \( (u_0, z_0) \).

Note that this theorem delivers a new proof of the former existence result [4, Thm. 6.1]. The condition (44) is automatically satisfied in the small strain limit for most of the physically meaningful energies. In the case of arbitrary deformations, condition (44) restricts the class of admissible polyconvex energies, see e.g. the discussion in [1]. In the case of Ogden materials (8), condition follows (44) by asking \( \gamma_i \geq 2 \) for some \( i \).

In order to prove the theorem we apply the general theory of the evolutionary \( \Gamma \)-convergence established in [52]. This relies on (separate) \( \Gamma \)-lim inf inequalities for energy and dissipation:
\[
\mathcal{E}_0(t, u, z) \leq \inf \left\{ \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \mid (u_\varepsilon, z_\varepsilon) \to (u, z) \text{ in } \mathcal{Y} \times \mathcal{Z} \right\},
\]
\[
\mathcal{D}_0(z, \dot{z}) \leq \inf \left\{ \liminf_{\varepsilon \to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \dot{z}_\varepsilon) \mid (z_\varepsilon, \dot{z}_\varepsilon) \to (z, \dot{z}) \text{ in } \mathcal{Z} \times \mathcal{Z} \right\},
\]
as well as on the existence, for any given \( (\tilde{u}_0, \tilde{z}_0) \) and \( (u_\varepsilon, z_\varepsilon) \to (u_0, z_0) \) with uniformly bounded energies, of a mutual recovery sequence \( (\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \) such that
\[
\limsup_{\varepsilon \to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \dot{z}_\varepsilon) \leq \mathcal{D}_0(z_0, \dot{z}_0),
\]
\[
\limsup_{\varepsilon \to 0} \left( \mathcal{E}_\varepsilon(t, \tilde{u}_\varepsilon, \tilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \right) \leq \left( \mathcal{E}_0(t, \tilde{u}_0, \tilde{z}_0) - \mathcal{E}_0(t, u_0, z_0) \right)
\]
We shall split the proof in lemmas and start by presenting a coercivity result corresponding to a version of Lemma 3.3 with respect to the rescaled variables.

**Lemma 4.2** (Coercivity of the energy 2). Under the assumptions of Theorem 4.1 we have that
\[
\|\nabla u\|_{L^2}^2 + \|\nabla \exp(2\varepsilon z)/\varepsilon\|_{L^2}^2 + \|z\|_{L^\infty} \leq c(1 + \psi_\varepsilon(u, z))
\]
for all \( (u, z) \in Q \).
Therefore, \( \liminf \) condition (44), yielding

\[
\int_{\Omega} \text{dist}^2(\nabla y, \text{SO}(3)) \, dx \leq c\varepsilon^2 (1+W_\varepsilon(u,z)).
\]

In particular, we can argue exactly as in [17].

The next two lemmas deliver the \( \Gamma \)-\( \liminf \) inequalities (45)-(46).

**Lemma 4.3** (\( \Gamma \)-\( \liminf \) inequality for \( E_\varepsilon \)). Under the assumptions of Theorem 4.1 we have that

\[
E_0(t,u,z) \leq \inf \left\{ \liminf_{\varepsilon \to 0} E_\varepsilon(t_\varepsilon,u_\varepsilon,z_\varepsilon) \mid (t_\varepsilon,u_\varepsilon,z_\varepsilon) \rightharpoonup (t,u,z) \text{ in } [0,T] \times Q_0 \right\}.
\]

**Proof.** The \( \liminf \)-inequality for the elastic term and gradient term can be proved as in [35]. The lower semicontinuity of the squared norm of the gradient ensures the respective \( \liminf \). Moreover, the inelastic variable \( z_\varepsilon \) is uniformly bounded in \( L^\infty \) and strongly \( L^2 \)-convergent. This ensures the weak convergence of the argument in the elastic potential.

Some care is to be taken with respect to the convergence of the inelastic part of the energy, namely the three contributions

\[
I^1_\varepsilon := \frac{1}{8\varepsilon^2} \int_{\Omega} |\exp(2\varepsilon z_\varepsilon) - I|^2 \, dx,
\]

\[
I^2_\varepsilon := \frac{\beta(\theta(t_\varepsilon))}{2\varepsilon} \int_{\Omega} |\exp(2\varepsilon z_\varepsilon) - I| \, dx,
\]

\[
I^3_\varepsilon := \int_{\Omega} I_\varepsilon \left( \frac{1}{2\varepsilon} |\exp(2\varepsilon z_\varepsilon) - I| \right) \, dx.
\]

Since \( |z|_{L^\infty} \leq c \), by writing \( \exp(2\varepsilon z_\varepsilon) - I = 2\varepsilon(z_\varepsilon + \varepsilon^2 L_\varepsilon) \), with \( \|L_\varepsilon\|_{L^\infty} \leq c \), and by taking into account the strong convergence \( z_\varepsilon \rightharpoonup z \) in \( L^2 \), one directly establishes

\[
I^1_\varepsilon \to \frac{1}{2} \int_{\Omega} |z|^2 \, dx \quad I^2_\varepsilon \to \beta(\theta(t)) \int_{\Omega} |z| \, dx.
\]

Let us hence consider the constraining term \( I^3_\varepsilon \). In case \( |z| \leq \epsilon_L \) almost everywhere, the \( \liminf \) trivially follows. On the contrary, suppose that

\[
\exists \eta > 0 \quad A_\eta := \{ x \in \Omega \mid |z(x)| \geq \epsilon_L + \eta \}, \quad |A_\eta| > 0
\]

Since \( z_\varepsilon \rightharpoonup z \) in \( L^2 \), there exists \( \varepsilon_0 > 0 \) sufficiently small such that, for any \( \varepsilon < \varepsilon_0 \),

\[
|z_\varepsilon| \geq \epsilon_L + \frac{\eta}{2} \text{ a.e. on } A_\eta, \quad \frac{1}{2\varepsilon} |\exp(2\varepsilon z_\varepsilon) - I| \geq \epsilon_L + \frac{\eta}{4} \text{ a.e. on } A_\eta.
\]

Therefore, \( \liminf_{\varepsilon \to 0} I^3_\varepsilon = +\infty \).

The following lemma is proved in [35]. We record its statement for the sake of completeness.

**Lemma 4.4** (\( \Gamma \)-\( \liminf \) inequality for \( D_\varepsilon \)). Under the assumptions of Theorem 4.1 we have that

\[
D_0(z,\bar{z}) \leq \inf \left\{ \liminf_{\varepsilon \to 0} D_\varepsilon(z_\varepsilon,\bar{z}_\varepsilon) \mid (z_\varepsilon,\bar{z}_\varepsilon) \rightharpoonup (z,\bar{z}) \text{ in } L^2(\Omega; (\mathbb{R}^{3\times3})^2) \right\}.
\]
Having established the \( \Gamma\)-lim inf inequalities (45)-(46), the next ingredient of the evolutive-\( \Gamma\)-convergence argument is the specification of a mutual recovery sequence. This is done within the following lemma.

**Lemma 4.5 (Mutual recovery sequence).** Under the assumptions of Theorem 4.1 let \((u_\varepsilon, z_\varepsilon) \rightharpoonup (u_0, z_0)\) in \(Q_0\) be given with \(\sup_{\varepsilon} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) < \infty\). Moreover, let

\[
(\tilde{u}_0, \tilde{z}_0) = (u_0, z_0) + (\tilde{u}, \tilde{z})
\]

where \((\tilde{u}, \tilde{z}) \in C^\infty_\varepsilon(\Omega; \mathbb{R}^3) \times C^\infty_\varepsilon(\Omega; \mathbb{R}^{3 \times 3})\). Then, there exists \((\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \in Q_0\) such that \((\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \rightharpoonup (\tilde{u}_0, \tilde{z}_0)\) in \(Q_0\) and (47)-(48) hold.

**Proof.** Note that the definition of the recovery sequence differs from that of [35]. We let

\[
\tilde{u}_\varepsilon := u_\varepsilon + \tilde{u} \circ (\text{id} + \varepsilon u_\varepsilon), \quad \tilde{z}_\varepsilon := (1 - \gamma \varepsilon) 2 \tilde{z}_0 \quad \text{for} \quad \gamma = |2\varepsilon_0|.
\]

The definition of \(\tilde{u}_\varepsilon\) can be rewritten in more intuitive terms as

\[
\tilde{y}_\varepsilon = \text{id} + \varepsilon u_\varepsilon + \varepsilon \tilde{u} \circ (\text{id} + \varepsilon u_\varepsilon) = \tilde{y} \circ y_\varepsilon,
\]

where now \(\tilde{y} = \text{id} + \varepsilon \tilde{u}\). In the following we will use the shorthand notations

\[
C_{\text{tr}, \varepsilon} := \exp(2\varepsilon z_\varepsilon), \quad \tilde{C}_{\text{tr}, \varepsilon} := \exp(2\varepsilon \tilde{z}_\varepsilon).
\]

Let us summarize some of the relevant properties of the recovery sequence. By the coercivity Lemma 4.2 and the convergence \(z_\varepsilon \rightharpoonup z_0\) in \(H^1\) we have the bound

\[
\| \nabla z_\varepsilon \|_{L^2}^2 + \| z_\varepsilon \|_{L^\infty} \leq c. \tag{50}
\]

From this, for any \(\alpha \in \mathbb{R}\) (in particular, \(\alpha = 1, -1/2\) are relevant to us), by expanding the exponential \(C^\alpha_{\text{tr}, \varepsilon} = \exp(2\alpha \varepsilon z_\varepsilon)\) we obtain that

\[
C^\alpha_{\text{tr}, \varepsilon} = I + 2\alpha \varepsilon z_\varepsilon + \varepsilon^2 L_\varepsilon \quad \text{with} \quad \| L_\varepsilon \|_{L^\infty} \leq c \tag{51}
\]

Passing to gradients in this expansion at \(\alpha = 1\) and using (50) we deduce

\[
\| \nabla C_{\text{tr}, \varepsilon} - 2 \nabla z_\varepsilon \|_{L^2} \leq c \varepsilon. \tag{52}
\]

We now establish the inequalities (47)-(48) in subsequent steps, by examining separately different contributions.

**Step 1:** The \(\lim \sup\) inequality for the dissipation. The explicit expression from (2.4) yields

\[
\mathcal{D}_\varepsilon(z_\varepsilon, \tilde{z}_\varepsilon) = \frac{\mu}{2\varepsilon} \int_{\Omega} \left| \log \left( \exp(-\varepsilon (1 - \gamma \varepsilon) 2 \tilde{z}_0) \exp(-2\varepsilon z_\varepsilon) \exp(-\varepsilon (1 - \gamma \varepsilon) 2 \tilde{z}_0) \right) \right| \, dx,
\]

so that the convergence \(z_\varepsilon \rightharpoonup z\) in \(L^2\) along with the uniform \(L^\infty\) bound on \(z_\varepsilon\) entail (47).

**Step 2:** The \(\lim \sup\) inequality for the gradient term. We aim at proving that

\[
\lim_{\varepsilon \to 0} \frac{\mu}{2\varepsilon^2} \left( \int_{\Omega} \| \nabla \exp(2\varepsilon \tilde{z}_\varepsilon) \|^2 dx - \int_{\Omega} \| \nabla \exp(2\varepsilon z_\varepsilon) \|^2 dx \right) \leq 2\mu \int_{\Omega} |\nabla \tilde{z}_0|^2 dx - 2\mu \int_{\Omega} |\nabla z_0|^2 dx. \tag{53}
\]

On the one hand, one trivially has

\[
\lim_{\varepsilon \to 0} \frac{\mu}{2\varepsilon^2} \left( \int_{\Omega} |\nabla \exp(2\varepsilon \tilde{z}_\varepsilon) |^2 d x \right) = 2\mu \int_{\Omega} |\nabla \tilde{z}_0|^2 d x
\]
On the other hand, \( \liminf_{\varepsilon \to 0} \| \nabla z_\varepsilon \|_{L^2} \geq \| \nabla z_0 \|_{L^2} \) as \( z_\varepsilon \to z_0 \) in \( H^1 \). By combining it with (52), we obtain

\[
\limsup_{\varepsilon \to 0} \frac{\mu}{2\varepsilon^2} \left( - \int_\Omega |\nabla \exp(2\varepsilon z_\varepsilon)|^2 \, dx \right) \leq -2\mu \int_\Omega |\nabla z_0|^2 \, dx
\]

and inequality (53) follows.

**Step 3: the \( \limsup \) inequality for the elastic energies.** Let us now check that

\[
\limsup_{\varepsilon \to 0} \int_\Omega \psi_{\varepsilon}^\text{el}(\varepsilon \nabla u_\varepsilon) \exp(-\varepsilon z_\varepsilon) \, dx - \int_\Omega \psi_0(\varepsilon \nabla u_\varepsilon) \exp(-\varepsilon z_\varepsilon) \, dx
\]

\[
\leq \frac{1}{2} \int_\Omega |\nabla u_0|_C^\text{sym} - \varepsilon z_0|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla u_0 - z_0|^2 \, dx,
\]

where we have used the short-hand notation \( \psi_{\varepsilon}(A) := \varepsilon^{-2} \psi_{\varepsilon}^\text{el}(\varepsilon + \varepsilon A) \) with

\[
A_\varepsilon := \varepsilon^{-1} [(\varepsilon + \varepsilon \nabla u_\varepsilon) \exp(-\varepsilon z_\varepsilon) - I], \quad \hat{A}_\varepsilon := \varepsilon^{-1} [(\varepsilon + \varepsilon \nabla u_\varepsilon) \exp(-\varepsilon z_0) - I].
\]

One can easily prove that (see [35]),

\[
\nabla u_\varepsilon - \varepsilon u_\varepsilon \overset{L^2}{\to} \nabla \hat{u}, \quad \varepsilon u_\varepsilon \overset{H^1}{\to} \hat{u}_0.
\]

Then, due to the convergence of \( z_\varepsilon \to z \) in \( L^2 \) and the bound (50), by expanding the exponentials in \( A_\varepsilon \) and \( \hat{A}_\varepsilon \), it follows that

\[
\hat{A}_\varepsilon - A_\varepsilon \overset{L^2}{\to} \nabla \hat{u} - \hat{z},
\]

\[
\hat{A}_\varepsilon + A_\varepsilon \overset{L^2}{\to} (\nabla u_0 - z_0) + (\nabla u_0 - z_0).
\]

From here on, one uses the quadratic behavior (43) and follows the argument in [53, 35].

**Step 4: The \( \limsup \) inequality for the transformation energy.** We aim at showing that

\[
\limsup_{\varepsilon \to 0} \left( \int_\Omega \psi_{\varepsilon}^\text{tr}(z_\varepsilon, \theta(t_\varepsilon)) \, dx - \int_\Omega \psi_0^\text{tr}(z_\varepsilon, \theta(t_\varepsilon)) \, dx \right)
\]

\[
\leq \int_\Omega \psi_{\varepsilon}^\text{tr}(z_0, \theta(t)) \, dx - \int_\Omega \psi_0^\text{tr}(z_0, \theta(t)) \, dx
\]

(54)

Convergence (49) entails the \( \limsup \) inequality for the first two contributions in \( \psi_{\varepsilon}^\text{tr} \), namely the hardening and thermomechanical coupling terms. As for the constraint, it suffices to use (51) for \( \varepsilon \) in order to find

\[
\frac{1}{2\varepsilon} |\exp(2\varepsilon z_\varepsilon) - I| = (1 - \gamma\varepsilon)|z_0| + \varepsilon|z_0|^2 + O(\varepsilon^2) = |\varepsilon z_0| \left( 1 - \frac{\varepsilon^2 \gamma}{2} \right) + O(\varepsilon^2).
\]

to check that, since \( |z_0| \leq \epsilon_L \) almost everywhere, one has

\[
I_{\epsilon_L} \left( \frac{1}{2\varepsilon} |\exp(2\varepsilon z_\varepsilon) - I| \right) = 0 \quad \text{a.e.},
\]

at least definitively for \( \varepsilon \to 0 \). This concludes the proof of the \( \limsup \) inequality (54). \( \square \)
Proof of Theorem 4.1. Having established Lemmas 4.3, 4.4, and 4.5, we are in the position of applying the abstract convergence theorem in [52, Thm. 3.1]. Although Lemma 4.5 deals with smooth and compactly supported perturbations only, the full strength of the recovery condition in [52] can be easily deduced by density.

The pointwise strong convergence of \((u_\varepsilon, z_\varepsilon)\) and the convergence of energies and dissipation follow at once from the uniform convexity of the linearized energy \(E_0\) along the same lines as in [53, Cor. 3.8 and Cor. 3.9].

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