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Time-discretization and global solution for a doubly nonlinear Volterra equation

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Abstract

This note addresses the analysis of an abstract doubly nonlinear Volterra equation with a nonsmooth kernel and possibly unbounded and degenerate operators. By exploiting a suitable implicit time-discretization technique, we obtain the existence of a global strong solution. As a by-product, the discrete scheme is proved to be conditionally stable and convergent.

MSC: 45K05; 35K55

Keywords: Doubly nonlinear; Volterra equation; Discretization; Existence

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be a bounded and open set with smooth boundary $\partial\Omega$. Consider the doubly nonlinear equation

$$\big(\beta(u)\big)_t - \operatorname{div}\big(\alpha(\nabla u) + k * \alpha(\nabla u)\big) \ni f \quad \text{in } \Omega, \tag{1.1}$$

where

$$\beta \subset \mathbb{R} \times \mathbb{R} \quad \text{and} \quad \alpha \subset \mathbb{R}^d \times \mathbb{R}^d \quad \text{are maximal monotone graphs}$$

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with $[0,0] \in \alpha$, the kernel

$$k \in BV(0,T), \tag{1.2}$$

and $f : \Omega \times [0,T] \to \mathbb{R}$ are given. Of course the convolution sign has to be understood in the sense of the standard product in $(0,t) \subset (0,T)$ where $T > 0$ denotes some reference final time. In particular, $(a * b)(t) := \int_0^t a(t-s)b(s)\,ds$ whenever it makes sense.

The main purpose of this paper is to provide an existence result for a suitable initial and boundary value problem for (1.1) in case of

$$\text{any graph } \beta \text{ and a suitably coercive and bounded graph } \alpha. \tag{1.3}$$

More precisely, we will ask for a (suitably big) constant $C > 0$ and an exponent $p \in (1,\infty)$ such that

$$\frac{1}{C}|\eta|^p - C \leqslant w \cdot \eta \quad \text{and} \quad |w|^q \leqslant C|\big(1 + |\eta|^p\big) \quad \forall [\eta, w] \in \alpha, \tag{1.4}$$

where q denotes the conjugate exponent of p and $|\cdot|$ is the euclidean norm.

Equations of the form of (1.1) arise as mathematical models of the evolution of materials with thermal memory and generally correspond to the energy balance of the body, possibly including nonlinear operators. In particular, the choice $\beta = id + H$ where id is the identity in $\mathbb{R}$ and H stands for the Heaviside graph, is connected with the Stefan problem in materials with memory (see [23,24] among others) and equations like (1.1) also arise in connection with some models for diffusion in fissured media [42,43]. Our interest in possibly unbounded and degenerate graphs β is however motivated by some recent contributions focusing on the entropy balance of a phase changing material with memory [16–20]. Referring the reader to the above mentioned papers for a thorough discussion of this modeling perspective as well as for some analytical results, we limit ourselves in observing that the evolution of a substance with thermal memory occupying the region Ω may be described (up to some approximation) by the *entropy balance*

$$s_t + \operatorname{div}(Q + k * Q) = f.$$

Here $u > 0$ is the absolute temperature, $s = s(u) \in \mathbb{R}$ is the entropy density, $Q + k * Q$ where $Q = Q(\nabla u) \in \mathbb{R}^d$ is the nonlocal entropy density flux, and f is a given external entropy source. Within the framework of continuum thermo-mechanics [29] the latter quantities are classically related to the free energy density $\psi : (0,+\infty) \to \mathbb{R}$ and the pseudo-potential of dissipation density of the medium $\phi : \mathbb{R}^d \to [0,+\infty)$ where the latter fulfill

$$\psi \text{ is concave and } \phi \text{ is convex.} \tag{1.5}$$

In particular, we have (see [31])

$$s = \partial(-\psi) \quad \text{and} \quad Q = -\partial\phi,$$

where the symbol ∂ stands for the subdifferential in the sense of convex analysis. Taking into account the latter positions and defining $\beta = s$ and $\alpha = -Q$, the latter entropy balance reduces to (1.1). Let us now stress that a fairly usual choice for the free energy density ψ is

$$\psi(u) = -c_0 u \ln u,$$

where c_0 denotes some specific heat density. In this concern, we shall be interested in studying problem (1.1) with the choice $\beta(u) = c_0 \ln u + c_0$ which is both unbounded and degenerate (see (1.3)). However, let us mention that, in the spirit of (1.3), our analysis allows full generality with respect to the choice of ψ, as long as (1.5) is fulfilled. On the other hand, some requirements on the possible growth and nondegeneracy of the pseudo-potential of dissipation density have to be accomplished (see (1.4)). In addition to (1.5), one should notice that the pseudo-potential of dissipation density is usually asked to attain its minimum in 0. Hence, the assumption $[0, 0] \in \alpha$ (which will however not be exploited in the subsequent analysis) is fully justified in connection with the latter modeling perspective.

Instead of focusing on the above introduced concrete equation, we will look for some more generality and address a suitable abstract version of (1.1). To this aim, let V be a reflexive Banach space and denote by V^* its dual. Moreover, let H be a Hilbert space such that $V \subset H$, and $A : V \to V^*$ and $B : H \to H$ be two possibly multivalued maximal monotone operators. The present analysis is concerned with the following abstract Cauchy problem

$$(Bu)' + Au + k * Au \ni f \quad (Bu)(0) \ni v^0, \tag{1.6}$$

where the prime denotes the derivative with respect to time, the equation is fulfilled in V^* almost everywhere with respect to time, and f and v^0 are given data.

The present contribution proves that problem (1.6) admits at least a solution whenever

$$B \text{ is a subdifferential,} \quad A \text{ is suitably coercive and bounded,} \tag{1.7}$$

and some compatibility condition is fulfilled. Indeed, (1.7) corresponds in the abstract setting to the former (1.3) and the above stated coercivity and boundedness will be strictly related to (1.4) (see (2.1)). The full generality of β in (1.3) has to be reconsidered here explicitly by stressing that

$$B \text{ is allowed to be unbounded and degenerate.}$$

Let us stress that some coercivity assumption on A seems mandatory since we aim to allow for strong degeneracy of B. In fact, our result includes the (noninteresting) case $B = 0$. In the latter situation, by letting V be a finite-dimensional space and A a subdifferential, it is easy to check that problem (1.6) has a solution for every f and $v^0 = 0$ if and only if A is coercive. Hence, some coerciveness assumption for A seems mandatory.

Let us now comment on the relation between (1.1) and problem (1.6). We shall start, for the sake of simplicity, from the case of homogeneous Dirichlet boundary conditions (other choices are of course admissible, see below) and choose (the reader is referred to [1] for definitions and properties of Sobolev spaces) $H = L^2(\Omega)$, $V = W_0^{1,p}(\Omega)$, $B : H \to H$, and $A : V \to V^*$ as

$$(Bu)(x) := \beta\big(u(x)\big) \quad \text{for a.e. } x \in \Omega \quad \text{and} \tag{1.8}$$

$$\forall w \in A(u) \quad (w, v) := \int_\Omega z \cdot \nabla v \quad \forall v \in V \text{ and some } z \in \alpha(\nabla u) \quad \text{a.e. in } \Omega. \tag{1.9}$$

The latter position makes sense owing to (1.4) which entails, in particular, (1.7) in the present case. Hence, relation (1.6) stems as a suitable abstract formulation of our original problem (1.1).

As mentioned above, in addition to (1.7), a suitable compatibility condition between A and B has to be imposed. Let us firstly comment the concrete case of relation (1.1). By choosing $\varepsilon \in (0, 1)$ and denoting by $\beta_\varepsilon := (id - (id + \varepsilon\beta)^{-1})/\varepsilon$ the standard Yosida approximation of β at level ε (see [21, p. 28]), one readily checks that

$$\int_\Omega \nabla\big(\beta_\varepsilon(u)\big) \cdot z \geqslant 0 \quad \forall u \in W_0^{1,p}(\Omega) \text{ and } z \in \alpha(\nabla u) \quad \text{a.e. in } \Omega. \tag{1.10}$$

Whenever (1.1) is complemented with boundary conditions such that

$$\int_{\partial\Omega} \beta_\varepsilon(u) z \cdot n \leqslant 0 \quad \text{for } z \in \alpha(\nabla u) \quad \text{a.e. in } \Omega, \tag{1.11}$$

and for admissible u (here n is the outward unit normal to $\partial\Omega$), the latter straightforward compatibility between the nonlinearities of (1.1) can be suitably reformulated within our abstract framework as

$$\big(B_\varepsilon(u), w\big) \geqslant 0 \quad \forall u \in V,\ w \in A(u) \cap H, \tag{1.12}$$

where now B_ε is the Yosida approximation of B at level ε and we used a standard notation for the scalar product in H. Of course, assumption (1.12) is not new and has to be traced back to Brezis and Pazy [22]. The reader may find an extensive discussion in the monograph [21].

As stated above, rather than restricting ourselves solely to Dirichlet homogeneous data, we are in the position of allowing some more general boundary conditions as well. The two issues that need to be checked in this direction are from the one hand the coercivity of the related operator A and from the other hand the validity of the side condition (1.11). We shall not go into details but mention that we could consider nonhomogeneous Dirichlet conditions with suitable constant data, mixed nonhomogeneous Dirichlet–Neumann conditions, and so-called third-type boundary conditions as well. On the other hand, homogeneous Neumann conditions are excluded since the coercivity of A would be lost (and one is forced to require B to be nondegenerate, see Remark 2.2 and Section 6 below).

The proof of our existence result relies on the possibility of implementing an effective implicit time-discretization procedure of (1.6). In particular, letting $\tau := T/N$ ($N \in \mathbb{N}$) denote the time-step and $\{k_i\}_{i=1}^N \in \mathbb{R}^N$, and $\{Au_i\}_{i=1}^N \in V^*$ be approximations of k and Au, respectively, we replace $k * Au$ by the quantities

$$\tau \sum_{j=1}^{i} k_{i-j+1} Au_j, \quad i = 1, \ldots, N.$$

This choice is especially well-suited for the aim of studying Volterra equations of convolution type (see also [48]). Indeed, it entails a useful discrete Young inequality (see Lemma 3.2 below) which turns out to be crucial in order to prove the conditional stability of the time-discretization scheme. Moreover, the above introduced discrete approximation of the convolution product $k * Au$ converges to its continuous counterpart as time-step τ tends to zero (see Corollary 3.4 below). In particular, the existence of a solution to (1.6) follows from the convergence of the approximation method.

A remarkable drawback of the above described time-discretization technique relies in the possibility of considering convolution kernels of bounded variation (see (1.2)). In particular, no continuity is assumed on k. We shall stress that the latter regularity requirement is especially justified within the applicative framework of materials with thermal memory. In fact, memory kernels may be surely expected to be nonnegative, nonincreasing, and bounded. We will however complement our existence theory with some comment on the possibility of considering kernels with unbounded variation. In particular, by exploiting our results in the case (1.2) and restricting our assumptions on the operators A and B, we will obtain some existence result for $k \in L^1(0,T)$ as well.

Let us now try to give some brief comment on the current literature for nonlinear Volterra equations of the type of (1.6). Of course the local-in-time case $k = 0$ has been deeply studied and is beyond our purposes to try to give here a survey of the current quite extensive literature. We limit ourselves in mentioning the classical references of Grange and Mignot [32], Barbu [13], Di Benedetto and Showalter [27], Alt and Luckhaus [6], and Bernis [15]. In connection with some possibly unbounded operators the reader may refer to Barbu [13], Hokkanen [34–36], Aizicovici and Hokkanen [4,5], Maitre and Witomski [39], and Gajewski and Skrypnik [30], among many others.

We now turn to the nonlocal case $k \neq 0$. We shall start by quoting a number of results for the situation of V being a Hilbert space, $k \in W^{1,1}(0,T)$, and A be linear. In this connection let us refer the reader to Aizicovici, Colli and Grasselli [2] and Barbu et al. [14] where B is assumed to be a subdifferential. By assuming further the operator B to be nondegenerate, Stefanelli [48] addresses a time-discretization of the problem. The latter turns out to be conditionally stable and convergent (the same discretization technique is indeed exploited in this paper in order to deal with the doubly nonlinear problem (1.6)). Moreover, an a priori error estimate is recovered and the analysis in [48] includes the treatment of some compact perturbation of the equation. In connection with the weaker assumption $k \in L^1(0,T)$, one should recall Aizicovici, Colli and Grasselli [3] and Colli and Grasselli [25,26]. Moreover, the situation of $k \in L^1(0,T)$ and of *positive type* [33, Section 20.2] has been considered by Stefanelli in both the Hilbert [49] and the reflexive Banach space case [50].

The case of a nonlinear operator A has recently attracted some attention. In this direction, one should mention Hokkanen [35] where the author devises an abstract theory which applies to (1.6) under some restrictive assumptions (see also [36] for some related results). In fact, B is assumed to be a subdifferential whose corresponding potential is finite in zero, A is asked to be a subdifferential complying with additional compatibility properties, and the kernel is assumed to be smooth, i.e., $k \in W^{2,1}(0,T)$ and such that $k(0) > 0$.

As for possibly less regular kernels, we refer the reader to Aizicovici, Colli and Grasselli [3] where the authors face the doubly nonlinear problem

$$(Bu)' + Au + k * Lu \in f, \tag{1.13}$$

where B is a linearly bounded and nondegenerate operator, A is a coercive and bounded subdifferential, $k \in W^{1,1}(0,T)$, and $L : V \to V^*$ is linear. Moreover, either B is a subdifferential or A decomposes into the sum of a linear and a compact operator. Finally, the doubly nonlinear situation for $k \in L^1(0,T)$ and of *positive type* is also addressed in [49,50] where the author focuses in particular on (1.13) with A coercive and linearly bounded, $L : V \to V^*$ linear continuous, positive, and self-adjoint, and B is suitably bounded in some intermediate space.

This is the plan of the paper. We will recall our assumptions and state the main existence result in Section 2. Some details in the direction of an effective implicit time-discretization procedure are given in Section 3. In particular, we recall a discrete convolution technique from [48] and address a useful discrete version of Young's theorem as well as some discrete resolvent theory. Next, some a priori estimates on the discrete solutions are obtained in Section 4, while Section 5 describes the passage to the limit in the approximation and concludes the existence proof. Finally, some existence result for the case $k \in L^1(0,T)$ is given in Section 6.

2. Main results

We start by listing our assumptions on data.

(A0) Let H be a real Hilbert space and V be a real reflexive Banach space densely and compactly embedded in H. Moreover, let $p, q > 1$ such that $1/p + 1/q = 1$.

The space H is identified with its dual H^*, whence $V \subset H \subset V^*$ with dense and compact embeddings. The symbol $(\cdot,\cdot)$ denotes both the duality pairing between V^* and V and the inner product in H. We let $\|\cdot\|, |\cdot|$, and $\|\cdot\|_*$ denote the norms in V, H, and V^*, respectively. Moreover, $\|\cdot\|_E$ stands for the norm in the generic normed space E.

The reader is referred to [7] for an extensive discussion on functions of bounded variation. Let us however recall some notation that will be exploited later on. First of all, for any $u \in L^1(0,T)$, let us set

$$\mathrm{Var}(u) := \sup\left\{ \int_0^T u\varphi' : \ \varphi \in C_c^1(0,T), \ \|\varphi\|_{L^\infty(0,T)} \leqslant 1 \right\},$$

and recall that

$$BV(0,T) := \left\{ u \in L^1(0,T): \ \mathrm{Var}(u) < +\infty \right\}.$$

The latter of course turns out to be a Banach space whenever endowed with the norm

$$\|u\|_{BV(0,T)} := \|u\|_{L^1(0,T)} + \mathrm{Var}(u).$$

For all $u \in BV(0,T)$ there exists a (unique) right-continuous function $\tilde{u}$ such that $\tilde{u} = u$ almost everywhere in $(0,T)$ and

$$\mathrm{Var}(u) = \mathrm{Var}(\tilde{u}) = \sup\left\{ \sum_{i=2}^N \left|\tilde{u}(t_i) - \tilde{u}(t_{i-1})\right| \text{ for } 0 < t_1 < \cdots < t_N < T \right\}$$

(see [7, Theorem 3.28, p. 136]). One should notice that $\tilde{u}$ is bounded and can be represented as the difference of two (bounded) monotone functions. In particular, $\tilde{u}$ turns out to admit right (left) limit in zero (T, respectively). Hence, by defining $\tilde{u}(0) := \tilde{u}(0_+)$ and $\tilde{u}(T) := \tilde{u}(T_-)$ with obvious notations, one readily has that

$$\operatorname{Var}(\tilde{u}) = \sup\left\{\sum_{i=1}^{N} \left|\tilde{u}(t_i) - \tilde{u}(t_{i-1})\right| \text{ for } 0 = t_0 < \cdots < t_N = T\right\},$$

as well. Owing to the latter notation we ask for

(A1) $k \in BV(0,T)$.

(A2) $A : V \to V'$ is a maximal monotone operator. Moreover, A is coercive and bounded in the sense that there exist positive constants α, λ_A, and Λ_A such that, for all $u \in V$ and $w \in A(u)$,

$$\alpha\|u\|^p - \lambda_A \leqslant (w,u), \quad \|w\|_*^q \leqslant \Lambda_A\big(1 + \|u\|^p\big). \tag{2.1}$$

(A3) $\psi : H \to (-\infty, +\infty]$ is a convex, proper, and lower semicontinuous function with $D(\psi) \cap V \neq \emptyset$, and $B = \partial\psi$.

(A4) Letting $\varepsilon \in (0,1)$ and B_ε be the Yosida approximation of B (see [21, p. 28]), one has

$$(v,w) \geqslant 0 \quad \forall u \in V,\ w \in A(u) \cap H, \quad v = B_\varepsilon(u). \tag{2.2}$$

(A5) $f \in L^q(0,T;V^*) \cap L^2(0,T;H)$, $v^0 \in D(\psi^*)$.

In the latter ψ^* is the conjugate of ψ [12, formula (2.3), p. 52]. We shall be concerned with the following abstract system:

$$v' + w + k * w = f \quad \text{a.e. in } (0,T), \tag{2.3}$$

$$v \in B(u) \quad \text{a.e. in } (0,T), \tag{2.4}$$

$$w \in A(u) \quad \text{a.e. in } (0,T), \tag{2.5}$$

$$v(0) = v^0. \tag{2.6}$$

Theorem 2.1. *Under assumptions* (A0)–(A5) *there exist* $u \in L^p(0,T;V)$, $v \in W^{1,q}(0,T;V^*) \cap L^\infty(0,T;H)$, *and* $w \in L^q(0,T;V^*)$ *fulfilling* (2.3)–(2.6).

Remark 2.2. The latter existence result may be generalized with little effort in many directions. First of all, our analysis may be readily extended to the situation where the first relation in (2.1) is replaced by the weaker

$$\alpha\|u\|^p - \lambda_A \leqslant (w,u) + \lambda_A^*\big(\psi^*(v) - (v,z)\big),$$

where $u \in V$, $w \in A(u)$, $v = B_\varepsilon(u)$ for $\varepsilon \in (0,1)$, and $\lambda_A^* \geqslant 0$ and $z \in D(\psi) \cap V$ are given. Moreover, whenever some nondegeneracy in B is assumed, one would be able to treat the situation of a weakly coercive operator A as well. This would allow us, for instance, to consider the case of homogeneous Neumann conditions within our applicative framework. As an example in this direction, let us refer to (1.8), (1.9) and consider the case of the p-laplacian operator with homogeneous Neumann conditions by letting $\alpha(\eta) = |\eta|^{p-2}\eta$ for all $\eta \in \mathbb{R}^d$, and choosing $V = W^{1,p}(\Omega)$ (we let $p > 2d(d+2)$ for $d \geqslant 3$ in order to have that $V \subset L^2(\Omega)$ compactly). Whenever the graph β is suitably nondegenerate (the case $\beta(r) = |r|^{p-2}r$ for all $r \in \mathbb{R}$ being admissible, for instance) our existence result could be extended to this situation.

Secondly, we shall point out that one could allow suitable compact/linear perturbations of the problem, even of nonlocal type. In particular, whenever B is nondegenerate, we could consider some additional term $G(u)$, where $G : L^2(0,T;H) \to L^2(0,T;H)$ is *causal*, i.e.,

$$\text{if } u_1, u_2 \in L^2(0,T;H),\ t \in (0,T), \quad \text{and} \quad u_1 = u_2 \quad \text{a.e. in } (0,t)$$
$$\text{then } G(u_1) = G(u_2) \quad \text{a.e. in } (0,t),$$

and Lipschitz continuous [48]. An example in this direction is given by the Volterra operator $G(u)(t) := \int_0^t h(t,s)\gamma(u(s))\,ds$ for all $t \in (0,T)$, where $h \in L^1((0,T)^2)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is smooth and linearly bounded.

Remark 2.3. Of course we tailored assumption (A4) to the specific example that we have in mind. Let us however comment that (2.2) could be weakened. For instance, one could ask for a positive constant C_{AB} such that, for all $u \in V$, $w \in A(u) \cap H$, and $v = B_\varepsilon(u)$, one has

$$(v, w) \geqslant -C_{AB}\bigl(1 + |v|^2 + \|u\|^p\bigr).$$

Moreover, we shall refer the reader to [46, Proposition 5.4, p. 199] where some list of equivalent conditions to (2.2) are presented and the reader finds some further discussion in the forthcoming Remark 3.7.

Remark 2.4. It is well known that strong nonuniqueness phenomena may occur for problem (2.3)–(2.6). Let us remark that nonuniqueness is not related to our Banach space setting nor to the nonlocal nature of the problem. No uniqueness is expected even in the case of local in time doubly nonlinear equations. In this concern, we refer the reader to [27, Section 5] where nonuniqueness for u is proved for $V = \mathbb{R}$, A, B subdifferentials and $k = 0$. Of course, whenever A is linear, continuous, and symmetric and the sum $A + B$ is strictly monotone, we easily extend to our nonlocal in time case the former uniqueness result [27, Theorem 4].

Remark 2.5. If the compatibility condition (2.2) fails, the component v of the solution to (2.3)–(2.6) is not to be expected to take values in H. In fact, let V be an infinite-dimensional Hilbert space, choose $A : V \to V^*$ to be the Riesz map, and let $u_0 \in V$ be such that $Au_0 \notin H$ (in particular, $u_0 \neq 0$). Assuming $f = 0$, $k = 0$, and ψ to be the indicator function of the singleton u_0 (i.e., $\psi(u_0) = 0$ and $\psi(u) = +\infty$ for all $u \neq u_0$), we readily check that

$$u(t) = u_0, \qquad v(t) = v^0 - tAu_0, \qquad w(t) = Au_0 \quad \forall t \in (0,T)$$

is the unique solution to (2.3)–(2.6) and $v(t) \notin H$ for all $t > 0$ (it can be easily checked that the compatibility condition (2.2) holds iff $u_0 = 0$).

3. Time-discretization

To the aim of proving our existence results, we shall consider a fully implicit time-discretization of problem (2.3)–(2.6). The latter is based on a discrete convolution procedure that has been originally presented in [48]. The very same approximation technique has been later exploited as the main existence tool in the analysis of [44,45]. As a first step, we recall from [48] some notation and results. Let us start by fixing a uniform partition of the time interval $[0, T]$ by choosing a constant time-step $\tau = T/N$, $N \in \mathbb{N}$.

3.1. Discrete convolution

The forthcoming material is to some extent imported from [48]. The reader shall be referred to [11,38,40,41] and references therein for some detailed discussion on discrete convolution procedures.

Definition 3.1. Let $a = \{a_i\}_{i=1}^N$ be a real vector and let $b = \{b_i\}_{i=1}^N \in E^N$, where E stands for a real linear space. Then, we define the vector $\{(a *_\tau b)_i\}_{i=0}^N \in E^{N+1}$ as

$$(a *_\tau b)_i := \begin{cases} 0 & \text{if } i = 0, \\ \tau \sum_{j=1}^i a_{i-j+1} b_j & \text{if } i = 1, \dots, N. \end{cases} \tag{3.1}$$

Let us list some properties of the latter discrete convolution product. First of all, we readily check that, for all $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N \in \mathbb{R}^N$, $\{c_i\}_{i=1}^N \in E^N$, one has

$$(a *_\tau b) = (b *_\tau a), \qquad \big((a *_\tau b) *_\tau c\big) = \big(a *_\tau (b *_\tau c)\big).$$

In the forthcoming discussion the following notation will be extensively used. Letting $z = \{z_i\}_{i=0}^N$ be a vector, we denote by z_τ and $\bar{z}_\tau$ two functions of the time interval $[0, T]$ which interpolate the values of the vector z piecewise linearly and backward constantly on the partition of diameter τ, respectively. Namely,

$$z_\tau(0) := z_0, \qquad z_\tau(t) := \gamma_i(t) z_i + \big(1 - \gamma_i(t)\big) z_{i-1},$$
$$\bar{z}_\tau(0) := z_0, \qquad \bar{z}_\tau(t) := z_i \quad \text{for } t \in \big((i-1)\tau, i\tau\big], \ i = 1, \dots, N,$$

where

$$\gamma_i(t) := \big(t - (i-1)\tau\big)/\tau \quad \text{for } t \in \big((i-1)\tau, i\tau\big], \ i = 1, \dots, N.$$

Let us also set

$$\delta z_i := \frac{z_i - z_{i-1}}{\tau} \quad \text{for } i = 1, \dots, N. \tag{3.2}$$

Then, of course δz stands for the vector $\{\delta z_i\}_{i=1}^N$. Owing to the previous notation, it is not difficult to check the following equality

$$\overline{(a *_\tau b)}_\tau(t) = (\bar{a}_\tau * \bar{b}_\tau)(i\tau) \quad \text{for } t \in \big((i-1)\tau, i\tau\big], \quad i = 1, \dots, N, \tag{3.3}$$

and that the function

$$\bar{a}_\tau * \bar{b}_\tau \quad \text{is piecewise affine on the time partition.} \tag{3.4}$$

The reader should notice that the above discussion yields, in particular,

$$(a *_\tau b)_\tau = \bar{a}_\tau * \bar{b}_\tau \quad \text{in } [0, T]. \tag{3.5}$$

Moreover, given $\{a_i\}_{i=0}^N \in \mathbb{R}^{N+1}$ and $\{b_i\}_{i=1}^N \in E^N$, we have

$$\begin{aligned} \delta(a *_\tau b)_i &= \sum_{j=1}^{i} a_{i-j+1} b_j - \sum_{j=1}^{i-1} a_{i-j} b_j = a_1 b_i + \sum_{j=1}^{i-1} \tau \delta a_{i-j+1} b_j \\ &= a_1 b_i + (\delta a *_\tau b)_i - \tau \delta a_1 b_i = a_0 b_i + (\delta a *_\tau b)_i \end{aligned} \tag{3.6}$$

for $i = 1, \dots, N$.

Finally, we recall a discrete version of Young's theorem.

Lemma 3.2 *(Discrete Young theorem). Let $\{a_i\}_{i=1}^N \in \mathbb{R}^N$, $\{b_i\}_{i=1}^N \in E^N$, where E denotes a real linear space endowed with the norm $\|\cdot\|_E$. Moreover, let $p, q, r \in [1, \infty]$ such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

along with the standard convention $1/\infty = 0$. Then the following inequality holds

$$\big\| \overline{(a *_\tau b)}_\tau \big\|_{L^r(0,T;E)} \leqslant \|\bar{a}_\tau\|_{L^p(0,T;E)} \|\bar{b}_\tau\|_{L^q(0,T;E)}. \tag{3.7}$$

Proof. This proof follows from adapting to the discrete setting the well-known Young theorem for continuous convolutions. Let us define $\alpha_i := |a_i|$ and $\beta_i := \|b_i\|_E$ for $i = 1, \dots, N$ and focus with no loss of generality on the proof of the following

$$\big\| \overline{(\alpha *_\tau \beta)}_\tau \big\|_{L^r(0,T)} \leqslant \|\bar{\alpha}_\tau\|_{L^p(0,T)} \|\bar{\beta}_\tau\|_{L^q(0,T)}. \tag{3.8}$$

First of all, owing to (3.4) one checks that

$$\|\overline{\alpha *_\tau \beta}\|_{L^\infty(0,T;E)} = \|\bar{\alpha}_\tau * \bar{\beta}_\tau\|_{L^\infty(0,T;E)}$$

and a straightforward application of Young's theorem for continuous convolutions entails (3.8) in the case $r = \infty$.

Let us turn to the case $r < \infty$ (where one has $p, q < \infty$ as well). A straightforward homogeneity argument ensures that (3.8) is equivalent to the following

$$\|\alpha *_\tau \beta\|_r \leqslant \|\alpha\|_p \|\beta\|_q, \tag{3.9}$$

where, for all vectors $\gamma = \{\gamma_i\}_{i=1}^N \in \mathbb{R}^N$, we have set

$$\|\gamma\|_s := \begin{cases} \left(\sum_{i=1}^N |\gamma_i|^s\right)^{1/s} & \text{if } s \in [1, \infty), \\ \max_{1 \leqslant i \leqslant N} |\gamma_i| & \text{if } s = \infty. \end{cases}$$

We directly check (3.9) in the case $p = q = r = 1$. This is indeed an easy computation since

$$\|\alpha *_\tau \beta\|_1 = \sum_{i=1}^N \sum_{j=1}^{N-i+1} \alpha_i \beta_j \leqslant \sum_{i,j=1}^N \alpha_i \beta_j = \|\alpha\|_1 \|\beta\|_1. \tag{3.10}$$

In the present proof, we will use the standard notation p' and q' in order to indicate the conjugate exponents of p and q, respectively. The next step will be to check (3.9) for $p = 1$ (equivalently for $q = 1$). In this case, one has that

$$\sum_{j=1}^i \alpha_{i-j+1}\beta_j = \sum_{j=1}^i \left(\alpha_{i-j+1}^{1/q}\beta_j\right)\alpha_{i-j+1}^{1/q'} \leqslant \left(\sum_{j=1}^i \alpha_{i-j+1}\beta_j^q\right)^{1/q} \left(\sum_{j=1}^i \alpha_{i-j+1}\right)^{1/q'}$$

for $i = 1, \ldots, N$. In the latter we used the finite-dimensional Hölder inequality. Hence, taking the q-power and the sum above we get that

$$\begin{aligned} \|\alpha *_\tau \beta\|_q^q &\leqslant \sum_{i=1}^N \left(\sum_{j=1}^i \alpha_{i-j+1}\beta_j^q\right) \|\alpha\|_1^{q/q'} \\ &\leqslant \|\alpha\|_1 \|\beta^q\|_1 \|\alpha\|_1^{q/q'} = \|\alpha\|_1^q \|\beta\|_q^q, \end{aligned}$$

where we used (3.10).

Finally, we turn to the case when $r < \infty$ and $p, q > 1$. One starts by observing that

$$\sum_{1 \leqslant j \leqslant i \leqslant N} \alpha_{i-j+1}^p \beta_j^q = \sum_{j=1}^N \beta_j^q \left(\sum_{i=j}^N \alpha_{i-j+1}^p\right) \leqslant \|\alpha\|_p^p \|\beta\|_q^q. \tag{3.11}$$

Then, since $\alpha_{i-j+1}\beta_j = (\alpha_{i-j+1}^p \beta_j^q)^{1/r} \alpha_{i-j+1}^{1-p/r} \beta_j^{1-q/r}$ and $1/r + 1/p' + 1/q' = 1$ we readily check that, for $i = 1, \ldots, N$,

$$\sum_{j=1}^i \alpha_{i-j+1}\beta_j \leqslant \left(\sum_{j=1}^i \alpha_{i-j+1}^p \beta_j^q\right)^{1/r} \left(\sum_{j=1}^i \alpha_{i-j+1}^{(1-p/r)q'}\right)^{1/q'} \left(\sum_{j=1}^i \beta_j^{(1-q/r)p'}\right)^{1/p'}.$$

Hence, also considering that

$$\left(1 - \frac{p}{r}\right)q' = p, \qquad \left(1 - \frac{q}{r}\right)p' = q, \qquad \frac{pr}{q'} = r - p, \qquad \frac{qr}{p'} = r - q,$$

one computes by means of (3.11) that

$$\|\alpha *_\tau \beta\|_r^r = \sum_{i=1}^N \left(\sum_{j=1}^i \alpha_{i-j+1}\beta_j\right)^r \leqslant \sum_{i=1}^N \left(\left(\sum_{j=1}^i \alpha_{i-j+1}^p \beta_j^q\right)\left(\sum_{j=1}^i \alpha_{i-j+1}^p\right)^{r/q'} \left(\sum_{j=1}^i \beta_j^q\right)^{r/p'}\right)$$
$$\leqslant \sum_{i=1}^N \left(\sum_{j=1}^i \alpha_{i-j+1}^p \beta_j^q\right)\|\alpha\|_p^{r-p}\|\beta\|_q^{r-q} \leqslant \|\alpha\|_p^r \|\beta\|_q^r,$$

and the assertion follows. □

Let us stress that, from now on, the exponents p and q will be considered to be fixed according to assumption (A0).

Finally, we point out an estimate which will play a crucial role in Section 5. The reader can check [48] for an analogous result in a somehow different setting.

Proposition 3.3. *Let $r \in [1, \infty]$, $\{a_i\}_{i=0}^{N+1} \in \mathbb{R}^{N+1}$, and $\{b_i\}_{i=1}^N \in E^N$, where E denotes a real Banach space. Then, we have*

$$\left\|\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\right\|_{L^r(0,T;E)} \leqslant \tau C_r\big(\mathrm{Var}(a_\tau) + |a_0|\big)\|\bar{b}_\tau\|_{L^r(0,T;E)}, \tag{3.12}$$

where $C_r := (1+r)^{-1/r} < 1$ for $r \in [1, \infty)$ and $C_\infty := 1$.

Proof. Let us start by checking (3.12) for $r \in [1, \infty)$. This argument was already sketched in [48, Proposition 4.4] for $r = 1$ and is here reported for the sake of completeness. The left-hand side of (3.12) may be easily controlled by virtue of relations (3.3) and (3.5) as follows

$$\left\|\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\right\|_{L^r(0,T;E)}^r = \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \left\|(\bar{a}_\tau * \bar{b}_\tau)(i\tau) - (\bar{a}_\tau * \bar{b}_\tau)(t)\right\|_E^r dt$$
$$= \frac{1}{1+r}\sum_{i=1}^N \tau^{1+r}\left\|(\bar{a}_\tau * \bar{b}_\tau)'\right\|_E^r = \frac{1}{1+r}\sum_{i=1}^N \tau^{1+r}\left\|\overline{\delta(a * b)}_\tau\right\|_E^r.$$

Hence, by making use of (3.6) and applying Lemma 3.2, we compute that

$$\left\|\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\right\|_{L^r(0,T;E)}^r = \frac{\tau^r}{1+r}\int_0^T \left\|\overline{(\delta a *_\tau b)}_\tau + a_0\bar{b}_\tau\right\|_E^r$$
$$\leqslant \frac{\tau^r}{1+r}\big(\mathrm{Var}(a_\tau) + |a_0|\big)^r \|\bar{b}_\tau\|_{L^r(0,T;E)}^r.$$

As for the case $r = \infty$, once more by Lemma 3.2 we check that

$$\left\|\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\right\|_{L^\infty(0,T;E)} = \tau\left\|(\bar{a}_\tau * \bar{b}_\tau)'\right\|_{L^\infty(0,T;E)} = \tau\left\|\overline{(\delta(a *_\tau b))}_\tau\right\|_{L^\infty(0,T;E)}$$
$$\leqslant \tau\Big(\left\|\overline{(\delta a *_\tau b)}_\tau\right\|_{L^\infty(0,T;E)} + |a_0|\|\bar{b}_\tau\|_{L^\infty(0,T;E)}\Big) \leqslant \tau\big(\mathrm{Var}(a_\tau) + |a_0|\big)\|\bar{b}_\tau\|_{L^\infty(0,T;E)}.$$

Before closing this proof we shall explicitly observe that the constant in (3.12) is sharp as the elementary example $E=\mathbb{R}$, $a_\tau=\bar{b}_\tau=1$ shows. □

Along the same lines above, starting from some $\{a_i\}_{i=1}^N \in \mathbb{R}^{N+1}$, and $\{b_i\}_{i=0}^N \in E^{N+1}$, one could also prove that, for all $r\in[1,\infty]$,

$$\big\|\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\big\|_{L^r(0,T;E)} \leqslant \tau C_r\big(\|b'_\tau\|_{L^1(0,T;E)} + \|b_0\|_E\big)\|\bar{a}_\tau\|_{L^r(0,T)}. \tag{3.13}$$

We shall use the latter result in order to suitably pass to limits within discrete convolution products. Namely, let us state for later reference the following:

Corollary 3.4. *Let $r\in[1,\infty]$ and E be a real reflexive Banach space. If $\bar{a}_\tau\to a$ strongly in $L^1(0,T)$, a_τ are equibounded in $BV(0,T)$, and $\bar{b}_\tau\to b$ weakly star (strongly) in $L^r(0,T;E)$, then $\overline{(a *_\tau b)}_\tau \to a*b$ weakly star (strongly, respectively) in $L^r(0,T;E)$.*

Proof. Let us start by claiming that, within the present framework, the values $a_\tau(0)$ turn out to be bounded in terms of $\|a_\tau\|_{BV(0,T)}$. In particular, $a_\tau(0)$ are equibounded. Then, it suffices to compute

$$\overline{(a *_\tau b)}_\tau - a*b = \big(\overline{(a *_\tau b)}_\tau - \bar{a}_\tau * \bar{b}_\tau\big) + (\bar{a}_\tau - a)*\bar{b}_\tau + a*(\bar{b}_\tau - b),$$

and exploit (3.12) and the assumptions in order to check that the three terms in the above right-hand side weakly star (strongly, respectively) converge to 0 in $L^r(0,T;E)$. □

3.2. Approximation of the kernel

Let us restrict ourselves to the case of a kernel $k:[0,T]\to\mathbb{R}$ such that

$$\operatorname{Var}(k) = \sup\left\{\sum_{i=0}^N \big|k(t_i)-k(t_{i-1})\big| \text{ for } 0=t_0<\cdots<t_N=T\right\}.$$

In fact, owing to the discussion of Section 2, this restriction entails no loss of generality with respect to the proof of Theorem 2.1. To the aim of introducing our approximation of problem (2.3)–(2.6) let us set

$$k_i := k(i\tau) \quad \text{for } i=0,1,\dots,N, \tag{3.14}$$

whence it is a standard matter to verify that (see (A1))

$$\|k-\bar{k}_\tau\|_{L^1(0,T)} \leqslant \tau\operatorname{Var}(k). \tag{3.15}$$

Moreover, we readily check that

$$\|k_\tau\|_{C[0,T]} \leqslant \|k\|_{L^\infty(0,T)} \quad \text{and} \quad \operatorname{Var}(k_\tau)\leqslant\operatorname{Var}(k), \tag{3.16}$$

independently of τ. For notational convenience, we will use the same symbol for the function k and the vector $k=\{k_i\}_{i=0}^N$ whenever the latter is involved in a discrete convolution product.

Let us now move in the direction of a discrete counterpart to the continuous resolvent theory [33] and remark that this discussion was not originally presented in [48]. Namely, we shall look for a vector $\{\rho_i\}_{i=0}^N \in \mathbb{R}^{N+1}$ such that

$$\rho_i + (k *_\tau \rho)_i = k_i \quad \text{for } i = 0, 1, \ldots, N. \tag{3.17}$$

The latter linear system may be solved whenever τ is small enough. Namely, by letting $\rho_0 = k_0 = k(0)$, it is straightforward to check that the remaining $N \times N$ linear system is lower-triangular and its determinant reads $(1 + \tau k_1)^N$. Hence, the latter is solvable whenever we have, for instance,

$$\tau |k_1| \leqslant \frac{1}{2}, \tag{3.18}$$

which, taking into account (A1) and definition (3.14), holds at least for small τ. We shall collect some properties of ρ in the following proposition.

Proposition 3.5. *Let* (A1) *and* (3.18) *hold and* $\{\rho_i\}_{i=0}^N \in \mathbb{R}^{N+1}$ *be defined as above. Then*

$$\rho_\tau \text{ are uniformly bounded in } BV(0,T) \text{ in terms of } \|k\|_{BV(0,T)}, \tag{3.19}$$

$$\rho_\tau \to \rho \quad \text{strongly in } L^1(0,T), \quad \rho \in BV(0,T), \text{ and } \rho + k * \rho = k \quad \text{a.e. in } (0,T). \tag{3.20}$$

Proof. It is a standard matter to check from (3.17) that

$$|\rho_i| \leqslant |k_i| + \tau \sum_{j=1}^{i} |k_{i-j+1}||\rho_j| \quad \text{for } i = 1, \ldots, N.$$

Hence, by exploiting (3.18), applying the discrete Gronwall lemma, i.e.,

$$\text{for all} \quad \{a_i\}_{i=0}^N \in [0,+\infty)^{N+1} \quad \text{and} \quad \{b_i\}_{i=1}^N \in [0,+\infty)^N,$$
$$a_i \leqslant a_0 + \sum_{j=1}^{i-1} b_j a_j \quad \text{for } i = 1, \ldots, N \quad \Longrightarrow \quad a_i \leqslant a_0 \exp\left(\sum_{j=1}^{i-1} b_j\right) \quad \text{for } i = 1, \ldots, N,$$

(see, e.g., [37, Proposition 2.2.1]) and recalling Lemma 3.2 and (3.16) one has that $\|\rho_\tau\|_{C[0,T]}$ is bounded in terms $\|k\|_{L^\infty(0,T)}$ independently of τ as soon as the partition is fine enough. In particular, we have that

$$\|\rho_\tau\|_{C[0,T]} = \sup_{i=1,\ldots,N} |\rho_i| \leqslant 2\|k\|_{L^\infty(0,T)} \exp\bigl(2T\|k\|_{L^\infty(0,T)}\bigr). \tag{3.21}$$

Next, taking (3.6) into account we deduce that

$$\delta\rho_i = \delta k_i - k_0 \rho_i - (\delta k *_\tau \rho)_i \quad \text{for } i = 1, \ldots, N. \tag{3.22}$$

Exploiting again Lemma 3.2 together with (3.6), (3.16), and (3.21) one gets that

$$\begin{aligned}\operatorname{Var}(\rho_\tau) &\leqslant \operatorname{Var}(k_\tau) + \bigl|k(0)\bigr| T \|\rho_\tau\|_{C[0,T]} + \operatorname{Var}(k_\tau)\|\rho_\tau\|_{C[0,T]} \\ &\leqslant \operatorname{Var}(k) + \bigl(\bigl|k(0)\bigr|T + \operatorname{Var}(k)\bigr) 2\|k\|_{L^\infty(0,T)} \exp\bigl(2T\|k\|_{L^\infty(0,T)}\bigr),\end{aligned}$$

and (3.19) follows.

Finally, we readily check that, owing to the compactness theorem in $BV(0,T)$ (see, e.g., [7, Theorem 3.23, p. 132]), for some not relabeled subsequence of diameters τ going to 0, ρ_τ converges strongly in $L^1(0,T)$ to some function $\rho \in L^1(0,T)$. It is easy to prove that also $\overline{\rho}_\tau$ converges strongly to ρ in $L^1(0,T)$ since

$$\begin{aligned}\|\rho_\tau - \overline{\rho}_\tau\|_{L^1(0,T)} &= \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \bigl|\gamma_i(t)\rho_i + \bigl(1-\gamma_i(t)\bigr)\rho_{i-1} - \rho_i\bigr|\,dt \\ &= \frac{\tau}{2}\sum_{i=1}^N |\rho_i - \rho_{i-1}| = \frac{\tau}{2}\operatorname{Var}(\rho_\tau).\end{aligned}$$

Next, we may recast (3.17) in the more compact form

$$\overline{\rho}_\tau + \overline{(k *_\tau \rho)}_\tau = \bar{k}_\tau \quad \text{a.e. in } (0,T).$$

Indeed, also owing to Corollary 3.4, we eventually pass to the limit in the above equation thanks to (3.15) obtaining

$$\rho + k * \rho = k \quad \text{a.e. in } (0,T).$$

Since the latter relation has at most one solution $\rho \in L^1(0,T)$ by means of Gronwall's lemma, the whole sequence ρ_τ converges to ρ. Finally, one readily recovers $\rho \in BV(0,T)$ [7, (3.11), p. 125]. □

Finally, owing to (3.17), we readily check that, given $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N \in E^N$ where E is some real linear space,

$$a_i + (k *_\tau a)_i = b_i \quad \text{for } i = 1,\dots,N \quad \Longleftrightarrow \quad b_i - (\rho *_\tau b)_i = a_i \quad \text{for } i = 1,\dots,N.$$

Namely, assume $a_i + (k *_\tau a)_i = b_i$ for $i = 1,\dots,N$. Then

$$\begin{aligned}b_i - (\rho *_\tau b)_i &= a_i + (k *_\tau a)_i - (\rho *_\tau a)_i - \bigl(\rho *_\tau (k *_\tau a)\bigr)_i \\ &= a_i + \bigl(\bigl(k - \rho - (k *_\tau \rho)\bigr) *_\tau a\bigr)_i = a_i \quad \text{for } i = 1,\dots,N.\end{aligned}$$

Conversely, let $b_i - (\rho *_\tau b)_i = a_i$ for $i = 1,\dots,N$. It is easy to check that

$$\begin{aligned}a_i + (k *_\tau a)_i &= b_i - (\rho *_\tau b)_i + (k *_\tau b)_i - \bigl(k *_\tau (\rho *_\tau b)\bigr)_i \\ &= b_i - \bigl(\bigl(\rho - k + (k *_\tau \rho)\bigr) *_\tau b\bigr)_i = b_i \quad \text{for } i = 1,\dots,N.\end{aligned}$$

Hence, let us conclude that, for all $a, b \in L^1(0,T;E)$, and $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N \in E^N$, we have the following

$$a + k * a = b \quad \text{a.e. in } (0,T) \quad \Longleftrightarrow \quad b - \rho * b = a \quad \text{a.e. in } (0,T), \tag{3.23}$$

$$\bar{a}_\tau + \overline{(k *_\tau a)}_\tau = \bar{b}_\tau \quad \text{a.e. in } (0,T) \quad \Longleftrightarrow \quad \bar{b}_\tau - \overline{(\rho *_\tau b)}_\tau = \bar{a}_\tau \quad \text{a.e. in } (0,T). \tag{3.24}$$

3.3. Discrete scheme

Taking (A5) into account, we will choose $f_i := \tau^{-1} \int_{(i-1)\tau}^{i\tau} f \in H$ for $i = 1, \dots, N$ so that

$$\bar{f}_\tau \to f \quad \text{strongly in } L^q\big(0,T;V^*\big). \tag{3.25}$$

Then, the discrete problem may be formulated as that of finding $\{u_i\}_{i=1}^N \in V^N$, $\{v_i\}_{i=0}^N \in H^{N+1}$, and $\{w_i\}_{i=1}^N \in H^N$ such that

$$\delta v_i + w_i + (k *_\tau w)_i = f_i \quad \text{for } i = 1, \dots, N, \tag{3.26}$$

$$v_i = B_\varepsilon(u_i) \quad \text{for } i = 1, \dots, N, \tag{3.27}$$

$$w_i \in A(u_i) \quad \text{for } i = 1, \dots, N, \tag{3.28}$$

$$v_0 = v^0. \tag{3.29}$$

Hence, let us observe in particular that relations (3.26)–(3.28) take place in H.

We are in the position of proving the following lemma.

Lemma 3.6. *Assuming* (3.18)*, problem* (3.26)–(3.29) *admits a solution.*

Proof. Thanks to the well-known Asplund's result [8,9], let us assume the space V to be equivalently renormed in such a way that V and its dual V^* are strictly convex and let $i : V \to H$ denote the injection and $id : H \to H$ be the identity in H.

By defining $I : i^* \circ id \circ i : V \to V^*$ and exploiting [12, Corollary 1.1, p. 39] it is straightforward to check that the sum $I + A : V \to V^*$ turns out to be an everywhere defined, maximal monotone operator. Moreover, taking into account (A2), we readily check that $I + A$ is coercive and onto [12, Theorem 1.3, p. 40]. In particular, for all $h \in H$ there exist $u \in V$ and $w \in A(u)$ such that

$$Iu + w = i^* h.$$

As a consequence, $w \in H$, the set $D(A_H) := \{u \in V\colon\ A(u) \cap H \neq 0\}$ is nonempty, and the operator $A_H : H \to H$ given by $A_H(u) := A(u) \cap H$ is maximal monotone.

Given now $\nu > 0$, we apply again [12, Corollary 1.1, p. 39] in order to check that $B_\varepsilon + \nu A_H : H \to H$ is maximal monotone. Moreover, again owing to (A2), we have that νA_H is coercive on H since

$$(w, u) \geqslant \alpha \nu \|u\|^p - \nu \lambda_A \quad \forall u \in D(A_H),\ w \in \nu A_H(u).$$

Hence, the sum $B_\varepsilon + \nu A_H$ is onto [12, Theorem 1.3, p. 40] and the equation

$$B_\varepsilon(u) + \big(\tau + \tau^2 k_1\big) A_H(u) \ni h,$$

has a solution $u \in V$ for any $h \in H$ and for all τ satisfying (3.18).

Then, we proceed by induction by observing that, at each level i, problem (3.26)–(3.28) reduces to

$$B_\varepsilon(u_i) + \big(\tau + \tau^2 k_1\big) A(u_i) \ni h_i,$$

where

$$h_i := \begin{cases} \tau f_1 + v^0 & \text{if } i = 1, \\ \tau f_i + v_{i-1} - \tau \sum_{j=1}^{i-1} \tau k_{i-j+1} w_j \in H & \text{if } i = 2, \dots, N, \end{cases}$$

and the assertion follows. □

Owing to the latter lemma, we are entitled to rewrite relations (3.26)–(3.29) in a more compact form as

$$v_\tau' + \overline{w}_\tau + \overline{(k *_\tau w)}_\tau = \bar{f}_\tau \quad \text{a.e. in } (0,T), \tag{3.30}$$

$$\bar{v}_\tau = B_\varepsilon(\bar{u}_\tau) \quad \text{a.e. in } (0,T), \tag{3.31}$$

$$\overline{w}_\tau \in A(\bar{u}_\tau) \quad \text{a.e. in } (0,T), \tag{3.32}$$

$$v_\tau(0) = v^0. \tag{3.33}$$

Let us once more stress that relations (3.30)–(3.33) actually take place in H.

Remark 3.7. We point that, by suitably passing to the limit as ε goes to 0 within assumption (2.2), one readily obtains

$$(v, w) \geqslant 0 \quad \forall u \in V,\ w \in A(u) \cap H,\ v \in B(u). \tag{3.34}$$

Nevertheless, one should consider that (3.34) is indeed weaker then (2.2) as it is shown by the elementary counterexample

$$H := \mathbb{R}, \qquad Au := -1/u \quad \forall u \in D(A) := (-\infty, 0), \quad B := \partial I_{[0,+\infty)}.$$

On the other hand, whenever (3.34) holds and

$$(id + \varepsilon B)^{-1}\big(D(A_H)\big) \subset D(A_H) \quad \text{or } (id + \varepsilon A_H)^{-1}\big(D(B)\big) \subset D(B)$$

for $\varepsilon \in (0, 1)$, one readily exploits the argument of [21, Lemma 4.4, p. 131] and deduces relation (2.2) as well.

4. Estimates

We now establish some estimates on the discrete solutions whose existence is proved above. To this aim let the symbol $C(\cdot)$ denote any constant with explicit dependencies. In particular, C will be independent of both ε and τ. Of course the value of C may vary from line to line and even in the same chain of inequalities.

4.1. First estimate

Let $z \in V$ denote any (fixed) element of the domain of ψ (such an element exists by means of (A3)) and denote by ψ_ε the Yosida approximation of ψ. Moreover, let ψ_ε^* be the conjugate of ψ_ε. Making use of the fact that $\psi_\varepsilon(u) \leqslant \psi(u)$ for all $u \in H$ [21, p. 39] we readily deduce that $\psi_\varepsilon^*(u) \geqslant \psi^*(u)$ for all $u \in H$ and

$$\psi_\varepsilon^*(v) - (v, z) \geqslant \psi^*(v) - (v, z) \geqslant -\psi(z) > -\infty \quad \forall v \in H. \tag{4.1}$$

Moreover, we readily check that

$$\psi_\varepsilon^*(v) = \psi^*(v) + \frac{\varepsilon}{2}|v|^2 \quad \forall v \in H. \tag{4.2}$$

First of all, let us test relation (3.26) by $\tau(u_i - z)$ in order to get that

$$(v_i - v_{i-1}, u_i - z) + \tau(w_i, u_i) = \tau(f_i, u_i - z) - \tau\big((k *_\tau w)_i, u_i - z\big) + \tau(w_i, z). \tag{4.3}$$

Moreover, owing to (3.27), one checks that

$$v_i = B_\varepsilon(u_i) = \partial\psi_\varepsilon(u_i) \quad \Longleftrightarrow \quad u_i \in \partial\psi_\varepsilon^*(v_i) \quad \Longrightarrow \quad \psi_\varepsilon^*(v_i) - \psi_\varepsilon^*(v_{i-1}) \leqslant (v_i - v_{i-1}, u_i). \tag{4.4}$$

Next, combining (4.3) and (4.4) and taking advantage of (A2), (A3), one can compute that

$$\begin{aligned}
&\psi_\varepsilon^*(v_i) - \psi_\varepsilon^*(v_{i-1}) - (v_i - v_{i-1}, z) + \alpha\tau\|u_i\|^p \\
&\quad \leqslant \tau\|f_i\|_*\big(\|u_i\| + \|z\|\big) + \tau\big\|(k *_\tau w)_i\big\|_*\big(\|u_i\| + \|z\|\big) + \tau\|w_i\|_*\|z\| + \tau\lambda_A \\
&\quad \leqslant \frac{\alpha\tau}{4}\|u_i\|^p + \tau C(\alpha, p)\big(\|f_i\|_*^q + \big\|(k *_\tau w)_i\big\|_*^q\big) \\
&\qquad + \tau\|z\|^p + \tau C(p, \Lambda_A)\big(1 + \|u_i\|^{p/q}\big)\|z\| + \tau\lambda_A \\
&\quad \leqslant \frac{\alpha\tau}{2}\|u_i\|^p + \tau C(\alpha, p, \lambda_A, \Lambda_A)\big(1 + \|f_i\|_*^q + \|z\|^p\big) + \tau C(\alpha, p)\|\bar{k}_\tau\|_{L^p(0,T)}^q \left(\sum_{j=1}^i \tau\|w_j\|_*^q\right) \\
&\quad \leqslant \frac{\alpha\tau}{2}\|u_i\|^p + \tau C\big(\alpha, p, \lambda_A, \Lambda_A, \|k\|_{L^\infty(0,T)}, T\big)\left(1 + \|f_i\|_*^q + \|z\|^p + \sum_{j=1}^i \tau\|u_j\|^p\right),
\end{aligned}$$

where we also exploited Lemma 3.2. Let us now take the sum above for $i = 1, \dots, m$ for arbitrary $m = 1, \dots, N$, and check that

$$\psi_\varepsilon^*(v_m) - (v_m, z) + \frac{\alpha}{2}\sum_{i=1}^m \tau\|u_i\|^p \leqslant \psi_\varepsilon^*\big(v^0\big) - \big(v^0, z\big) + C\big(\alpha, p, \lambda_A, \Lambda_A, \|z\|, \|k\|_{L^\infty(0,T)}, T\big) \times \sum_{i=1}^m \tau\left(1 + \|f_i\|_*^q + \sum_{j=1}^i \tau\|u_j\|^p\right).$$

Finally, owing also to (4.1), by applying Lemma 3.2 and the discrete Gronwall lemma (the diameter τ being small) and using (3.16) we have deduced that

$$\psi_\varepsilon^*(v_m) - (v_m, z) + \sum_{i=1}^m \tau\|u_i\|^p \leqslant C\big(\alpha, p, \lambda_A, \Lambda_A, \|z\|, \|k\|_{L^\infty(0,T)}, T, \psi^*\big(v^0\big), \big|v^0\big|, \|f\|_{L^q(0,T;V^*)}\big).$$

Hence, we have obtained that

$$\bar{u}_\tau \text{ is bounded in } L^p(0, T; V) \text{ independently of } \varepsilon \text{ and } \tau. \tag{4.5}$$

Moving from (4.5), some consequent estimates follow from (A2) and a comparison in (3.30), namely,

$$\overline{w}_\tau \text{ and } v_\tau' \text{ are bounded in } L^q(0, T; V^*) \text{ independently of } \varepsilon \text{ and } \tau. \tag{4.6}$$

4.2. Second estimate

It is a standard matter to exploit the argument of Section 3.2 (see (3.24)) in order to rewrite Eq. (3.26) as

$$w_i = f_i - \delta v_i - (\rho *_\tau f)_i + (\rho *_\tau \delta v)_i \quad \text{for } i = 1, \dots, N.$$

By exploiting (3.6) we deduce that

$$\delta v_i + w_i = f_i - (\rho *_\tau f)_i + (\delta\rho *_\tau v)_i + \rho_0 v_i - \rho_i v^0 \quad \text{for } i = 1, \dots, N. \tag{4.7}$$

Let us test the latter equation by τv_i, recall that $\rho_0 = k_0 = k(0)$, and obtain

$$\frac{1}{2}|v_i|^2 + \frac{1}{2}|v_i - v_{i-1}|^2 - \frac{1}{2}|v_{i-1}|^2 + \tau(w_i, v_i) = \tau(f_i, v_i) - \tau\big((\rho *_\tau f)_i, v_i\big) + \tau\big((\delta\rho *_\tau v)_i, v_i\big) + \tau k(0)|v_i|^2 - \tau\rho_i\big(v^0, v_i\big).$$

Owing to (2.2) and Lemma 3.2, we take the sum above for $i = 1, \dots, m$ for any $m = 1, \dots, N$, and check that

$$\begin{aligned}\frac{1}{2}|v_m|^2 &\leqslant \frac{1}{2}\left|v^0\right|^2 + \sum_{i=1}^{m} \tau\big(1 + \left|k(0)\right|\big)|v_i|^2 + \frac{1}{2} m\tau \|\bar{\rho}_\tau\|^2_{L^\infty(0,T)}|v_0|^2 \\ &\quad + C\sum_{i=1}^{m} \tau\big(|f_i|^2 + \left|(\rho *_\tau f)_i\right|^2 + \left|(\delta\rho *_\tau v)_i\right|^2\big) \\ &\leqslant C\big(\left|k(0)\right|, \mathrm{Var}(\rho_\tau)\big)\left(\sum_{i=1}^{m} \tau|v_i|^2\right) + C\big(\left|v^0\right|, \|\bar{\rho}_\tau\|_{L^\infty(0,T)}, T, \|f\|_{L^2(0,T;H)}\big).\end{aligned}$$

Finally, the bound in (3.19) and an application of the discrete Gronwall lemma entails, for all diameters τ small enough,

$$v_\tau \text{ is bounded in } C([0,T]; H) \text{ independently of } \varepsilon \text{ and } \tau.$$

5. Limit

We shall now pass to the limit in (3.30)–(3.33) as ε and τ go to zero. Since the above established estimates are independent of both ε and τ, we are entitled to fix, for instance, $\varepsilon = \tau$ and, thanks to well-known compactness results (see, e.g., [47, Corollary 4]), to find three functions u, v, and w such that, for some not relabeled subsequence of diameters,

$$\bar{u}_\tau \to u \quad \text{weakly in } L^p(0,T;V), \tag{5.1}$$

$$v_\tau \to v \quad \text{weakly star in } W^{1,q}\big(0,T;V^*\big) \cap L^\infty(0,T;H) \text{ and strongly in } C\big([0,T];V^*\big), \tag{5.2}$$

$$\bar{v}_\tau \to v \quad \text{weakly star in } L^\infty(0,T;H) \text{ and strongly in } L^\infty\big(0,T;V^*\big), \tag{5.3}$$

$$\overline{w}_\tau \to w \quad \text{weakly in } L^q\big(0,T;V^*\big). \tag{5.4}$$

Of course, v_τ and $\bar{v}_\tau$ have the same limit since

$$\|v_\tau - \bar{v}_\tau\|^q_{L^\infty(0,T;V^*)} \leqslant \sum_{i=1}^{N} \|v_i - v_{i-1}\|^q_* = \tau^{q-1}\left\|v'_\tau\right\|^q_{L^q(0,T;V^*)},$$

and (4.6) holds. In view of (5.2), this leads to the second convergence in (5.3). These convergences, (3.15), (3.16), and Corollary 3.4 are sufficient in order to pass to the limit in (3.30) and obtain (2.3) as well as (2.6).

On the other hand, the functional $\psi_\varepsilon = \psi_\tau$ converges to ψ in the sense of Mosco in H (see, e.g., [10]). The first consequence of this fact is that indeed $B_\varepsilon = B_\tau$ converges to B in the graph sense in $H \times H$ and inclusion (2.4) follows from (3.27), the convergences (5.1) and (5.3), and standard results on maximal monotone operators (see, e.g., [12, Proposition 1.2, p. 42]). A second drawback of the stated Mosco convergence is that

$$\psi^*(x) \leqslant \liminf_{\tau\to 0} \psi^*_\tau(x_\tau) \quad \text{for all } x_\tau \to x \text{ weakly in } H. \tag{5.5}$$

In particular, we shall use (5.5) in order to check (2.5). Let us start from exploiting (3.24) and obtain from (3.30)

$$\overline{\delta v}_\tau + \overline{w}_\tau = \bar{f}_\tau - \overline{(\rho *_\tau f)}_\tau + \overline{(\rho *_\tau \delta v)}_\tau. \tag{5.6}$$

Testing the latter by $\bar{u}_\tau$, taking the integral over $(0, T)$, and using (4.4), one has that

$$\int_0^T (\overline{w}_\tau, \bar{u}_\tau) \leqslant -\psi_\tau^*\big(\bar{v}_\tau(T)\big) + \psi_\tau^*\big(v^0\big) + \int_0^T \big((\bar{f}_\tau, \bar{u}_\tau) - \big(\overline{(\rho *_\tau f)}_\tau, \bar{u}_\tau\big) + \big(\overline{(\rho *_\tau \delta v)}_\tau, \bar{u}_\tau\big)\big). \tag{5.7}$$

Our next aim will be to pass to the lim sup as τ goes to 0 in the above relation. To this end, let us firstly exploit (5.5) in order to check that

$$\limsup_{\tau\to 0}\big(-\psi_\tau^*\big(\bar{v}_\tau(T)\big) + \psi_\tau^*\big(v^0\big)\big) \leqslant -\psi^*\big(v(T)\big) + \psi^*\big(v^0\big). \tag{5.8}$$

In fact, (5.2), (5.3) imply that v is continuous in T (in fact everywhere) with respect to the weak topology of H. On the other hand, $\bar{v}_\tau$ obviously has the same property. Hence, the first convergence in (5.3) implies that $v_\tau(T)$ converges to $v(T)$ weakly in H, whence (5.5) can be used.

As for the terms containing v_τ' in the right-hand side of (5.7), we exploit Proposition 3.3 and obtain

$$\overline{(\rho *_\tau \delta v)}_\tau - \bar{\rho}_\tau * v_\tau' \to 0 \quad \text{strongly in } L^q\big(0, T; V^*\big).$$

On the other hand, by recalling that $\rho_\tau(0) = k(0)$ one readily computes that

$$\begin{aligned}
\bar{\rho}_\tau * v_\tau' &= \rho_\tau * v_\tau' + (\bar{\rho}_\tau - \rho_\tau) * v_\tau' \\
&= \rho_\tau' * v_\tau + k(0)v_\tau - \rho_\tau v^0 + (\bar{\rho}_\tau - \rho_\tau) * v_\tau' \\
&= \rho_\tau' * v + \rho_\tau' * (v_\tau - v) + k(0)v_\tau - \rho_\tau v^0 + (\bar{\rho}_\tau - \rho_\tau) * v_\tau' \\
&= \rho_\tau * v' + \rho_\tau v^0 - k(0)v + \rho_\tau' * (v_\tau - v) + k(0)v_\tau - \rho_\tau v^0 + (\bar{\rho}_\tau - \rho_\tau) * v_\tau' \\
&= \rho_\tau * v' + k(0)(v_\tau - v) + \rho_\tau' * (v_\tau - v) + (\bar{\rho}_\tau - \rho_\tau) * v_\tau'.
\end{aligned}$$

By recalling (3.20) and (5.2), it is a standard matter to check that the above right-hand side converges strongly to $\rho * v'$ in $L^q(0, T; V^*)$. Therefore, we readily conclude that

$$\overline{(\rho *_\tau \delta v)}_\tau \;\to\; \rho * v' \quad \text{strongly in } L^q\big(0, T; V^*\big). \tag{5.9}$$

Again, by (3.19), (3.20), (3.25), and Corollary 3.4 we easily check that

$$\bar{f}_\tau - \overline{(\rho *_\tau f)}_\tau \;\to\; f - \rho * f \quad \text{strongly in } L^q\big(0, T; V^*\big). \tag{5.10}$$

Thus, by collecting (5.8)–(5.10) and exploiting (5.1) in (5.7), we have proved that

$$\limsup_{\tau\to 0}\int_0^T(\overline{w}_\tau,\bar{u}_\tau)\leqslant -\psi^*\big(v(T)\big)+\psi^*\big(v^0\big)+\int_0^T\big((f-\rho*f,u)+(\rho*v',u)\big)=\int_0^T(w,u).$$

In the latter we exploited the already established relations (2.3), (2.4) along with the classical chain rule for convex functions (see [21, Lemma 3.3, p. 73] for $p=2$, the present situation $p>1$ being completely analogous). Hence, relation (2.5) follows from standard properties of maximal monotone operators (see, e.g., [12, Proposition 1.2, p. 42]).

Remark 5.1. The above detailed analysis of the discrete scheme could be generalized with no particular intricacy to the situation of a family of approximating initial data $v_\tau^0\in H$ such that

$$v_\tau^0\to v^0\quad\text{weakly in } H\quad\text{and}\quad\psi_\tau^*\big(v_\tau^0\big)\to\psi^*\big(v^0\big).$$

On the other hand, some different approximation $\{f_i\}_{i=1}^N\in H^N$ of the function f could be considered as long as

$$\bar{f}_\tau\to f\quad\text{weakly in } L^2(0,T;H)\text{ and strongly in } L^q\big(0,T;V^*\big).$$

6. Kernels with unbounded variation

Let us conclude our analysis by explicitly observing that it is possible to argue along the lines of Theorem 2.1 in order to recover some existence results for problem (1.6) in the case of kernels with possibly unbounded variation, i.e.,

(A6) $k\in L^1(0,T)$.

In fact, by assuming A to be a subdifferential and B to be bi-Lipschitz continuous (i.e., Lipschitz continuous along with its inverse) one can prove that (1.6) admits a solution in case of (A6). Note that in this setting the compatibility assumption (A4) plays no role and one is entitled not to ask for (1.11). Moreover, since B is nondegenerate, homogeneous Neumann conditions may be considered. To be precise, let us ask for the following (compare with (A2), (A3) and (A5)):

(A7) $\phi:V\to[0,+\infty]$ is a convex, proper, and lower semicontinuous function and $A:=\partial\phi:V\to V'$. Moreover, A is coercive and bounded in the sense of (2.1).

(A8) $\psi:H\to(-\infty,+\infty]$ is a convex, proper, and lower semicontinuous function with $B:=\partial\psi:H\to H$ Lipschitz continuous and strongly monotone. Namely, there exist two positive constants c_B, C_B such that

$$c_B|u_1-u_2|^2\leqslant\big(B(u_1)-B(u_2),u_1-u_2\big),$$
$$\big|B(u_1)-B(u_2)\big|\leqslant C_B|u_1-u_2|\quad\forall u_1,u_2\in H.$$

(A9) $f\in L^q(0,T;V^*)\cap L^2(0,T;H)$, $u^0\in V$, $v^0=Bu^0$.

Let us stress that clearly the latter assumptions are stronger than the corresponding ones of Theorem 2.1. In particular, (A8), (A9) entail that $v^0 \in H = D(\psi^*)$. We are in the position of stating the following:

Theorem 6.1. *Under assumptions* (A0), (A6)–(A9) *there exist* $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $v \in H^1(0,T;H) \cap W^{1,q}(0,T;V^*)$, *and* $w \in L^2(0,T;H) \cap L^q(0,T;V^*)$ *fulfilling* (2.3)–(2.6).

The latter result is aimed to suggest some application and it is not intended to be the best possible in any sense. Let us observe that, whenever A is a subdifferential fulfilling (2.1), the corresponding potential φ is coercive as well. In particular, there exist two positive constants $\bar{\alpha}$ and $\bar{\lambda}_A$ such that

$$\bar{\alpha}\|u\|^p - \bar{\lambda}_A \leqslant \phi(u) \quad \forall u \in V. \tag{6.11}$$

Proof. We will not provide here a full proof but rather sketch the argument and refer to the above analysis for the details. One shall start by approximating the kernel $k \in L^1(0,T)$ by suitable kernels $k_\varepsilon \in W^{1,1}(0,T)$ depending on the approximation parameter $\varepsilon \in (0,1)$ which will later tend to zero. The latter approximation may of course be accomplished in such a way that

$$k_\varepsilon \to k \quad \text{and} \quad \rho_\varepsilon \to \rho \quad \text{strongly in } L^1(0,T), \tag{6.12}$$

where ρ_ε and ρ are the resolvents of k_ε and k, respectively (see Section 3.2). Let us now solve the evolution problem at some fixed level ε. To this end, one cannot simply exploit Theorem 2.1 since we are missing the compatibility assumption (A4). Hence, one has to adapt our time-discretization technique to the present situation. The operator B is now Lipschitz continuous and we can assume with no loss of generality that $\psi(0) = 0$ so that the corresponding conjugate ψ^* is everywhere nonnegative. Moreover, there is actually no need to perform the Yosida approximation of B in order to tackle the discrete problem on the uniform partition $\{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$ with diameter τ. In particular, letting $k_{\varepsilon,\tau}$ be defined via $k_{\varepsilon,i} := k_\varepsilon(t_i)$ for $i = 0, 1, \ldots, N$, one readily checks that

$$\|\bar{k}_{\varepsilon,\tau}\|_{L^1(0,T)} \quad \text{and} \quad \|\bar{\rho}_{\varepsilon,\tau}\|_{L^1(0,T)} \quad \text{are bounded independently of } \tau,$$

where $\bar{\rho}_{\varepsilon,\tau}$ is of course the discrete resolvent of $\bar{k}_{\varepsilon,\tau}$ in the sense of (3.17). Then, the discussion of Section 3.3 entails the validity of Lemma 3.6, i.e., the existence of suitable discrete solutions $u_{\varepsilon,\tau}, v_{\varepsilon,\tau}, w_{\varepsilon,\tau}$ to relations (3.30)–(3.33) (where B_ε is now replaced by B). Moreover, the estimates (4.5), (4.6) (independent of the time-step τ but depending on ε) may be readily recovered. The second estimate of Section 4 is no longer valid since we are not assuming (A4) (nor the necessary regularity in time of the ingredients). One has to test instead the discrete equation (5.6) by $u'_{\varepsilon,\tau}$, take the integral in time, and exploit (A8). By letting $\bar{g}_\tau := \bar{f}_\tau - \overline{(\rho *_\tau f)}_\tau$ and making use of Lemma 3.2, we get for all $t \in (0,T)$ that

$$\int_0^t \big(v'_{\varepsilon,\tau}, u'_{\varepsilon,\tau}\big) + \phi\big(u_{\varepsilon,\tau}(t)\big) - \phi\big(u^0\big)$$

$$\leqslant \int_0^t \big(\bar{g}_\tau, u'_{\varepsilon,\tau}\big) + \int_0^t \big(\overline{(\rho *_\tau \delta v)}_\tau, u'_{\varepsilon,\tau}\big)$$

$$\leqslant \frac{c_B}{2}\int_0^t \left|u'_{\varepsilon,\tau}\right|^2 + C(c_B)\left(\|\overline{g}_\tau\|^2_{L^2(0,t;H)} + \int_0^t \left|\overline{(\rho *_\tau \delta v)}_\tau\right|^2\right)$$

$$\leqslant \frac{c_B}{2}\int_0^t \left|u'_{\varepsilon,\tau}\right|^2 + C\big(c_B, \|\overline{\rho}_{\varepsilon,\tau}\|_{L^1(0,T)}, \|\bar{f}_\tau\|_{L^2(0,T;H)}\big)$$

$$+ C\big(c_B, C_B, \|\overline{\rho}_{\varepsilon,\tau}\|_{L^2(0,T)}\big)\int_0^t \left\|u'_{\varepsilon,\tau}\right\|^2_{L^2(0,s;H)}\,ds.$$

We readily conclude by Gronwall's lemma that, for all fixed ε,

$$u_{\varepsilon,\tau} \text{ and } v_{\varepsilon,\tau} \text{ are bounded in } H^1(0,T;H) \text{ independently of } \tau. \tag{6.13}$$

Note that the latter bound depends on ε since the norms $\|\overline{\rho}_{\varepsilon,\tau}\|_{L^2(0,T)}$ need not be bounded independently of ε.

We now pass to the limit as τ goes to zero. Arguing exactly as in Section 5, one readily finds a triplet $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ such that, at least for some (not relabeled) subsequence, the (suitable analogue of) convergences (5.1)–(5.4) hold. Hence, since the kernels k_ε and ρ_ε have a bounded variation, we pass to the limit in the discrete problem and obtain that

$$v'_\varepsilon + w_\varepsilon + k_\varepsilon * w_\varepsilon - f = v'_\varepsilon + w_\varepsilon - \rho_\varepsilon * v'_\varepsilon - f + \rho_\varepsilon * f = 0 \quad \text{a.e. in } (0,T), \tag{6.14}$$

$$v_\varepsilon = B(u_\varepsilon) \quad \text{and} \quad w_\varepsilon \in A(u_\varepsilon) \quad \text{a.e. in } (0,T), \qquad v_\varepsilon(0) = v^0. \tag{6.15}$$

Let $\mu \geqslant 1$ to be chosen later, test (6.14) by $t \mapsto e^{-\mu t}u_\varepsilon(t)$, and take the integral over $(0,t)$ for $t \in (0,T]$. One has that

$$\int_0^t e^{-\mu s}\big(v'_\varepsilon(s), u_\varepsilon(s)\big)\,ds + \int_0^t e^{-\mu s}\big(w_\varepsilon(s), u_\varepsilon(s)\big)\,ds$$

$$= -\int_0^t e^{-\mu s}\big((k_\varepsilon * w_\varepsilon)(s), u_\varepsilon(s)\big)\,ds + \int_0^t e^{-\mu s}\big(f(s), u_\varepsilon(s)\big)\,ds. \tag{6.16}$$

As for the first term in the left-hand side of (6.16), one readily gets by integration by parts that

$$\int_0^t e^{-\mu s}\big(v'_\varepsilon(s), u_\varepsilon(s)\big)\,ds = \int_0^t e^{-\mu s}\frac{d}{ds}\psi^*\big(v_\varepsilon(s)\big)\,ds$$

$$= e^{-\mu t}\psi^*\big(v_\varepsilon(t)\big) - \psi^*\big(v^0\big) + \mu\int_0^t e^{-\mu s}\psi^*\big(v_\varepsilon(s)\big)\,ds$$

$$\geqslant e^{-\mu t}\psi^*\big(v_\varepsilon(t)\big) - \psi^*\big(v^0\big).$$

Next, we exploit (2.1) and $\mu \geqslant 1$ and get that

$$\int_0^t e^{-\mu s}\big(w_\varepsilon(s), u_\varepsilon(s)\big)\, ds \geqslant \alpha \int_0^t e^{-\mu s}\big\|u_\varepsilon(s)\big\|^p\, ds - \frac{\lambda_A}{\mu}\big(1 - e^{-\mu t}\big)$$
$$\geqslant \alpha \int_0^t e^{-\mu s}\big\|u_\varepsilon(s)\big\|^p\, ds - \lambda_A.$$

By letting

$$k_\mu(t) := e^{-\mu t/q} k_\varepsilon(t), \qquad w_\mu(t) := e^{-\mu t/q} w_\varepsilon(t), \qquad u_\mu(t) := e^{-\mu t/p} u_\varepsilon(t),$$

we handle the convolution term in (6.16) by means of Young's theorem as follows

$$\begin{aligned}
&-\int_0^t e^{-\mu s}\big((k_\varepsilon * w_\varepsilon)(s), u_\varepsilon(s)\big)\, ds \\
&\quad = -\int_0^t (k_\mu * w_\mu, u_\mu) \leqslant \frac{\alpha}{4}\int_0^t \|u_\mu\|^p + C(\alpha, p)\|k_\mu\|^q_{L^1(0,T)} \int_0^t \|w_\mu\|^q_* \\
&\quad \leqslant \left(\frac{\alpha}{4} + C(\alpha, p, \Lambda_A)\|k_\mu\|^q_{L^1(0,T)}\right) \int_0^t e^{-\mu s}\big\|u_\varepsilon(s)\big\|^p\, ds + C\big(\alpha, p, \Lambda_A, \|k_\varepsilon\|_{L^1(0,T)}, T\big).
\end{aligned} \tag{6.17}$$

Finally, one readily checks that

$$\int_0^t e^{-\mu s}\big(f(s), u_\varepsilon(s)\big)\, ds \leqslant \frac{\alpha}{4}\int_0^t e^{-\mu s}\big\|u_\varepsilon(s)\big\|^p\, ds + C(\alpha, p)\|f\|^q_{L^q(0,T;V^*)}.$$

Let us now observe that $\lim_{\mu\to+\infty}\|k_\mu\|_{L^1(0,T)} = 0$ uniformly with respect to ε. Indeed since k_ε are equi-integrable by the Dunford–Pettis criterion [28], for all $\delta > 0$ one may find $\nu \in (0, T)$ such that

$$\sup_{\varepsilon\in(0,1)} \int_0^\nu \big|k_\varepsilon(t)\big|\, dt \leqslant \delta.$$

Hence, one readily computes that

$$\sup_{\varepsilon\in(0,1)} \|k_\mu\|_{L^1(0,T)} = \sup_{\varepsilon\in(0,1)} \int_0^T e^{-\mu t/q}\big|k_\varepsilon(t)\big|\, dt$$

$$= \sup_{\varepsilon\in(0,1)} \left(\int_0^\nu e^{-\mu t/q} \left|k_\varepsilon(t)\right| dt + \int_\nu^T e^{-\mu t/q} \left|k_\varepsilon(t)\right| dt \right)$$
$$\leqslant \delta + e^{-\mu\nu/q} \sup_{\varepsilon\in(0,1)} \|k_\varepsilon\|_{L^1(0,T)}.$$

Now, letting μ be such that the coefficient of the integral in the right-hand side of (6.17) is smaller than $\alpha/4$, we easily deduce from (6.16) that

$$e^{-\mu t}\psi^*\big(v_\varepsilon(t)\big) + \frac{\alpha}{4}\int_0^t e^{-\mu s}\left\|u_\varepsilon(s)\right\|^p$$
$$\leqslant C\big(\alpha, p, \lambda_A, \Lambda_A, \psi^*\big(v^0\big), \|k_\varepsilon\|_{L^1(0,T),} \|f\|_{L^q(0,T;V^*)}, T\big).$$

In particular, we have recovered the estimates (see (4.5), (4.6))

$$u_\varepsilon \text{ is bounded in } L^p(0,T;V), \qquad \psi^*_\varepsilon(v_\varepsilon) \text{ is bounded in } L^\infty(0,T), \quad \text{and}$$
$$w_\varepsilon \text{ and } v'_\varepsilon \text{ are bounded in } L^q(0,T;V^*) \text{ independently of } \varepsilon. \tag{6.18}$$

The latter estimates have indeed a discrete equivalent that could have been exploited in the first place in Section 4. We preferred however to sketch it here at the continuous level in order to avoid technicalities.

Next, we test (6.14) by the function $t \mapsto e^{-\kappa t}u'_\varepsilon(t)$ where $\kappa \geqslant 1$ has to be chosen later, take the integral in time, and obtain, for all $t \in (0,T)$,

$$\int_0^t e^{-\kappa s}\big(v'_\varepsilon(s), u'_\varepsilon(s)\big)\,ds + \int_0^t e^{-\kappa s}\big(w_\varepsilon(s), u'_\varepsilon(s)\big)\,ds$$
$$= \int_0^t e^{-\kappa s}\big((f-\rho_\varepsilon * f)(s), u'_\varepsilon(s)\big)\,ds + \int_0^t e^{-\kappa s}\big(\big(\rho_\varepsilon * v'_\varepsilon\big)(s), u'_\varepsilon(s)\big)\,ds. \tag{6.19}$$

Owing to (A7), (A8), we easily compute that

$$\int_0^t e^{-\kappa s}\big(v'_\varepsilon(s), u'_\varepsilon(s)\big)\,ds \geqslant c_B \int_0^t e^{-\kappa s}\left|u'_\varepsilon(s)\right|^2 ds,$$
$$\int_0^t e^{-\kappa s}\big(w_\varepsilon(s), u'_\varepsilon(s)\big)\,ds = e^{-\kappa t}\phi\big(u_\varepsilon(t)\big) - \phi\big(u^0\big) + \kappa\int_0^t e^{-\kappa s}\phi\big(u_\varepsilon(s)\big)\,ds$$
$$\geqslant e^{-\kappa t}\phi\big(u_\varepsilon(t)\big) - \phi\big(u^0\big).$$

The term containing f in the right-hand side of (6.19) can be handled as follows

$$\begin{aligned}&\int_0^t e^{-\kappa s}\big((f-\rho_\varepsilon * f)(s), u_\varepsilon'(s)\big)\,ds\\&\quad\leqslant \frac{c_B}{4}\int_0^t e^{-\kappa s}\big|u_\varepsilon'(s)\big|^2\,ds + C(c_B)\int_0^t e^{-\kappa s}\big|(f-\rho_\varepsilon * f)(s)\big|^2\,ds\\&\quad\leqslant \frac{c_B}{4}\int_0^t e^{-\kappa s}\big|u_\varepsilon'(s)\big|^2\,ds + C\big(c_B, \|\rho_\varepsilon\|_{L^1(0,T)}, \|f\|_{L^2(0,T;H)}\big).\end{aligned}$$

As for the term containing v_ε' in the right-hand side of (6.19), we introduce the functions

$$\rho_\kappa(t) := e^{-\kappa t/2}\rho_\varepsilon(t), \qquad v_\kappa'(t) := e^{-\kappa t/2}v_\varepsilon'(t), \qquad u_\kappa'(t) := e^{-\kappa t/2}u_\varepsilon'(t),$$

and use Young's theorem in order to deduce that

$$\begin{aligned}&\int_0^t e^{-\kappa s}\big(\big(\rho_\varepsilon * v_\varepsilon'\big)(s), u_\varepsilon'(s)\big)\,ds\\&\quad= \int_0^t \big(\rho_\kappa * v_\kappa', u_\kappa'\big) \leqslant \frac{c_B}{4}\int_0^t \big|u_\kappa'\big|^2 + C(c_B, C_B)\|\rho_\kappa\|^2_{L^1(0,t)}\int_0^t \big|u_\kappa'\big|^2.\end{aligned} \tag{6.20}$$

Since ρ_ε are equi-integrable by the Dunford–Pettis criterion, for all $\delta > 0$ one may find $\nu \in (0, T)$ such that

$$\sup_{\varepsilon\in(0,1)}\int_0^\nu \big|\rho_\varepsilon(t)\big|\,dt \leqslant \delta,$$

and we have that

$$\begin{aligned}&\sup_{\varepsilon\in(0,1)} \|\rho_\kappa\|_{L^1(0,T)}\\&\quad= \sup_{\varepsilon\in(0,1)}\int_0^T e^{-\kappa t/2}\big|\rho_\varepsilon(t)\big|\,dt = \sup_{\varepsilon\in(0,1)}\left(\int_0^\nu e^{-\kappa t/2}\big|\rho_\varepsilon(t)\big|\,dt + \int_\nu^T e^{-\kappa t/2}\big|\rho_\varepsilon(t)\big|\,dt\right)\\&\quad\leqslant \delta + e^{-\kappa\nu/2}\sup_{\varepsilon\in(0,1)}\|\rho_\varepsilon\|_{L^1(0,T)}.\end{aligned}$$

We now choose δ and κ in such a way that the coefficient of the second term in the right-hand side of (6.20) is bounded by $c_B/4$. Then, relation (6.19) entails that (recall also (6.11))

$$u_\varepsilon \text{ and } v_\varepsilon \text{ are bounded in } H^1(0,T;H) \quad \text{and}$$
$$u_\varepsilon \text{ is bounded in } L^\infty(0,T;V) \text{ independently of } \varepsilon. \tag{6.21}$$

Let us now pass to the limit as ε goes to zero. Taking the above bounds into account and by suitably extracting (not relabeled) subsequences, we get a triplet (u, v, w) such that (2.3), (2.4), (2.6), and the convergences (5.1)–(5.4) hold. Moreover, owing to (6.21), we also get the extra convergences

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \quad \text{weakly in } H^1(0,T;H) \text{ and strongly in } C\big([0,T];H\big). \tag{6.22}$$

Finally, in order to check for (2.5), we argue once again as in (5.6). In particular, let us test (6.14) by u_ε and take the integral over $(0, T)$ getting

$$\int_0^T (w_\varepsilon, u_\varepsilon) = -\psi^*\big(v_\varepsilon(T)\big) + \psi^*\big(v^0\big) + \int_0^T \big((f - \rho_\varepsilon * f, u_\varepsilon) + \big(\rho_\varepsilon * v_\varepsilon', u_\varepsilon\big)\big). \tag{6.23}$$

Our next aim is to pass to the lim sup as ε goes to zero in the latter relation. To this end, we shall exploit the lower semicontinuity of ψ^*, convergences (6.12) and (6.22), and observe that

$$\rho_\varepsilon * v_\varepsilon' \to \rho * v' \quad \text{weakly in } L^2(0,T;H),$$

in order to obtain

$$\limsup_{\varepsilon\to 0} \int_0^T (w_\varepsilon, u_\varepsilon) \leqslant -\psi^*\big(v(T)\big) + \psi^*\big(v^0\big) + \int_0^T (f - \rho * f, u) + \int_0^T (\rho * v', u) = \int_0^T (w, u).$$

Finally, relation (2.5) follows from [12, Proposition 1.2, p. 42]. □

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