EXISTENCE RESULT FOR A NONLINEAR MODEL RELATED TO IRREVERSIBLE PHASE CHANGES

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This paper deals with an initial and boundary value problem for a nonlinear evolution system which may be used to describe irreversible phase transition phenomena. The existence of a global solution is established via a regularization — a priori estimate — passage to the limit procedure. Moreover, uniqueness is also discussed under additional regularity assumptions on data.

1. Introduction
The present analysis is concerned with a nonlinear system of partial differential equations describing the evolution of two unknown fields $\theta$ and $\chi$. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial \Omega$, and $T$ be some final time, we deal with the following relations

$$
\partial_t \theta + \theta \partial_t \chi - \Delta \theta = 0 \quad \text{a.e. in } \Omega \times ]0, T[, \tag{1.1}
$$

$$
\partial_t \chi + \alpha(\partial_t \chi) - \Delta \chi + \beta(\chi) \ni \theta - \theta_c \quad \text{a.e. in } \Omega \times ]0, T[, \tag{1.2}
$$

where $\alpha$ and $\beta$ are maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ and $\theta_c$ stands for a positive constant. Indeed, $\beta$ is assumed completely arbitrary, while we prescribe $\alpha$ to be the subdifferential of the indicator function $I_{[0, +\infty]}$ of the set $[0, +\infty]$. Namely, we have

$$
\alpha(r) = \begin{cases} 
| - \infty, 0 | & \text{if } r = 0, \\
0 & \text{if } r > 0. 
\end{cases} \tag{1.3}
$$

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The system (1.1)–(1.2) has a relevant interest within applications. Indeed, it is expected to describe the behavior of a material which may undergo irreversible phase transitions. The latter phenomena are to be found in several different situations including, possibly, thermal hardening of glues, glass formation, and food cooking.

The original model, firstly proposed in Bonfanti–Frémond–Luterotti, is based on the consideration that the microscopic movements can entail macroscopic effects on the phase change. Here we want to recall the main features of the model introduced in that paper. Towards this aim, we have to consider a two-phase substance contained in a domain $\Omega \subset \mathbb{R}^3$ and a given final time $T > 0$. Dealing with the heat diffusion inside the body, we choose the volume fraction of one of the phases as a state quantity and denote it by $\chi = \chi(x,t)$, for $x \in \Omega$ and $t \in [0,T]$. Thus, the order parameter $\chi$ satisfies the relation $0 \leq \chi \leq 1$, and, assuming that no voids appear in the mixture, the volume proportion of the other phase is simply given by $1 - \chi$. Of course, the absolute temperature $\theta = \theta(x,t)$ is the other state variable for the thermodynamical system, and it has to be non-negative.

Then, taking into account the power of the microscopic movements, we derive the following energy balance equation

$$\partial_t \epsilon + \text{div} \mathbf{q} = B \partial_t \chi + \mathbf{H} \cdot \nabla \partial_t \chi,$$

where $\epsilon$ denotes the internal energy and $\mathbf{q} = -\lambda \nabla \theta$ stands for the heat flux (i.e. the Fourier law is assumed with a constant thermal conductivity $\lambda$). Indeed, $B$ and $\mathbf{H}$ are respectively a scalar quantity and a vector related to the microscopic interior forces and obeying the following consequence of the virtual power principle

$$-\text{div} \mathbf{H} + B = A,$$

where $A$ collects the amount of external forces.

Now, in order to state the constitutive relations for $B$ and $\mathbf{H}$, we have to choose an explicit expression for the free energy $\Psi$ as

$$\Psi(\chi, \nabla \chi, \theta) = -c_s \theta \log \theta - \frac{L}{\theta_c}(\theta - \theta_c)\chi + I_{[0,1]}(\chi) + \frac{k}{2} |\nabla \chi|^2,$$

where $L > 0$ stands for the latent heat at the critical transition temperature $\theta_c > 0$, $c_s > 0$ represents the specific heat, the parameter $k \geq 0$ is the factor of the interfacial energy term, and the indicator function $I_{[0,1]}$ plays the role of the constraint for the phase proportion $\chi$. As far as the pseudo-potential of dissipation is concerned, we set

$$\Phi(\partial_t \chi) = \frac{\mu}{2} (\partial_t \chi)^2 + I_{[0, +\infty]}(\partial_t \chi),$$

for small coefficient $\mu > 0$. This position accounts for the irreversibility of the phase change because of the presence of the term $I_{[0, +\infty]}(\partial_t \chi)$ that forces $\partial_t \chi$ to attain solely positive values.
Now, the straightforward choices for the constitutive laws for $B$ and $H$

$$B = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial (\partial \chi)} \quad \text{and} \quad H = \frac{\partial \Psi}{\partial (\nabla \chi)} \quad (1.8)$$

turn out to ensure the basic laws of thermodynamics. Moreover, the internal energy $e$ is related to the free energy $\Psi$ and to the entropy $S = -\frac{\partial \Psi}{\partial \theta}$ by the rather classical relation

$$e = \Psi + S \theta = \Psi - \theta \frac{\partial \Psi}{\partial \theta}. \quad (1.9)$$

Setting for simplicity $A = 0$ and noting that $\partial I_{[0, \infty]}(\partial \chi)\partial \chi = 0$, the balance and constitutive laws — together with (1.6) and (1.7) — yield the following system

$$c_s \partial_t \theta + L \partial_t \chi - \lambda \Delta \theta = -\frac{L}{\theta_c} (\theta - \theta_c) \partial_t \chi + \mu (\partial_t \chi)^2, \quad (1.10)$$

$$\mu \partial_t \chi - k \Delta \chi + \partial I_{[0, \infty]}(\partial \chi) + \partial I_{[0, 1]}(\chi) \gtrless L \frac{1}{\theta_c} (\theta - \theta_c). \quad (1.11)$$

As far as we know, no existence results have been obtained for initial-boundary problems related to the system (1.10)–(1.11): it contains various nonlinearities which look very difficult to be handled. Indeed, in Bonfanti–Frémond–Luterotti,\cite{4} the well-posedness analysis is performed by assuming that

$$\frac{\theta}{\theta_c} \approx 1 \quad \text{and} \quad \mu (\partial_t \chi)^2 \approx 0. \quad (1.12)$$

Namely, the first smallness assumption above is justified provided that the temperature $\theta$ is close to the critical temperature $\theta_c$. More in detail, they consider the relation

$$c_s \partial_t \theta + L \partial_t \chi - \lambda \Delta \theta = 0, \quad (1.13)$$

which is the standard expression of the energy balance equation in phase transition phenomena with Fourier heat flux law.

Here we want to address a deeper study of the model. Thus, we only assume $\mu (\partial_t \chi)^2 \approx 0$ (allowing indeed the temperature of being far from the critical value $\theta_c$) and obtain

$$c_s \partial_t \theta + \frac{L}{\theta_c} \theta \partial_t \chi - \lambda \Delta \theta = 0. \quad (1.14)$$

Then (1.1)–(1.2) can be recovered at once from (1.14), (1.11) by taking most of the physical constants equal to 1 and setting $\partial I_{[0, 1]} = \beta$ and $\partial I_{[0, \infty]} = \alpha$. Let us remark that our result still holds when $\beta$ is an arbitrary maximal monotone graph; on the contrary, we limit ourselves to the original model for the choice of $\alpha$.

Clearly, for the purpose of an analytical study, the system (1.1)–(1.2) has to be complemented with some initial and boundary conditions. We prescribe

$$\theta(\cdot, 0) = \theta_0, \quad \chi(\cdot, 0) = \chi_0 \quad \text{a.e. in } \Omega, \quad (1.15)$$

$$\partial_n \theta = \partial_n \chi = 0 \quad \text{a.e. on } \partial \Omega \times ]0, T[, \quad (1.16)$$

where $n$ stands for the outer unit normal vector to the boundary $\partial \Omega$. 
We want to stress that we cannot adapt here the arguments used in the linear — as far as the energy balance equation is concerned — case considered in Bonfanti–Frémond–Luterotti. Indeed, one has to perform a direct investigation, consisting of a regularization procedure (different from the one in the quoted paper), a fixed point argument, some a priori estimates and a passage to limit. It must be noted that, thanks to the irreversibility, the energy balance equation yields the uniform positivity and boundedness of the temperature $\theta$ via maximum principle considerations.

Such a key argument does not apply to the fully nonlinear case of (1.10). Indeed, although the positivity of $\theta$ may still be achieved arguing as in Lemma 2.5, no upper bound for the temperature can be proven. Moreover, the highly nonlinear term $(\partial_t \chi)^2$ looks rather difficult to handle.

Now we give a brief overview concerning some contributions close to the present analysis. First, we have to mention the paper Bonfanti–Frémond–Luterotti, which provides the introduction of the original model. The authors obtain an existence result for an initial–boundary problem related to a system close to (1.2), (1.13), by using an approximation procedure. The latter consists of a suitable regularization combined with a time discretization. Such a result also holds when $\alpha$ is a general maximal monotone graph and when $\partial_t \chi$ does not occur explicitly in (1.2) (i.e. $\mu = 0$ is taken in (1.11)).

In Bonfanti–Frémond–Luterotti, the authors consider the fully nonlinear energy balance Eq. (1.10), coupling it with the inclusion (1.11) where a dissipation term is added and it is also assumed $\alpha \equiv 0$ (i.e. only the reversible case is considered). The local existence of a solution to the resulting system is achieved, by using a regularization procedure together with a fixed point technique (exploiting the dissipation effects through the gain of regularity on $\partial_t \chi$).

We also have to mention the former paper Blanchard–Damlamian–Ghidouche, where a system essentially consisting of (1.13), together with (1.11) with $k = 0$ is studied. These authors deal with the typical inclusion of the phase relaxation model with irreversible evolution (introduced in Frémond–Visintin). Existence and uniqueness results are proved by using regularization and monotonicity arguments.

Recently, in Colli–Luterotti–Schimperna–Stefanelli, the authors show an existence result for a system consisting of (1.10) and (1.11) with $k = 0$ (hence, close to the phase relaxation model of Frémond–Visintin with irreversible evolution). The proof is performed by exploiting a regularization procedure combined with a truncation. The derivation of some bounds for the unknowns is a key step and is obtained via maximum principle and monotonicity arguments, which work thanks to the absence of the Laplacian despite the loss of the spatial regularity on $\chi$.

The remainder of the paper is organized as follows. In Sec. 2, we formulate the assumptions on the data and state the existence of a global solution. Besides, we prove a maximum principle that implies, in particular, positivity and boundedness of $\theta$. Section 3 is concerned with the detailed proof of the existence result. Starting with a procedure of regularization of the maximal monotone graph $\beta$...
we actually obtain a family of approximating problems. Then, the local (in time) well-posedness of such problems is established through a fixed point argument. The subsequent estimates — independent of the regularization parameter — entail that the approximating solutions are defined on the whole time interval and, using semicontinuity and compactness, allow us to pass to the limit and recover a solution of the original problem.

2. Statement of the Problem and Main Result

We start by fixing some notation. Let

\[ H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and} \quad W := \{ u \in H^2(\Omega), \text{such that } \partial_n u = 0 \text{ on } \partial\Omega \} \]

and identify \( H \) with its dual space \( H' \), so that

\[ W \subset V \subset H \subset V' \subset W' \]

with dense and continuous embeddings. Besides, let the symbol \( \| \cdot \| \) denote the norm either of \( H \) or \( H^N \).

We make the following assumptions on data. Let

\[ 0 \leq \theta_c \leq \theta_* \quad \text{be assigned constants}, \quad (2.1) \]

\[ \alpha = \partial I_{[0, +\infty]}, \quad (2.2) \]

\[ j : \mathbb{R} \to [0, +\infty] \quad \text{be proper, convex, and lower semicontinuous} \]

such that \( \beta = \partial j \) and \( j(0) = 0 \), \quad (2.3)

\[ \theta_0 \in V \quad \text{and} \quad 0 \leq \theta_0 \leq \theta_* \quad \text{a.e. in } \Omega, \quad (2.4) \]

\[ \chi_0 \in W, \quad \chi_0 \in D(\beta) \quad \text{a.e. in } \Omega, \quad (2.5) \]

\[ \exists \xi_0 \in H \text{ such that } \xi_0 \in \beta(\chi_0) \quad \text{a.e. in } \Omega \]

and

\[ (1 + \alpha)^{-1}(\theta_0 - \theta_c + \Delta \chi_0 - \xi_0) \in H, \quad (2.6) \]

where we recall that \( I_{[0, +\infty]} \) stands for the indicator function of the set \([0, +\infty]\), \( D(\beta) \) denotes the effective domain of \( \beta \), and 1 is the identity in \( \mathbb{R} \).

**Remark 2.1.** We emphasize that assumptions (2.5)–(2.6) entail in particular the following property:

\[ \chi_0 \in D(j) \quad \text{a.e. in } \Omega \quad \text{and} \quad j(\chi_0) \in L^1(\Omega). \quad (2.7) \]

Indeed, setting

\[ J(v) := \int_\Omega j(v) \, dx \quad \text{for } v \in H, \quad (2.8) \]

\( J \) turns out to be a convex, lower semicontinuous, and proper functional on \( H \); moreover, one can verify that (2.5)–(2.6) imply \( \chi_0 \in D(\partial J) \) (cf. Prop. 2.16, pp. 47–48 in Brezis\(^4\)), where \( \partial J \) is seen as a maximal monotone operator from \( H \) to \( 2^H \), of course. Hence, we also have \( \chi_0 \in D(J) \), i.e. (2.7).
Now, we are able to state the main result of the paper, which reads as follows:

**Theorem 2.2.** (Existence) Let assumptions (2.1)–(2.6) hold. Then, there exists at least a quadruple \((\theta, \chi, \xi, \eta)\) such that

\[
\begin{align*}
\theta &\in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \\
\chi &\in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \\
\xi &\in L^\infty(0, T; H), \\
\eta &\in L^\infty(0, T; H), \\
\partial_t \theta + \theta \partial_t \chi - \Delta \theta &= 0 \quad \text{a.e. in } \Omega \times ]0, T[, \\
\partial_t \chi + \eta - \Delta \chi + \xi &= \theta - \theta_c \quad \text{a.e. in } \Omega \times ]0, T[, \\
\xi &\in \beta(\chi) \quad \text{a.e. in } \Omega \times ]0, T[, \\
\eta &\in \alpha(\partial_t \chi) \quad \text{a.e. in } \Omega \times ]0, T[, \\
\theta(., 0) &= \theta_0, \quad \chi(., 0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{align*}
\]

Moreover, it turns out that

\[
0 \leq \theta \leq \theta_* \quad \text{a.e. in } \Omega \times ]0, T[.
\]

**Remark 2.3.** Note that, at a first survey, the high regularity of \(\theta\) provided by (2.9) could be surprising, since the nonlinear term \(\theta \partial_t \chi\) appears in (2.13). However, owing to (2.10) and (2.18), one can easily check that \(\theta \partial_t \chi \in L^2(Q)\) (actually much more is true); hence, (2.9) is justified.

The proof of this result will be carried out throughout the remainder of the paper by exploiting a regularization procedure. Indeed, we replace \(\beta\) with its Yosida approximation \(\beta_\varepsilon\) and solve the regularized problem. Then, proper \textit{a priori} estimates independent of \(\varepsilon\) are established and passage to the limit is reached via compactness and monotonicity arguments. Moreover, it will be clear that the regularized problem admits a unique solution. In this concern, we are able to prove the following uniqueness result

**Proposition 2.4.** (Uniqueness) Let (2.1)–(2.6) hold. Moreover, assume that

\[
\beta \text{ is Lipschitz continuous.}
\]

Then the quadruple \((\theta, \chi, \xi, \eta)\), whose existence is stated in Theorem 2.2, is unique.

Note that this result goes in the same direction of Theorem 2.2 of Bonfanti–Frémond–Luterotti,\(^4\) where, nevertheless, the authors deal with the term \(\partial_t \chi\) instead of \(\theta \partial_t \chi\) (cf. (2.13)).

Since positivity and boundedness of \(\theta\) (see (2.18)) play a crucial role in the remainder of the paper, we state and prove a maximum principle which applies to (2.13).
Lemma 2.5. (Maximum principle) Let assumptions (2.1) and (2.4) hold and \( u, v \) fulfill the following

\[
\begin{align*}
    u & \in H^1(0,T;H) \cap L^2(0,T;W), \\
    v & \in L^2(0,T;H) \quad \text{and} \quad v \geq 0 \ \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack, \\
    \partial_t u - \Delta u & = -vu \quad \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack, \\
    u(t,0) & = \theta_0 \quad \text{a.e. in} \ \Omega.
\end{align*}
\]

Then, we have

\[
0 \leq u \leq \theta_* \quad \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack.
\] (2.24)

Proof. We multiply Eq. (2.22) by the function \(-u^-\), where

\[
u^- := \max\{0,-u\} \in L^2(0,T;H)\]

and take the integral over \( \Omega \times \lbrack 0,T \rbrack \), for \( t \in \lbrack 0,T \rbrack \). Accounting for (2.4), and observed that (2.20) and a comparison in (2.22) ensure that \( vu \in L^2(0,T;H)\), we obtain

\[
\frac{1}{2} \| u^-(t) \|^2 + \int_0^t \int_\Omega \| \nabla (u^-) \|^2 \, dx \, ds = -\int_0^t \int_\Omega (u^-)^2 v \, dx \, ds,
\]

and the integral on the right-hand side of the previous equation is non-positive since we have \( v \geq 0 \ \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack \). Then, one infers that \( \| u^-(t) \| = 0 \) for all \( t \in \lbrack 0,T \rbrack \), hence \( u \geq 0 \ \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack \).

Next, we multiply (2.22) by the function

\[
(u - \theta_*)^+ := \max\{0,u - \theta_*\} \in L^2(0,T;H)
\]

and take the integral over \( \Omega \times \lbrack 0,t \rbrack \), for \( t \in \lbrack 0,T \rbrack \).

By virtue of (2.4), we may infer that

\[
\frac{1}{2} \| (u - \theta_*)^+(t) \|^2 + \int_0^t \int_\Omega \| \nabla (u - \theta_*)^+ \|^2 \, dx \, ds = -\int_0^t \int_\Omega (u - \theta_*)^+ v \, dx \, ds.
\]

Owing to the non-negativity of \( v \) (see (2.21)), the right-hand side above is once again non-positive, that is \( u \leq \theta_* \ \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack \), and (2.24) holds. \( \square \)

As a first application of the previous result, note that (2.18) may be simply inferred from (2.1)–(2.2), (2.4), (2.9)–(2.10), (2.13), and (2.16)–(2.17). Indeed, note that relation (2.16) forces \( \partial_t \chi \) to take value in \( D(\alpha) \ \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack \). Hence, we have that

\[
\partial_t \chi \geq 0 \quad \text{a.e. in} \ \Omega \times \lbrack 0,T \rbrack.
\] (2.25)
3. Proof of Theorem 2.2

3.1. Approximation

For the sake of proving Theorem 2.2, we apply a regularization procedure to the maximal monotone graph $\beta$. Namely, let $\beta_\varepsilon$ be the Yosida approximation of $\beta$ (we refer to Brezis\textsuperscript{6} for details) and, consequently, denote by $j_\varepsilon$ the unique primitive of $\beta_\varepsilon$ verifying $j_\varepsilon(0) = 0$. Note that (see p. 28 of Brezis\textsuperscript{6}) one has

$$0 \leq j_\varepsilon(r) \leq j(r) \quad \text{for all } \varepsilon > 0 \quad \text{and} \quad r \in \mathbb{R}. \quad (3.1)$$

We are now able to state and prove an existence and uniqueness result for the regularized problem.

**Theorem 3.1.** (Regularized problem) Let assumptions $(2.1)$--$(2.6)$ hold. Then, there exists a unique quadruple $(\theta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)$ such that

$$\theta_\varepsilon \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \quad (3.2)$$

$$\chi_\varepsilon \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \quad (3.3)$$

$$\xi_\varepsilon \in W^{1,\infty}(0,T;H), \quad (3.4)$$

$$\eta_\varepsilon \in L^\infty(0,T;H), \quad (3.5)$$

$$\partial_t \theta_\varepsilon + \theta_\varepsilon \partial_t \chi_\varepsilon - \Delta \theta_\varepsilon = 0 \quad \text{a.e. in } \Omega \times ]0,T[, \quad (3.6)$$

$$\partial_t \chi_\varepsilon + \eta_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon = \theta_\varepsilon - \theta_c \quad \text{a.e. in } \Omega \times ]0,T[, \quad (3.7)$$

$$\xi_\varepsilon = \beta_\varepsilon(\chi_\varepsilon) \quad \text{a.e. in } \Omega \times ]0,T[, \quad (3.8)$$

$$\eta_\varepsilon \in \alpha(\partial_t \chi_\varepsilon) \quad \text{a.e. in } \Omega \times ]0,T[, \quad (3.9)$$

$$\theta_\varepsilon(\cdot,0) = \theta_0, \quad \chi_\varepsilon(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \quad (3.10)$$

Moreover, it turns out that

$$0 \leq \theta_\varepsilon \leq \theta_\ast \quad \text{a.e. in } \Omega \times ]0,T[. \quad (3.11)$$

**Remark 3.2.** (Proof of Proposition 2.4) Let us stress that it is possible to prove Theorem 3.1 for any Lipschitz continuous function $\beta_c$. Indeed, our proof does not rely on the particular choice of the Yosida approximation of $\beta$. Thus, assuming $(2.19)$, no regularization on $\beta$ is necessary and Theorem 3.1 provides a proof of Proposition 2.4 as well.

Our aim is to prove this result by devising a fixed point argument. To this end, let us state here two preliminary lemmas which will be crucial in the sequel.

**Lemma 3.3.** Let $(2.1)$ and $(2.4)$ hold and choose

$$v \in L^2(0,T;H) \quad \text{and} \quad v \geq 0 \text{ a.e. in } \Omega \times ]0,T[. \quad (3.12)$$
Then, there exists a unique function $u$ such that
\begin{align}
  u & \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W) \\
  \begin{cases}
    \partial_t u - \Delta u = -vu & \text{a.e. in } \Omega \times [0,T[ \\
    u(\cdot,0) = \theta_0 & \text{a.e. in } \Omega.
  \end{cases}
\end{align}

Moreover, we have that
\begin{equation}
0 \leq u \leq \theta_* \text{ a.e. in } \Omega \times [0,T[.
\end{equation}

This lemma may be proved by referring to standard results on parabolic PDEs and performing a simple fixed point argument. We point out that the assumption on the non-negativity of $v$ plays a crucial role in order to achieve the regularity (3.13). Indeed, by virtue of a maximum principle analogous to that of Lemma 2.5, the hypothesis $v \geq 0$ ensures that every solution to (3.14) satisfies (3.15), so that $-vu \in L^2(0,T;H)$, which leads exactly to the regularity (3.13). Of course, some care should be taken in a rigorous proof, since in this setting the boundedness (3.15) and the regularity (3.13) are to be obtained together (compare with Lemma 2.5, where (2.20) was a hypothesis instead).

**Lemma 3.4.** Let (2.1)–(2.3) and (2.5)–(2.6) hold and choose
\begin{equation}
  u \in H^1(0,T;H).
\end{equation}

Then, there exists a unique solution
\begin{align}
v & \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \\
  \begin{cases}
    \partial_t v + \alpha(\partial_t v) - \Delta v + \beta_\varepsilon(v) \ni u - \theta_c & \text{a.e. in } \Omega \times [0,T[ \\
    v(\cdot,0) = \chi_0 & \text{a.e. in } \Omega.
  \end{cases}
\end{align}

Moreover, if we assume that (cf. (2.6))
\begin{equation}
(1 + \alpha)^{-1}(\theta_0 - \theta_c + \Delta \chi_0 - \xi_0) \in V,
\end{equation}
we achieve the further regularity
\begin{equation}
v \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V).
\end{equation}

**Outline of the Proof of Lemma 3.4.** We prefer not to go into details of this argument since it is quite close to the investigations devised in Bonfanti–Frémond–Luterotti. According to the above cited paper, one substitutes $\alpha$ with its Yosida approximation $\alpha_\delta$ and studies the time-discrete scheme
\begin{align}
  \begin{cases}
    (1 + \alpha_\delta) \left( \frac{v_i - v_{i-1}}{\tau} \right) - \Delta v_i + \beta_\varepsilon(v_i) = u_i - \theta_c \\
    \text{a.e. in } \Omega, \quad \text{for } i = 1, \ldots, N, \\
    v_0 = \chi_0 & \text{a.e. in } \Omega,
  \end{cases}
\end{align}
where $N \in \mathbb{N}$, $\tau := T/N$ denotes the time step and \( \{u_i\}_{i=1}^N \) is a suitable approximation of the function $u$, namely (cf. (3.16))

$$u_i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} u(s) \, ds \quad \text{for} \quad i = 1, \ldots, N.$$  

Then, the unique solution to (3.17) is achieved by establishing suitable \textit{a priori} estimates for the solution to (3.20) which are independent of both $\delta$ and $\tau$ (not on $\varepsilon$, anyway), and passing to the limit firstly as $\tau \to 0$ and then as $\delta \to 0$.

The regularity (3.19) may be proved under further assumption (3.18) by approximating (3.17) (with $\alpha_3$ instead of $\alpha$) in a completely different manner. Indeed, it is possible to apply the well-known Faedo–Galerkin method, and then pass to the limit firstly in the dimension of the approximating space and then as $\delta \to 0$.

\textbf{Proof of Theorem 3.1.} Consider the sets

$$X(\tau) = \{ w \in L^2(0, \tau; H) \text{ such that } w \geq 0 \text{ a.e. in } \Omega \times [0, \tau) \},$$

$$Y(\tau) = \{ w \in H^1(0, \tau; H) \text{ such that } 0 \leq w \leq \theta, \text{ a.e. in } \Omega \times [0, \tau) \},$$

where $\tau \in [0, T]$ is to be fixed later. First, we define the operator $T_1 : X(\tau) \to Y(\tau)$ mapping $v \mapsto T_1(v)$, where $T_1(v)$ stands for the unique solution (with datum $v$ and up to time $\tau$) of the problem (3.14). Indeed, we have that

$$\partial_t T_1(v) - \Delta T_1(v) = -v T_1(v) \quad \text{a.e. in } \Omega \times [0, \tau].$$

Next, we denote by $T_2 : Y(\tau) \to X(\tau)$ the operator that maps $u \mapsto T_2(u)$, where $T_2(u)$ represents the derivative in time of the unique solution (up to time $\tau$) to problem (3.17) where $u$ is assumed. Letting $*$ denote the standard convolution product on $]0, T[$, namely $(a * b)(t) := \int_0^t a(t-s)b(s) \, ds$, and fixed

$$w := \chi_0 + 1 * T_2(u) \quad \text{for } t \in [0, \tau],$$

one infers that

$$T_2(u) + \alpha(T_2(u)) - \Delta w + \beta_\varepsilon(w) \ni u - \theta \quad \text{a.e. in } \Omega \times [0, \tau].$$

By virtue of Lemmas 3.3 and 3.4, the operator

$$\mathcal{T} := T_2 \circ T_1 : X(\tau) \to X(\tau),$$

can be defined. Our next aim is to prove its local (hence depending on $\tau$) contraction character. To this end, choose $v_1, v_2 \in X(\tau)$ and let $u_1 := T_1(v_1)$, $u_2 := T_1(v_2)$, and (see (3.24)) $w_1 := \chi_0 + 1 * T_2(u_1)$, $w_2 := \chi_0 + 1 * T_2(u_2)$. Set also

$$\tilde{v} := v_1 - v_2, \quad \tilde{u} := u_1 - u_2, \quad \tilde{T} := T(v_1) - T(v_2), \quad \tilde{w} := w_1 - w_2.$$

By writing (3.25) for both $u_1$ and $u_2$, and taking the difference, we obtain

$$\tilde{T} + \eta_1 - \eta_2 - \Delta \tilde{w} + \beta_\varepsilon(w_1) - \beta_\varepsilon(w_2) = \tilde{u} \quad \text{a.e. in } \Omega \times [0, \tau],$$

$$\eta_i \in \alpha(\mathcal{T}(v_i)) \quad \text{a.e. in } \Omega \times [0, \tau], \quad \text{for } i = 1, 2.$$
Theorem 2.1 in Baiocchi thus, an application of Gronwall lemma (see, e.g. the version reported in following).

On the other hand, take the difference between (3.23) written for $v_1$ and the same equation in $v_2$, multiply by $\tilde{u}$, and take the integral over $\Omega \times [0, \tau]$, for $t \in [0, \tau]$. Since $\tilde{u}(\cdot, 0) = 0$ a.e. in $\Omega$, Young theorem yields

$$\|\tilde{u}\|_{L^2(0, \tau; H)}^2 = \|1 - T\|_{L^2(0, \tau; H)}^2 \leq \tau^2 \|\tilde{T}\|_{L^2(0, \tau; H)}^2.$$  

Finally, reporting in (3.28), and choosing

$$\tau < \varepsilon/2,$$

one infers that

$$\|\tilde{T}\|_{L^2(0, \tau; H)}^2 \leq 4\|\tilde{u}\|_{L^2(0, \tau; H)}^2.$$  

Now, multiply (3.26) by $\tilde{T}$ and take the integral over $\Omega \times [0, \tau]$ (recall that $\tau$ is chosen in $[0, T]$). By virtue of the monotonicity of $\alpha$ and the Lipschitz continuity of $\beta_\varepsilon$ (with Lipschitz constant equal to $1/\varepsilon$), we infer that

$$\|\tilde{T}\|_{L^2(0, \tau; H)}^2 + \frac{1}{2} \int_0^\tau \|\nabla \tilde{w}(\tau)\|^2 d\tau$$

$$\leq \frac{1}{\varepsilon} \int_0^\tau \int_\Omega |\tilde{w}| |\tilde{T}| d\tau d\sigma + \int_0^\tau \int_\Omega |\tilde{u}| |\tilde{T}| d\tau d\sigma$$

$$\leq \frac{1}{2} \|\tilde{T}\|_{L^2(0, \tau; H)}^2 + \frac{1}{\varepsilon^2} \|\tilde{u}\|_{L^2(0, \tau; H)}^2 + \|\tilde{u}\|_{L^2(0, \tau; H)}^2.$$  

Then, since $\tilde{w}(\cdot, 0) = 0$ a.e. in $\Omega$, Young theorem yields

$$\|\tilde{u}\|_{L^2(0, \tau; H)}^2 = \|1 - T\|_{L^2(0, \tau; H)}^2 \leq \tau^2 \|\tilde{T}\|_{L^2(0, \tau; H)}^2.$$  

Owing to (3.21), it is now straightforward to deduce that $I_1(t)$ is non-positive for all $t \in [0, \tau]$. As regards $I_2(t)$, the uniform boundedness of $u_2$ (see (3.15)) allows the following

$$I_2(t) \leq \frac{\theta^2}{2} \|\tilde{v}\|_{L^2(0, t; H)}^2 + \frac{1}{2} \|\tilde{u}\|_{L^2(0, t; H)}^2,$$

thus, an application of Gronwall lemma (see, e.g. the version reported in Theorem 2.1 in Baiocchi) ensures that, for all $t \in [0, \tau]$,

$$\|\tilde{u}(t)\|^2 \leq \theta^2 e^t \|\tilde{v}\|_{L^2(0, t; H)}^2.$$  

The previous relation along with (3.30) implies that

$$\|\tilde{T}\|_{L^2(0, \tau; H)}^2 \leq 4\tau \|\tilde{u}\|_{C^0([0, \tau]; H)}^2 \leq 4\theta^2 \tau e^\tau \|\tilde{v}\|_{L^2(0, \tau; H)}^2.$$


Finally, with the choice (cf. (3.29))
\[
\tau \in \min \left\{ \varepsilon, \frac{1}{2\epsilon^2} \right\},
\]
\(\mathcal{T}\) turns out to be a contraction mapping from the closed subset \(X(\tau)\) of \(L^2(0, \tau; H)\) into itself. This proves that there exists a unique quadruple \((\theta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)\) fulfilling (up to time \(\tau\)) relations (3.2)–(3.11) (for the proof of (3.3)–(3.4), you can proceed by comparison in the equation in (3.17); the other regularities are part of Lemmas 3.3 and 3.4). Furthermore, owing to the next subsection, it is possible to find \textit{a priori} estimates for \((\theta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)\) independent of \(\tau\) and \(\varepsilon\), which allow us to extend our local unique solution to a global one, defined on the whole interval \([0, T]\). Thus, Theorem 3.1 will be completely proved. In this concern, we prefer, for the sake of clarity, to establish the following \textit{a priori} estimates directly for a global solution on \([0, T]\) instead of carrying on with the parameter \(\tau\).

\[\square\]

3.2. \textit{A priori} estimates

We are now interested in deducing some estimates for \((\theta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon)\) independent of the parameter \(\varepsilon\). Henceforth, let \(C\) denote any constant, possibly depending on \(\theta_\varepsilon, \chi_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \|j(\chi_0)\|_{L^1(\Omega)}, \|\theta_0\|_{V},\) and \(T\) but not on \(\varepsilon\). Of course, \(C\) may vary from line to line.

First estimate. Multiplying (3.6) by \(\partial_t \theta_\varepsilon\) and taking the integral on \(\Omega \times [0, T]\), for \(t \in [0, T]\), it is straightforward to check that
\[
\frac{1}{2}\|\theta_\varepsilon(t)\|^2 + \|\nabla \theta_\varepsilon\|_{L^2(0, t; L^2(\Omega))^N}^2 \leq \frac{1}{2}\|\theta_0\|^2 - \int_0^t \int_\Omega (\theta_\varepsilon)^2 \partial_t \chi_\varepsilon \, dx \, ds,
\]
and the last integral is non-positive due to the fact that \(\partial_t \chi_\varepsilon \in D(\alpha)\) a.e. in \(\Omega \times [0, T]\). Thus, owing to (2.4), we conclude that
\[
\|\theta_\varepsilon\|_{L^\infty(0, T; H) \cap L^2(0, T; V)} \leq C.
\]

Second estimate. Multiply (3.6) by \(\partial_t \theta_\varepsilon\) and take the integral over \(\Omega \times [0, T]\), for \(t \in [0, T]\). By virtue of (3.11), we infer
\[
\|\partial_t \theta_\varepsilon\|_{L^2(0, t; H)}^2 + \frac{1}{2}\|\nabla \theta_\varepsilon\|^2 \leq \frac{1}{2}\|\nabla \theta_\varepsilon\|^2 + C \int_0^t \|\partial_t \theta_\varepsilon(s)\| \|\partial_t \chi_\varepsilon(s)\| \, ds.
\]

On the other hand, multiply (3.7) by \(\partial_t \chi_\varepsilon\) and integrate in space and time. Since relations (3.9) and (3.11) hold, by exploiting the monotonicity of \(\alpha\), one has
\[
\|\partial_t \chi_\varepsilon\|_{L^2(0, t; H)}^2 + \frac{1}{2}\|\nabla \chi_\varepsilon\|^2 + \int_\Omega j_\varepsilon(\chi_\varepsilon(t)) \, dx 
\leq \frac{1}{2}\|\nabla \chi_0\|^2 + \int_\Omega j_\varepsilon(\chi_0) \, dx + \int_0^t ||(\theta_\varepsilon - \theta_\varepsilon)(s)|| \|\partial_t \chi_\varepsilon(s)\| \, ds.
\]
Thus, recalling (2.3), (2.7), (3.1) and (3.11), we conclude that
\[
\| \partial_t \chi_\varepsilon \|_{L^2(0,T;H)} + \| \nabla \chi_\varepsilon \|_{L^\infty(0,T;(L^2(\Omega))^N)} + \| J_\varepsilon(\chi_\varepsilon) \|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{3.34}
\]

So, easy calculations and (3.33) allow the following estimate as well
\[
\| \partial_t \theta_\varepsilon \|_{L^2(0,T;H)} + \| \nabla \theta_\varepsilon \|_{L^\infty(0,T;(L^2(\Omega))^N)} \leq C. \tag{3.35}
\]

**Consequent estimate.** By a comparison in (3.6) one deduces that
\[
\| \Delta \theta_\varepsilon \|_{L^2(0,T;H)} \leq C,
\]
which, along with standard results on elliptic regularity, entails \( \theta \in C^0([0,T];V) \) and
\[
\| \theta_\varepsilon \|_{L^2(0,T;W)} \leq C. \tag{3.36}
\]

**Third estimate.** As far as we are interested in deducing some estimates for \( \xi_\varepsilon \) and \( \eta_\varepsilon \), we multiply (3.7) by \( \partial_t (\Delta \chi_\varepsilon + \xi_\varepsilon) \). We stress that, for the present, this choice is just formal. However, the reader is referred to the forthcoming Remark 3.5 for a motivation. Integrating on \( \Omega \times [0,T] \) for \( t \in [0,T] \), one has
\[
\| \nabla (\partial_t \chi_\varepsilon) \|_{L^2(0,T;H)}^2 + \int_0^t \int_\Omega \beta'_\varepsilon(\chi_\varepsilon)(\partial_t \chi_\varepsilon)^2 \, dx \, ds + \frac{1}{2} \| (\Delta \chi_\varepsilon + \xi_\varepsilon)(t) \|^2 \]
\[
- \int_0^t \int_\Omega \eta_\varepsilon \Delta (\partial_t \chi_\varepsilon) \, dx \, ds + \int_0^t \int_\Omega \eta_\varepsilon \beta'_\varepsilon(\chi_\varepsilon) \partial_t \chi_\varepsilon \, dx \, ds
\]
\[
= \frac{1}{2} \| \Delta \chi_0 + \beta_\varepsilon(\chi_0) \|^2 + \int_0^t \int_\Omega (\theta_\varepsilon - \theta_0)(-\Delta \chi_\varepsilon + \xi_\varepsilon) \, dx \, ds, \tag{3.37}
\]
where \( \beta'_\varepsilon \) stands obviously for the derivative of \( \beta_\varepsilon \). Note that the monotonicity of \( \beta_\varepsilon \) ensures that \( \beta'_\varepsilon(\chi_\varepsilon)(\partial_t \chi_\varepsilon)^2 \geq 0 \) a.e. on \( \Omega \times [0,T] \). For the sake of dealing with the two integrals containing \( \eta_\varepsilon \) on the left hand side of (3.37), we observe that relation (3.9) implies that the set \( \{ \eta_\varepsilon \neq 0 \} \) (with obvious notation) is essentially contained in \( \{ \partial_t \chi_\varepsilon = 0 \} \). Thus, the quantity \( \eta_\varepsilon \partial_t \chi_\varepsilon \) vanishes a.e. in \( \Omega \times [0,T] \), and both the referred integrals are equal to zero. Moreover, by performing an integration by parts and easy calculations, we deduce that
\[
\int_0^t \int_\Omega (\theta_\varepsilon - \theta_0)(\partial_t (\Delta \chi_\varepsilon + \xi_\varepsilon)) \, dx \, ds \leq \int_\Omega (\theta_\varepsilon(t) - \theta_0)(-\Delta \chi_\varepsilon + \xi_\varepsilon)(t) \, dx
\]
\[
+ \int_\Omega (\theta_0 - \theta_\varepsilon)(-\Delta \chi_\varepsilon + \beta_\varepsilon(\chi_0)) \, dx + \int_0^t \int_\Omega \partial_t \theta_\varepsilon(-\Delta \chi_\varepsilon + \xi_\varepsilon) \, dx \, ds
\]
\[
\leq \| \theta_\varepsilon(t) - \theta_\varepsilon \|^2 + \frac{4}{2} \| (\Delta \chi_\varepsilon + \xi_\varepsilon)(t) \|^2 + \frac{1}{2} \| \theta_0 - \theta_\varepsilon \|^2 + \frac{1}{2} \| -\Delta \chi_0 + \beta_\varepsilon(\chi_0) \|^2
\]
\[
+ \frac{1}{2} \int_0^t \| (\Delta \chi_\varepsilon + \xi_\varepsilon)(s) \|^2 \, ds + \frac{1}{2} \| \partial_t \theta_\varepsilon \|_{L^2(0,T;H)}^2.
\]
By reporting in (3.37), accounting for (2.4)–(2.5), (3.35), and applying Gronwall lemma, we deduce that

\[ \| \nabla (\partial_t \chi) \|_{L^2(0,T;H)} + \| -\Delta \chi + \xi \|_{L^\infty(0,T;H)} \leq C. \]

Hence, by using the monotonicity of \( \beta_\varepsilon \) along with standard elliptic estimates, we infer that

\[ \| \chi_\varepsilon \|_{L^\infty(0,T;W)} + \| \xi_\varepsilon \|_{L^\infty(0,T;H)} \leq C. \]  

Finally, writing (3.7) as

\[ \partial_t \chi_\varepsilon = (1 + \alpha)^{-1}(\theta_\varepsilon - \theta_c + \Delta \chi_\varepsilon - \xi_\varepsilon) \]

and owing to (2.1), (3.32), (3.38), and the Lipschitz continuity of \((1 + \alpha)^{-1}\), we achieve that

\[ \| \partial_t \chi_\varepsilon \|_{L^\infty(0,T;H)} \leq C. \]  

Hence, a comparison in (3.7) yields the estimate

\[ \| \eta_\varepsilon \|_{L^\infty(0,T;H)} \leq C. \]  

\textbf{Remark 3.5.} As we have already noted, the previous estimate has to be made rigorous; the point is the (lack of) regularity of the factor \(-\partial_t \Delta \chi_\varepsilon\). We give here the main steps of a regularization procedure which, although classical, is clearly displayed in the last paragraph of Colli–Gilardi–Grasselli.\(^7\) In particular, let us set for the sake of simplicity \( u = -\Delta \chi_\varepsilon \). Then, for any \( \delta > 0 \), we define the function \( u_\delta \) as the unique solution to the problem

\[ u_\delta(t) \in V \quad \text{and} \quad u_\delta(t) + \delta^2 \mathcal{J} u_\delta(t) = u(t) \quad \text{for a.e.} \ t \in [0,T], \]

where \( \mathcal{J} \) stands for the Riesz isomorphism between \( V \) and \( V' \). According to Colli–Gilardi–Grasselli\(^7\) and relation (3.3), we obtain that \( u_\delta \in H^2(0,T;V) \) (actually much more is true). Hence, one may multiply (3.7) by the function

\[ \partial_t(\xi_\varepsilon + u_\delta) \in L^2(0,T,H), \]

and integrate in space and time. By devising some calculation in the direction of (3.38) and taking advantage of Proposition 6.1 of Colli–Gilardi–Grasselli,\(^7\) we can eventually pass to the limit as \( \delta \to 0 \) and establish rigorously the desired estimate.

\textbf{Remark 3.6.} We wish to further emphasize that the above described troubles, concerning the derivation of space regularity estimates for Eq. (2.14), are due to the presence of the term \( \alpha(\partial_t \chi) \) (i.e. to the irreversibility). Indeed, if that relation is multiplied by \(-\Delta \chi_\varepsilon\), which is an admissible test function, we note that the product \(-\alpha(\partial_t \chi_\varepsilon)\Delta \chi \) does not necessarily give a non-negative contribution. Hence, we are forced to use a test function containing a higher order derivative (i.e. \(-\partial_t \Delta \chi_\varepsilon\)) and some form of approximation has to be performed in order that the procedure makes sense in a rigorous setting.
3.3. Passage to the limit

Thanks to well-known compactness results, the estimates (3.11), (3.34)–(3.36), and (3.38)–(3.40) allow us to deduce (possibly taking subsequences) the following convergences

\[
\theta_{\varepsilon} \rightarrow \theta \quad \text{weakly in } H^1(0, T; H) \cap L^2(0, T; W),
\]

\[
\theta_{\varepsilon} \rightarrow \theta \quad \text{weakly star in } L^\infty(0, T; V) \cap L^\infty(\Omega \times ]0, T[),
\]

\[
\chi_{\varepsilon} \rightarrow \chi \quad \text{weakly in } H^1(0, T; V),
\]

\[
\chi_{\varepsilon} \rightarrow \chi \quad \text{weakly star in } L^\infty(0, T; W) \cap W^{1,\infty}(0, T; H),
\]

\[
\xi_{\varepsilon} \rightarrow \xi \quad \text{weakly in } L^\infty(0, T; H),
\]

\[
\eta_{\varepsilon} \rightarrow \eta \quad \text{weakly star in } L^\infty(0, T; H).
\]

Moreover, from (3.32)–(3.35) and the generalized Ascoli theorem (see, e.g., Corollary 4 in Simon\textsuperscript{10}), we may also infer the convergences

\[
\theta_{\varepsilon} \rightarrow \theta \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V),
\]

\[
\chi_{\varepsilon} \rightarrow \chi \quad \text{strongly in } C^0([0, T]; V).
\]

Besides, from (3.44) and (3.47), we achieve that

\[
\theta_{\varepsilon} \partial_t \chi_{\varepsilon} \rightarrow \theta \partial_t \chi \quad \text{weakly star in } L^\infty(0, T; H).
\]

The above convergences suffice to write the limit equations (2.13)–(2.14), and to ensure relations (2.9)–(2.12) along with the Cauchy conditions (2.17). Thus, we only have to prove inclusions (2.15) and (2.16). First of all, note that (3.45) and (3.48) allow the following convergence for every \( t \in ]0, T[\)

\[
\int_0^t \int_\Omega \xi_{\varepsilon} \chi_{\varepsilon} \, dx \, ds \rightarrow \int_0^t \int_\Omega \xi \chi \, dx \, ds.
\]

Whence, owing to the classical result Proposition 2.5, p. 27 in Brezis,\textsuperscript{6} we are able to conclude for (2.15).

On the other hand, multiply Eq. (3.7) by \( \partial_t \chi_{\varepsilon} \) and integrate over \( \Omega \times ]0, T[ \). It is straightforward to deduce that

\[
\int_0^T \int_\Omega \eta_{\varepsilon} \partial_t \chi_{\varepsilon} \, dx \, dt = \int_0^T \int_\Omega (-\partial_t \chi_{\varepsilon} - \xi_{\varepsilon} + \Delta \chi_{\varepsilon} + \theta_{\varepsilon} - \theta_{\varepsilon}) \partial_t \chi_{\varepsilon} \, dx \, dt
\]

\[
= -\|\partial_t \chi_{\varepsilon}\|_{L^2(0, T; H)}^2 - \int_\Omega j_{\varepsilon}(\chi_{\varepsilon}(T)) \, dx + \int_\Omega j_{\varepsilon}(\chi_{\varepsilon}(0)) \, dx
\]

\[
- \int_0^T \int_\Omega \nabla \chi_{\varepsilon} \cdot \nabla \partial_t \chi_{\varepsilon} \, dx \, dt + \int_0^T \int_\Omega (\theta_{\varepsilon} - \theta_{\varepsilon}) \partial_t \chi_{\varepsilon} \, dx \, dt.
\]
Next, we take the lim sup as $\varepsilon \to 0$ on both sides of (3.49). By explicit integration in time, we have that

$$
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega} j_\varepsilon(\varepsilon(T)) \, dx = -\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega} j_\varepsilon(\varepsilon(T)) \, dx \leq - \int_{\Omega} j(\varepsilon(T)) \, dx \tag{3.49}
$$

and the last inequality holds since the functional induced by $j_\varepsilon$ on $H$, namely

$$
J_\varepsilon(v) := \int_{\Omega} j_\varepsilon(v) \, dx \quad \text{for } v \in H,
$$

converges to $J(v)$ (cf. (2.8) for the definition of $J$) in the sense of Mosco (see Proposition 3.56, p. 354 in Attouch$^1$) and, furthermore, by virtue of (3.43) we have

$$
\chi_\varepsilon(T) \rightharpoonup \chi(T) \text{ strongly in } H.
$$

Thus, thanks to (3.1), (3.43), (3.47)–(3.49) we conclude for

$$
\limsup_{\varepsilon \to 0} \int_0^T \int_{\Omega} \eta_\varepsilon \partial_t \chi_\varepsilon \, dx \, dt \leq -\|\partial_t \chi\|_{L^2(0,T;H)}^2 - \int_{\Omega} j(\chi(T)) \, dx + \int_{\Omega} j(\chi_0) \, dx
$$

$$
- \int_0^T \int_{\Omega} \nabla \chi \cdot \nabla \partial_t \chi \, dx \, dt + \int_0^T \int_{\Omega} (\theta - \theta_c) \partial_t \chi \, dx \, dt.
$$

Recalling (2.14), the right-hand side of the previous equation may be rewritten as

$$
\int_0^T \int_{\Omega} (\partial_t \chi - \xi + \Delta \chi + \theta - \theta_c) \partial_t \chi \, dx \, dt = \int_0^T \int_{\Omega} \eta \partial_t \chi \, dx \, dt,
$$

hence, one infers that

$$
\limsup_{\varepsilon \to 0} \int_0^T \int_{\Omega} \eta_\varepsilon \partial_t \chi_\varepsilon \, dx \, dt \leq \int_0^T \int_{\Omega} \eta \partial_t \chi \, dx \, dt.
$$

Finally, owing to convergences (3.43), (3.46) and applying again the Proposition 2.5, p. 27 in Brezis,$^6$ relation (2.16) is established and Theorem 2.2 is completely proved.

References