Existence results for a phase transition model based on microscopic movements

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Abstract

A nonlinear system of PDEs accounting for phase transition phenomena with strong dissipation is considered. The global existence of solutions to a Cauchy-Neumann problem is proved in the one dimensional space setting, using a regularization - a priori estimates - passage to the limit procedure. An asymptotic analysis is also provided, when the dissipation vanishes.

Key words: phase transitions, microscopic movements, existence result.

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1 Introduction and preliminaries

In this paper we study a Cauchy-Neumann problem related to the following system

\begin{align*}
\partial_t \theta + \theta \partial_t \chi - \partial_{xx} \theta &= (\partial_t \chi)^2 + k(\partial_{xt} \chi)^2, \quad \text{in } Q \tag{1.1} \\
\partial_t \chi - k \partial_{xt} \chi - \partial_{xx} \chi + \beta(\chi) &\ni \theta - \theta_c, \quad \text{in } Q \tag{1.2}
\end{align*}

\begin{tabular}{l}
\textbf{energy} \hspace{1cm} \\
\textbf{phase}
\end{tabular}
for some $\ell, T > 0$, where $k$ and $\theta_\ast$ are positive constants, $Q := (0, \ell) \times (0, T)$, and $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

We set
\[ \Omega := [0, \ell[, \quad Q_t := ]0, \ell[ \times ]0, t[ \quad \forall t \in ]0, T], \quad Q := Q_T. \]

Next, we let
\[ H := L^2(\Omega), \quad V := H^1(\Omega), \]
\[ W := \{ u \in H^2(\Omega) \text{ such that } \partial_x u(0) = \partial_x u(\ell) = 0 \}, \tag{1.3} \]
and identify $H$ with its dual space $H'$, so that
\[ V \subset H \subset V', \]
with dense, compact, and continuous embeddings. Besides, we let the symbol $\| \cdot \|$ denote the standard norm of $H$, while $\| \cdot \|_E$ stands for the norm of the general normed space $E$. Moreover, we denote by $< \cdot, \cdot >$ the duality pairing between $V'$ and $V$, by $(\cdot, \cdot)$ the scalar product in $H$, and by $J : V \to V'$ the Riesz isomorphism of $V$ onto $V'$.

We note that, thanks to the one dimensional framework of our problems, we have the continuous injections
\[ L^1(\Omega) \subset V', \quad V \subset L^\infty(\Omega). \tag{1.4} \]
Hence, there exist two positive constants $c_1$ and $c_2$ such that the following relations hold
\[ \| u \|_{V'} \leq c_1 \| u \|_{L^1(\Omega)}, \quad \forall u \in L^1(\Omega), \]
\[ \| u \|_{L^\infty(\Omega)} \leq c_2 \| u \|_V, \quad \forall u \in V. \tag{1.5} \]

Now, we recall an elementary inequality which will be useful in the sequel
\[ ab \leq (\delta/2)a^2 + (2\delta)^{-1}b^2 \quad \forall a, b \in \mathbb{R}, \quad \delta > 0. \tag{1.6} \]

Finally, we also remark that there exists a positive constant $c_3$ depending only on $T$ such that the following holds for any $u \in H^1(0, T; H)$
\[ \| u \|_{L^2(0, t; H)}^2 \leq c_3 \left( \| u(0) \|^2 + \int_0^t \| \partial_t u \|_{L^2(0, s; H)}^2 ds \right) \quad \forall t \in (0, T]. \tag{1.7} \]

We give here the precise statement of our problem, introducing the following assumptions on the data.

\[ \theta_\ast > 0 \text{ are assigned constant}, \tag{1.8} \]
\[ \varphi : \mathbb{R} \to [0, +\infty] \text{ is proper, convex, and lower semicontinuous} \]
and there exists $c_4, c_5 > 0$ such that
\[ \varphi(r) \geq c_4 r^2 - c_5 \quad \forall r \in D(\varphi) \]
\[ \beta := \partial \varphi, \tag{1.9} \]
\[ \theta_0 \in V \quad \text{and} \quad \theta_0 \geq \theta_\ast \quad \text{in } \bar{\Omega}, \tag{1.10} \]
\[ \chi_0 \in W, \tag{1.11} \]
\[ \chi_0 \in D(\beta) \text{ a.e. in } Q, \quad \text{and} \quad \beta^0(\chi_0) \in H, \tag{1.12} \]
where $D(\varphi)$ and $D(\beta)$ denote the effective domain of $\varphi$ and $\beta$, respectively, and $\beta^0(\chi_0)$ stands for the element of minimal norm of the set $\beta(\chi_0)$.

la frase seguente e’ da modificare: bisogna introdurre $\Phi(u) = \int_{\Omega} \varphi(x) \, dx$\forall u \in L^1(\Omega) e \eta \in \partial_{V^\prime} = \beta_{V^\prime}$?

Now we introduce $\beta_{V^\prime}$, the maximal monotone operator acting from $V$ to $V'$ induce by the graph $\beta$, since we are going to deal with a weak functional setting (see (1.14) and (1.17) below).

**Problem (P).** Find a triplet $(\theta, \chi, \eta)$ such that

\begin{align*}
\theta &\in H^1(0,T; V') \cap C^0([0,T]; H) \cap L^2(0,T; V), \\
\chi &\in W^{1,\infty}(0,T; V), \\
\eta &\in L^\infty(0,T; V'), \\
< \partial_t \theta + \theta \partial_t \chi, v > + (\partial_x \theta, \partial_x v) &= < (\partial_t \chi)^2 + k(\partial_t x)^2, v > \\
\forall v &\in V \quad \text{a.e. in } ]0,T[, \\
< \partial_t \chi + \eta, v > + (k\partial_x \chi + \partial_x \chi, \partial_x v) &= < \theta - \theta_c, v > \\
\forall v &\in V \quad \text{a.e. in } ]0,T[, \\
\exists \theta_0 &> 0 \quad \text{such that } \theta \geq \theta_0 \quad \text{a.e. in } Q_T, \\
\theta(\cdot,0) & = \theta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{align*}

**Remark 1.1.** Let us stress that the coercivity assumption on $\varphi$ in (1.9) is perfectly motivated in our framework since $I_{[0,1]}(r) \geq r^2 - 1$ for all $r \in [0,1]$. Moreover, the hypothesis $\varphi(0) = 0$ may be replaced by the weaker $0 \in D(\varphi)$ without any particular intricacy.

Now, we are able to state the main result of the paper.

**Theorem 1.2.** Let assumptions (1.8)-(1.12) hold. Then Problem 1 admits at least a solution.

**Remark 1.3.** It will be clear that the assumption $\theta_0 \in H$ is sufficient in order that Thm. 1.2 holds. Nevertheless we should need $\theta_0 \in V$ to establish the well-posedness of the approximating problems and for the asymptotic analysis. Hence, for the sake of simplicity we assume $\theta_0 \in V$ in the whole paper instead of considering some suitable approximation.

The proof of this result will be carried out throughout the remainder of the paper by exploiting an approximation procedure. Indeed, we replace $\beta$ with its Yosida approximation $\beta_\varepsilon$ and solve locally (in time) the regularized problem by the means of fixed point techniques. Then, global $a$ priori estimates independent of $\varepsilon$ are established and the passage to the limit is obtained via compactness and monotonicity arguments.
Theorem 1.4. Let \((\theta^k, \chi^k, \eta^k)\) be a solution for the Problem (P). Then there exists a triplet \((\theta^0, \chi^0, \eta^0)\) such that the following convergences hold

\[
\begin{align*}
\theta_k \rightharpoonup \theta & \quad \text{in } H^1(0, T; V') \cap L^2(0, T; V) \cap L^\infty(0, T; H), \\
\chi_k \rightharpoonup \chi & \quad \text{in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^2(0, T; W), \\
k \chi_k \rightharpoonup \chi & \quad \text{in } W^{1,\infty}(0, T; V) \cap L^\infty(0, T; W), \\
\eta_k \rightharpoonup \eta & \quad \text{in } L^\infty(0, T; V').
\end{align*}
\]

and

\[
\begin{align*}
\theta^0 & \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \\
\chi^0 & \in W^{1,\infty}(0, T; V) \cap L^2(0, T; W), \\
\eta^0 & \in L^2(0, T; H), \\
\partial_t \theta^0 + \theta^0 \partial_x \chi^0 - \partial_{xx} \theta^0 = (\partial_t \chi^0)^2 & \quad \text{a.e. in } Q, \\
\partial_t \chi^0 + \eta^0 - \partial_{xx} \chi^0 = \theta - \theta_c & \quad \text{a.e. in } Q, \\
\eta^0 & \in \beta(\chi^0) \quad \text{in } L^2(0, T; H), \\
\exists \theta_0 > 0 \text{ such that } \theta & \geq \theta_0, \quad \text{a.e. in } Q, \\
\theta^0(\cdot, 0) = \theta_0, & \quad \chi^0(\cdot, 0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{align*}
\]

2 Approximation

For the sake of proving Theorem 1.2, we apply a regularization procedure to the maximal monotone graph \(\beta\). Namely, let \(\beta_{\varepsilon}\) be the Yosida approximation of \(\beta\) (we refer to [9] for details) and, consequently, denote by \(\varphi_{\varepsilon}\) the unique primitive of \(\beta_{\varepsilon}\) verifying \(\varphi_{\varepsilon}(0) = 0\). Note that (see [9, page 28]) one has

\[
|\beta_{\varepsilon}(r)| \leq |\beta^0(r)| \quad \text{for all } \varepsilon > 0 \text{ and } r \in D(\beta).
\]

Moreover, it is straightforward to check that

\[
\varphi_{\varepsilon}(r) = \min_{s \in D(\varphi)} \left( \frac{1}{2\varepsilon} |r - s|^2 + \varphi(s) \right). \tag{2.2}
\]

Thus, we readily have that

\[
\varphi_{\varepsilon}(r) \leq \varphi(r) \quad \forall r \in D(\varphi). \tag{2.3}
\]

Moreover, the function \(\varphi_{\varepsilon}\) is defined in all \(\mathbb{R}\) and, taking into account the coercivity assumption in (1.9), it turns out to be coercive as well. Namely, we have that

\[
\varphi_{\varepsilon}(r) \geq \frac{c_1}{2} r^2 - c_3 \quad \forall r \in \mathbb{R}, \quad \forall \varepsilon \in (0, (2c_4)^{-1}). \tag{2.4}
\]

Indeed, let us consider \(r \in \mathbb{R}, s \in D(\varphi), \) and \(\varepsilon \in (0, (2c_4)^{-1})\). Then

\[
\begin{align*}
\varphi_{\varepsilon}(r) & \geq \frac{c_1}{2} r^2 - c_3 \\
& \geq \frac{c_4}{2} r^2 - c_3 + c_5 \leq \frac{1}{2\varepsilon} |r - s|^2 + \varphi(s) + c_5,
\end{align*}
\]
and the assertion (2.4) is proved.

Let us introduce the approximating problems (the regularization parameter \( \varepsilon > 0 \) being fixed).

**Problem** \((P_\varepsilon)\). Find a couple \((\theta_\varepsilon, \chi_\varepsilon)\) such that

\[
\begin{align*}
\theta_\varepsilon &\in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \\
\chi_\varepsilon &\in H^2(0,T;W), \\
\partial_t \theta_\varepsilon + \theta_\varepsilon \partial_t \chi_\varepsilon - \partial_{xx} \theta_\varepsilon = (\partial_t \chi_\varepsilon)^2 + k(\partial_{xx} \chi_\varepsilon)^2, &\text{a.e. in } Q_T, \\
\partial_t \chi_\varepsilon - k \partial_{xxt} \chi_\varepsilon - \partial_{xx} \chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon) = \theta_\varepsilon - \theta_c, &\text{a.e. in } Q_T, \\
\exists \theta_\ast > 0 &\text{ independent of } \varepsilon \text{ such that } \theta_\varepsilon > \theta_\ast \text{ a.e. in } Q_T, \\
\theta_\varepsilon(\cdot,0) = \theta_0, \quad \chi_\varepsilon(\cdot,0) = \chi_0 &\text{ a.e. in } \Omega.
\end{align*}
\]

**Theorem 2.1.** Let assumptions (1.8)-(1.12) hold. Then Problem \((P_\varepsilon)\) admits one and only one solution.

Without loss of generality we will take in the sequel of this Section \( k = 1 \).

We start with the proof on the existence part of Theorem 2.1. To this aim, we apply the Schauder theorem to a suitable operator \( T \) constructed as it will be specified in a moment. For the sake of brevity we will not detail the whole procedure.

For \( R > 0 \), let us consider \( Y(\tau, R) \) the closed ball of \( H^1(0,\tau;W^{1,4}(\Omega)) \) with center 0 and radius \( R \), i.e.

\[
Y(\tau, R) = \{ v \in H^1(0,\tau;W^{1,4}(\Omega)) \text{ such that } \| v \|_{H^1(0,\tau;W^{1,4}(\Omega))} \leq R \},
\]

where \( \tau \in [0,T] \) will be determined later in such a way that \( T : Y(\tau, R) \rightarrow Y(\tau, R) \) is a compact and continuous operator.

We consider the following auxiliary problems whose well-posedness is guaranteed by standard arguments (hence, for the sake of brevity, we omit any detail).

Let \( \hat{x} \in Y(\tau, R) \) be fixed and let \( \bar{\theta} := T_1(\hat{x}) \) be the unique solution to the following problem.

**Problem 1.** Given \( \hat{x} \in Y(\tau, R) \), find \( \bar{\theta} \) such that

\[
\begin{align*}
\bar{\theta} &\in [W^{1,1}(0,\tau;H) + H^1(0,\tau;V')] \cap C^0([0,\tau];H) \cap L^2(0,\tau;V), \\
< \partial_t \bar{\theta}, v > + (\partial_x \partial_x \hat{x}, v) + (\partial_x \bar{\theta}, \partial_x v) = ((\partial_t \hat{x})^2 + (\partial_{xx} \hat{x})^2, v) &\text{a.e. in } ]0,\tau[, \\
\bar{\theta}(\cdot,0) = \theta_0 &\text{ a.e. in } \Omega.
\end{align*}
\]

Now, given such \( \bar{\theta} \), let \( \bar{x} \), with \( \bar{x} := T_2(\bar{\theta}) \), be the unique solution of the following problem.
Problem 2. Given $\overline{v} \in L^2(0, \tau; H)$, find $\overline{\chi}$ such that
\begin{align}
\overline{\chi} &\in H^1(0, \tau; W), \tag{2.15} \\
\partial_t \overline{\chi} - \partial_{xx} \partial_t \overline{\chi} - \partial_{xx} \overline{\chi} + \beta_c(\overline{\chi}) &= \overline{v} - \theta_c \quad \text{a.e. in } Q_\tau, \tag{2.16} \\
\overline{\chi}(\cdot, 0) &= \chi_0 \quad \text{a.e. in } \Omega. \tag{2.17}
\end{align}

Finally, we define the operator $T$ as the composition $T_2 \circ T_1$. Our aim is to show that, at least for small times, the Schauder theorem applies to the map $T$ from $Y(\tau, R)$ into itself. Thus, we will prove that there exists $\tau > 0$ such that $T$ satisfies the following properties

- $T$ maps $Y(\tau, R)$ into itself;
- $T$ is compact;
- $T$ is continuous.

We start by deriving some \textit{a priori} bounds on $\overline{v}$ and $\overline{\chi}$. We warn that in the proofs we employ the same symbol $c$ for different constants (independent of $\tau$ and $R$), even in the same formula, in regard of simplicity. Now, in order to obtain \textit{a priori} bounds on $\overline{v}$, we choose $v = \overline{v}$ in (2.13) and we integrate from 0 to $t$, with $0 < t < \tau$. Owing to (1.6), the Hölder inequality, and the continuous injection $V \hookrightarrow L^4(\Omega)$, we have

\begin{align*}
\frac{1}{2} \|\overline{v}(t)\|^2 + \|\partial_t \overline{v}\|_{L^2(0, t, H)}^2 &\leq \frac{1}{2} \|\theta_0\|^2 + \\
+ c \int_0^t &\|\overline{v}(s)\|_{L^4(\Omega)} \|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)} \|\overline{\theta}(s)\| \, ds + \\
+ \int_0^t &\left( \|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 + \|\partial_x \partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 \right) \|\overline{\theta}(s)\| \, ds \leq \\
&\leq \frac{1}{2} \|\theta_0\|^2 + c \int_0^t \left( \|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)} \|\overline{\theta}(s)\|_{V} + \\
+ &\|\partial_x \partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 \|\overline{\theta}(s)\| \right) \, ds. \tag{2.18}
\end{align*}

Next, in order to recover the full $V$-norm of $\overline{v}$ in the left hand side, we add to (2.18) $\|\overline{\theta}\|_{L^2(0, t, H)}^2$. Then, we use (1.6) and get

\begin{align*}
\frac{1}{2} \|\overline{v}(t)\|^2 + \|\overline{v}\|_{L^2(0, t, V)}^2 &\leq \frac{1}{2} \|\theta_0\|^2 + \frac{1}{2} \|\overline{\theta}\|_{L^2(0, t, V)}^2 + \\
+ c &\int_0^t \left( 1 + \|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 \right) \|\overline{\theta}(s)\|^2 \, ds + \\
+ c &\int_0^t \left( \|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 + \|\partial_x \partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2 \right) \|\overline{\theta}(s)\| \, ds. \tag{2.19}
\end{align*}

Recalling that, by the definition of $Y(\tau, R)$, $\|\partial_t \overline{\chi}(s)\|_{L^4(\Omega)}^2$ and $\left( \|\partial_t \overline{\chi}\|_{L^4(\Omega)}^2 + \|\partial_x \partial_t \overline{\chi}\|_{L^4(\Omega)}^2 \right)$ belong to $L^1(0, \tau)$, we apply to (2.19) a generalized version of the Gronwall lemma introduced in [1] and we deduce that there exists a positive constant $c$ depending on $T$, $\Omega$, and $R$ such that

\begin{align*}
\|\overline{v}\|_{L^\infty(0, \tau; H) \cap L^2(0, \tau, V)} &\leq c. \tag{2.20}
\end{align*}
Next, in order to obtain a priori bounds on $\chi$, we multiply (2.16) by $\partial_t \chi$ and we integrate over $Q_t$. Using the Lipschitz continuity of $\beta$, the Hölder inequality, and relations (1.7)-(1.6), we have

$$
\|\partial_t \chi\|_{L^2(0,t;H)}^2 + \|\partial_x \partial_t \chi\|_{L^2(0,t;H)}^2 + \frac{1}{2}\|\partial_x \chi(t)\|^2 \leq \frac{1}{2}\|\partial_x \chi_0\|^2 + 
+ c \int_{Q_t} (|\chi| + 1)|\partial_t \chi| \int_{Q_t} |(\bar{\theta} - \theta_c)\partial_t \chi| \leq c + \frac{1}{2}\|\partial_x \chi_0\|^2 + 
+ c \int_{0}^{t} \|\partial_x \chi\|^2_{L^2(0,s;H)} ds + c\|\bar{\theta} - \theta_c\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x \chi_0\|^2_{L^2(0,t;H)}.
$$

(2.21) \hfill \text{s4}

Thanks to (2.20), we deduce that there exists a positive constant $c$ such that

$$
\|\chi\|_{H^1(0,\tau;V)} \leq c.
$$

(2.22) \hfill \text{s5}

On account of (1.4), from (2.22) it follows

$$
\|\chi\|_{L^\infty(Q_t)} \leq c,
$$

(2.23) \hfill \text{s6}

for some positive constant $c$.

Next, we multiply (2.16) by $-\partial_{xx} \partial_t \chi$, we integrate over $Q_t$, and, thanks to the Lipschitz continuity of $\beta$, we obtain

$$
\|\partial_x \partial_t \chi\|_{L^2(0,t;H)}^2 + \|\partial_{xx} \partial_t \chi\|_{L^2(0,t;H)}^2 + \frac{1}{2}\|\partial_{xx} \chi(t)\|^2 \leq \n\leq \frac{1}{2}\|\partial_{xx} \chi_0\|^2 + \int_{Q_t} \|\beta_k(\chi)\|_{L^2(0,t;H)} + \int_{Q_t} |(\bar{\theta} - \theta_c)\partial_{xx} \chi| \leq \n\leq \frac{1}{2}\|\partial_{xx} \chi_0\|^2 + c(|\chi|_{L^\infty(\Omega)} + 1) + c\|\bar{\theta} - \theta_c\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_{xx} \partial_t \chi\|^2_{L^2(0,t;H)}.
$$

(2.24) \hfill \text{s7}

Thus, on account of (2.20) and (2.23) from (2.24) we deduce the further bound

$$
\|\chi\|_{H^1(0,\tau;V)} \leq c,
$$

(2.25) \hfill \text{s8}

for some positive constant $c$.

Next, taking the $H$-norm of both sides of (2.16), we have $J\partial_t \chi = \partial_{xx} \chi - \beta_k(\chi) + \bar{\theta} - \theta_c$ ($J$ being the Riesz isomorphism between $V$ and $V'$) and then the sup in $[0, \tau]$, using (2.22), (2.23), (2.20), we have the bound

$$
\|\chi\|_{W^1,\infty(0,\tau;W)} \leq c,
$$

(2.26) \hfill \text{s9}

for some positive constant $c$.

Finally, we differentiate with respect to $t$ both sides of (2.16); thanks to (2.25), (2.12), and the Lipschitz continuity of $\beta_k$, a comparison in the resulting relations give the bound

$$
\|\partial_{tt} \chi\|_{L^1(0,\tau;V)} \leq c
$$

(2.27) \hfill \text{s10}
Now, our aim is to find $\tau > 0$ such that the operator $T : Y(\tau, R) \to Y(\tau, R)$ turns out to be well-defined. Exploiting the previous estimates (cf. (2.26)), we have
\[
\|\chi\|_{W^{1,\infty}(0,\tau;W^{1,4}(\Omega))} \leq c.  \tag{2.28}\]
Thus, by the Hölder inequality, we find
\[
\|\chi\|_{H^1(0,\tau;W^{1,4}(\Omega))} \leq c\sqrt{\tau}\|\chi\|_{W^{1,\infty}(0,\tau;W^{1,4}(\Omega))} \leq c\sqrt{\tau}. \tag{2.29}\]
Hence, we can take $\tau$ so small that
\[
c\sqrt{\tau} \leq R \tag{2.30}\]
and ensure that $\chi$ belongs to $Y(\tau, R)$.

Next, we observe that the above arguments (cf. (2.25) and (2.27)) lead to
\[
\|\chi\|_{W^{2,1}(0,\tau;V) \cap W^{1,\infty}(0,\tau;W)} \leq c, \tag{2.31}\]
for some positive constant $c$ independent of the choice of $\hat{\chi}$ in $Y(\tau, R)$, which ensures that $T$ is a compact operator. Hence, in order to apply the Schauder theorem, it remains to show that $T$ is continuous with respect to the natural topology induced in $Y(\tau, R)$ by $H^1(0, \tau; W^{1,4}(\Omega))$. To this aim, we consider a sequence $\hat{\chi}_n$ in $Y(\tau, R)$ such that
\[
\hat{\chi}_n \to \hat{\chi} \text{ strongly in } Y(\tau, R), \tag{2.32}\]
as $n \to +\infty$. Now, we denote by $\bar{\theta}_n$ the sequence of the solutions to Problem 1 once $\hat{\chi}$ is substituted by $\hat{\chi}_n$, i.e.,
\[
\bar{\theta}_n := T_1(\hat{\chi}_n). \tag{2.33}\]
Arguing as in the derivation of (2.20), we can find a positive constant $c$ not depending on $n$ such that
\[
\|\bar{\theta}_n\|_{L^\infty(0,\tau;H) \cap L^2(0,\tau;V)} \leq c. \tag{2.34}\]
By well-known compactness results, there exists a subsequence of $n$, still denoted by $n$ for the sake of brevity, such that
\[
\bar{\theta}_n \rightharpoonup \bar{\theta} \text{ in } L^\infty(0,\tau;H) \cap L^2(0,\tau;V), \tag{2.35}\]
as $n \to +\infty$. In order to show that $\bar{\theta}$ in (2.35) is the solution to Problem 1 related to $\hat{\chi}$, i.e., $\bar{\theta} = T_1(\hat{\chi})$, we can pass to the limit in (2.13) written at the step $n$ as $n \to +\infty$. Remark that for the nonlinear term we have $\bar{\theta}_n \partial_t \hat{\chi}_n \to \bar{\theta} \partial_t \hat{\chi}$ in $L^2(0,\tau;H)$, thanks to the previous, respectively, weak and strong convergences. Actually, thanks to the uniqueness result holding for Problem 1, we deduce that the whole sequence $\bar{\theta}_n$ converges to $\bar{\theta}$, as $n \to +\infty$.

Second step: we consider the sequence $\chi_n$ of the solutions to Problem 2 once $\bar{\theta}$ is substituted by $\bar{\theta}_n$, i.e., we consider, in particular
\[
\chi_n := T_2(\bar{\theta}_n) = T_2 \circ T_1(\hat{\chi}_n) = T(\hat{\chi}_n). \tag{2.36}\]
Proceeding as in the previous estimates (cf. (2.25) and (2.27)), we can find a positive constant $c$ not depending on $n$ such that
\[
\|\chi_n\|_{W^{2,1}(0,\tau;V) \cap W^{1,\infty}(0,\tau;W)} \leq c, \tag{2.37}\]
Hence, there exists a subsequence of $n$, still denoted by $n$ for the sake of brevity, such that
\[ \chi_n \rightharpoonup^* \chi \quad \text{in} \quad W^{1,\infty}(0, \tau; W) \quad (2.38) \]
as $n \to +\infty$. Moreover, by compactness (see [22]) from (2.27), we can deduce that
\[ \chi_n \to \chi \quad \text{in} \quad H^1(0, \tau; W^{1,4}(\Omega)). \quad (2.39) \]
The above convergences and also (2.35) allows us to pass to the limit in the relation (2.16). Thus, thanks to the uniqueness result holding for Problem 2, we have that the whole sequence $\chi_n$ converges to $T_2(\tilde{\theta}) = T_2 \circ T_1(\hat{\chi}) = T(\hat{\chi})$ and we can identify $\chi$ with $T(\hat{\chi})$.

Thus, by (2.39), we have
\[ T(\hat{\chi}_n) \to T(\hat{\chi}) \quad \text{in} \quad H^1(0, \tau; W^{1,4}(\Omega)) \quad (2.40) \]
which concludes the proof of the continuity of the operator $T$. Thus, we have proved that $T$ has a fixed point in $Y(\tau, R)$, i.e., there exists a local in time solution of the system (2.7)-(2.10), defined on the interval $[0, \tau]$. Note that, at the moment, (2.7) is satisfied only in a weak sense (cf. (2.13)). Actually, performing some standard parabolic estimates, taking advantage of (1.10) and (2.15), we can prove the further regularity for $\theta$ specified in (2.5) so that (2.7) is satisfied a.e. in $Q_\tau$.

Now, we have to discuss the extension of the latter solution to the whole interval $[0, T]$ as well. To this end, we derive some additional a priori estimates which yield suitable global bounds on the solution.

**Positivity of the temperature** A straightforward application of [21, Thm. 1] (in the case where of the viscous phase relaxation) gives the lower bound (1.19) for $\theta$.

### 3 A priori estimates and passage to the limit

Since we are also interested in an asymptotic analysis when $k \to 0$, we are going to deduce some a priori estimates for $(\theta_\varepsilon, \chi_\varepsilon)$ independent of the regularization parameter $\varepsilon$, paying attention at the role of the dissipation terms. Henceforth, let $C$ denote any constant, possibly depending on the data, but not on $\varepsilon$ and $k$. Of course, $C$ may vary from line to line.

#### 3.1 First estimate

Let us integrate (2.7) over $Q_\tau$. Moreover, we multiply (2.8) by $\partial_t \chi_\varepsilon$ and integrate over $Q_\tau$. Taking the sum of the resulting expressions and performing some cancelations, we obtain
that
\[
\int_{\Omega} \theta_\varepsilon(t) + \frac{1}{2} \|\partial_x \chi_\varepsilon(t)\|^2 + \int_{\Omega} \varphi_\varepsilon(\chi_\varepsilon(t)) \\
\leq \int_{\Omega} \theta_0 + \frac{1}{2} \|\partial_x \chi_0\|^2 + \int_{\Omega} \varphi_\varepsilon(\chi_0) + \theta_\varepsilon \left| \int_{\Omega} (\chi_\varepsilon(t) - \chi_0) \right| \\
\leq \int_{\Omega} \varphi(\chi_0) + \theta_\varepsilon \|\chi_\varepsilon(t)\|_{L^1(\Omega)} + C, \tag{3.1}
\]
where we also used assumptions (1.10)-(1.11) and the definition of $\varphi_\varepsilon$. In order to control the right hand side of (3.1) it suffices to recall (1.12) and observe that (see (2.4))
\[
\theta_\varepsilon \|\chi_\varepsilon(t)\|_{L^1(\Omega)} \leq \frac{c_1}{4} \|\chi_\varepsilon(t)\|^2 + C \leq \frac{1}{2} \int_{\Omega} \varphi(\chi_\varepsilon(t)) + C,
\]
whenever $\varepsilon$ is small enough.

Hence, moving from (2.9), we readily deduce that
\[
\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{3.2}
\]
\[
\|\chi_\varepsilon\|_{L^\infty(0,T;V)} \leq C, \tag{3.3}
\]
\[
\|\varphi_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{3.4}
\]
at least for sufficiently small $\varepsilon$.

3.2 Second estimate

Let us multiply equation (2.7) by the function $-\theta_\varepsilon^{-1}$. The latter choice turns out to be admissible due to (2.9) since $-\theta_\varepsilon^{-1} \in L^\infty(Q_T)$. Moreover, we integrate on $Q_I$, and exploit (1.10) and (3.2) in order to get
\[
-\int_{\Omega} \ln \left(\theta_\varepsilon(t)\right) + \int_{\Omega} \int_{Q_I} \left(\frac{\partial_x \theta_\varepsilon}{\theta_\varepsilon^2} + \frac{\partial_t \chi_\varepsilon^2}{\theta_\varepsilon} + \frac{k(\partial_x \partial_t \chi_\varepsilon^2)}{\theta_\varepsilon} \right) = -\int_{\Omega} \ln(\theta_0) + \int_{\Omega} \int_{Q_I} \partial_t \chi_\varepsilon \\
\leq -\int_{\Omega} \ln(\theta_0) + \frac{1}{2} \int_{\Omega} \int_{Q_I} \left(\frac{\partial_x \chi_\varepsilon^2}{\theta_\varepsilon} + \theta_\varepsilon \right) \leq C + \frac{1}{2} \int_{\Omega} \int_{Q_I} \frac{\partial_t \chi_\varepsilon^2}{\theta_\varepsilon}.
\]
Of course the first term in the above left hand side is bounded from below by means of (3.2). Then, the bound (3.2) and the continuity of the inclusion $W^{1,1}(\Omega) \subset L^\infty(\Omega)$ entail in particular that
\[
\int_{0}^{T} \|\theta_\varepsilon\|_{L^\infty(\Omega)} = \int_{0}^{T} \|\theta_\varepsilon^{1/2}\|_{L^\infty(\Omega)}^2 \leq C \int_{0}^{T} \left(\|\partial_x (\theta_\varepsilon^{1/2})\|_{L^1(\Omega)}^2 + \|\theta_\varepsilon\|_{L^1(\Omega)}^2 \right) \\
\leq C \left(1 + \int_{0}^{T} \left(\int_{\Omega} \left|\frac{\partial_x \theta_\varepsilon}{\theta_\varepsilon^{1/2}}\right| \right)^2 \right) \leq C \left(1 + \int_{0}^{T} \left(\|\partial_x \theta_\varepsilon/\theta_\varepsilon\| \|\theta_\varepsilon^{1/2}\| \right)^2 \right) \\
\leq C \left(1 + \int_{0}^{T} \|\partial_x \theta_\varepsilon/\theta_\varepsilon\|^2 \right) \leq C. \tag{3.5}
\]
Hence,
\[ \|\theta_\varepsilon\|_{L^1(0,T;L^\infty(\Omega))} \leq C, \]  
(3.6) \textbf{boundbis}
and finally, by interpolation with (3.2),
\[ \|\theta_\varepsilon\|_{L^2(0,T;H)} \leq C. \]  
(3.7) \textbf{boundtris}

### 3.3 Third estimate

Taking into account (3.7), it is now a standard matter to multiply (2.8) by \( \partial_t \chi_\varepsilon \), integrate on \( Q_t \), exploit the relation \( \beta_\varepsilon = \partial \varphi_\varepsilon \), and obtain the bound
\[ \|\chi_\varepsilon\|_{H^1(0,T;H)} + \sqrt{k}\|\chi_\varepsilon\|_{H^1(0,T;V)} \leq C. \]  
(3.8) \textbf{boundquater}

### 3.4 Fourth estimate

We multiply (2.7) by \( \theta_\varepsilon + \partial_t \chi_\varepsilon \); we differentiate (2.8) with respect to \( t \) and then we multiply by \( \partial_t \chi_\varepsilon \). We add the resulting equations and we integrate over \( Q_t \); thanks to some cancellations, we obtain
\[
\frac{1}{2} \|\theta_\varepsilon(t)\|^2 + \|\partial_x \theta_\varepsilon\|_{L^2(0,T;H)}^2 + \frac{1}{2} \|\partial_t \chi_\varepsilon(t)\|^2 + \|\partial_x \partial_t \chi_\varepsilon\|_{L^2(0,T;H)}^2 + \frac{k}{2} \|\partial_x \partial_t \chi_\varepsilon(t)\|^2 + \int_0^t \int_\Omega \beta'_\varepsilon(\chi_\varepsilon)(\partial_t \chi_\varepsilon)^2 = \\
= \frac{1}{2} \|\theta_0\|^2 + \frac{1}{2} \|\partial_t \chi_\varepsilon(0)\|^2 + \frac{k}{2} \|\partial_x \partial_t \chi_\varepsilon(0)\|^2 + \sum_{i=5}^{9} I_i,  
\]
(3.9) \textbf{schi1}

where
\[
I_5 := k \int \int_{Q_t} (\partial_x \partial_t \chi_\varepsilon)^2 \theta_\varepsilon,  
\]
(3.10)
\[
I_6 := - \int \int_{Q_t} (\theta_\varepsilon)^2 \partial_t \chi_\varepsilon,  
\]
(3.11)
\[
I_7 := \int \int_{Q_t} (\partial_t \chi_\varepsilon)^3,  
\]
(3.12)
\[
I_8 := k \int \int_{Q_t} (\partial_x \partial_t \chi_\varepsilon)^2 \partial_t \chi_\varepsilon,  
\]
(3.13)
\[
I_9 := - \int \int_{Q_t} \partial_x \theta_\varepsilon \partial_x \partial_t \chi_\varepsilon.  
\]
(3.14)

We set \( t = 0 \) in (2.8), obtaining
\[ \partial_t \chi_\varepsilon(0) - k \partial_{xx} \partial_t \chi_\varepsilon(0) = \theta_0 - \theta_c + \partial_{xx} \chi_0 - \beta_c(\chi_0). \]  
(3.15) \textbf{schi2}

We multiply (3.15) by \( \partial_t \chi_\varepsilon(0) \) and integrate over \( \Omega \); we obtain that
\[
\frac{1}{2} \|\partial_t \chi_\varepsilon(0)\|^2 + \frac{k}{2} \|\partial_x \partial_t \chi_\varepsilon(0)\|^2 \leq \frac{1}{2} \|\theta_0 - \theta_c + \partial_{xx} \chi_0 - \beta_c(\chi_0)\|^2;  
\]
(3.16) \textbf{schi3}
and hence the first three terms in the right hand side of (3.9) are bounded by a constant independent of $\varepsilon$, thanks to (1.8), (1.10), (1.11), and (1.12). As for the integrals terms, using Hölder inequality and (1.5), we have that

$$
|I_5| \leq k \int_0^t \| \partial_x \partial_t \chi \|_2^2 \| \theta \|_{L^\infty(\Omega)} ;
$$

(3.17) schi4

$$
|I_6| \leq \int_0^t \| \theta \|_2^2 \| \partial_t \chi \|_{L^\infty(\Omega)} \leq c_1 \int_0^t \| \theta \|_2^2 \| \partial_t \chi \|_V \leq \frac{1}{8} \| \partial_t \chi \|_{L^2(0,t;V)} + c \int_0^t \| \theta \|_2^2 \| \theta \|_2^2 ;
$$

(3.18) schi5

$$
|I_7| \leq \int_0^t \| \partial_t \chi \|_2^2 \| \partial_t \chi \|_{L^\infty(\Omega)} \leq c_1 \int_0^t \| \partial_t \chi \|_2^2 \| \partial_t \chi \|_V \leq \frac{1}{8} \| \partial_t \chi \|_{L^2(0,t;V)} + c \int_0^t \| \partial_t \chi \|_2^2 \| \partial_t \chi \|_2^2 ;
$$

(3.19) schi6

$$
|I_8| \leq k \int_0^t \| \partial_x \partial_t \chi \|_2^2 \| \partial_t \chi \|_{L^\infty(\Omega)} \leq kc_1 \int_0^t \| \partial_x \partial_t \chi \|_2^2 \| \partial_t \chi \|_V ;
$$

(3.20) schi7

$$
|I_9| \leq \frac{1}{2} \| \partial_x \partial_t \chi \|_{L^2(0,t;H)}^2 + \frac{1}{2} \| \partial_x \partial_t \chi \|_{L^2(0,t;H)}^2 .
$$

(3.21) schi8

Thanks to (3.8) and (3.6), we have that $\| \partial_x \partial_t \chi \|_{L^2(0,t;H)}^2$ is bounded independent of $\varepsilon$, $\| \theta \|_{L^\infty(\Omega)} \in L^1(0,T)$, $\| \theta \| \in L^2(0,T)$, and $\| \partial_t \chi \| \in L^2(0,T)$; hence, we can apply an extended version of Gronwall lemma to the function $\| \theta \|_2(t)^2 + \| \partial_t \chi \|_2(t)^2 + k \| \partial_x \partial_t \chi \|_2(t)^2$, obtaining

$$
\| \theta \|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C ,
$$

(3.22) boundschi1

and

$$
\| \chi \|_{H^1(0,T;V) \cap W^{1,\infty}(0,T;H)} + \sqrt{k} \| \chi \|_{W^{1,\infty}(0,T;V)} \leq C .
$$

(3.23) boundschi2

### 3.5 Fifth estimate

Multiply (2.8) by $-\partial_{xx} \chi$, integrate on $Q_t$, exploit the monotonicity of $\beta \varepsilon$, and obtain the bound

$$
\| \chi \|_{L^2(0,T;W)} + \sqrt{k} \| \chi \|_{L^\infty(0,T;W)} \leq C .
$$

(3.24) boundsuppl

### 3.6 Sixth estimate

By comparison in (2.8) and in (2.7), we also obtain

$$
\| \beta \varepsilon \chi \|_{L^\infty(0,T;V')} \leq C ,
$$

(3.25) boundschi3

$$
\| \theta \varepsilon \|_{H^1(0,T;V')} \leq C .
$$

(3.26) boundschi4
3.7 Passage to the limit

We show here the passage to the limit as $\varepsilon \to 0$; most of the arguments can be adapted for the subsequent asymptotic analysis as $k \to 0$. Taking into account well-known compactness results, the bounds (3.22)-(3.26), allow us to deduce the existence of a triplet of functions $(\theta, \chi, \eta)$ such that (possibly passing to not relabeled subsequences) the following convergences hold

\begin{align*}
\theta_\varepsilon &\rightharpoonup^* \theta & \text{in } H^1(0,T;V') \cap L^2(0,T;V) \cap L^\infty(0,T;H), \\
\chi_\varepsilon &\rightharpoonup^* \chi & \text{weakly star in } W^{1,\infty}(0,T;V) \cap L^2(0,T;W), \\
\beta_\varepsilon(\chi_\varepsilon) &\rightharpoonup^* \eta & \text{weakly in } L^2(0,T;V').
\end{align*}

Moreover, from (3.26), (3.22), (3.23), and the generalized Ascoli theorem (see, e.g., [22, Cor. 4]), we may also infer the convergences

\begin{align*}
\theta_\varepsilon &\longrightarrow \theta & \text{strongly in } C^0([0,T];V') \cap L^2(0,T;H), \\
\chi_\varepsilon &\longrightarrow \chi & \text{strongly in } C^0([0,T];H).
\end{align*}

Let us stress that, owing to (2.9), the convergence (3.30) entail that (1.19) holds. Hence, we can pass to the limit in (2.8), and see that the properties (1.13)-(1.15) along with (1.17) are fulfilled by the triplet $(\theta, \chi, \eta)$.

In order to prove (1.18), we multiply (2.8) by $\chi_\varepsilon$ in the duality pairing between $V'$ and $V$ and integrate from 0 to $t$. We find

\begin{align*}
\int_0^t \langle \beta_\varepsilon(\chi_\varepsilon), \chi_\varepsilon \rangle &= -\int \int_{Q_t} \chi_\varepsilon \partial_t \chi_\varepsilon - \int \int_{Q_t} |\partial_x \chi_\varepsilon|^2 + \\
&\quad - \frac{k}{2} \|\partial_x \chi_\varepsilon(t)\|^2 + \frac{k}{2} \|\partial_x \chi_0\|^2 + \int \int_{Q_t} (\theta_\varepsilon - \theta_c) \chi_\varepsilon.
\end{align*}

We take the limsup as $\varepsilon \to 0$ of both sides of (3.32). Thanks to previous strong and weak convergences and owing to l.s.c. properties, we get

\begin{align*}
\limsup_{\varepsilon \to 0} \int_0^t \langle \beta_\varepsilon(\chi_\varepsilon), \chi_\varepsilon \rangle &\leq -\int \int_{Q_t} \chi \partial_t \chi - \int \int_{Q_t} |\partial_x \chi|^2 + \\
&\quad - \frac{1}{2} \|\partial_x \chi(t)\|^2 + \frac{1}{2} \|\partial_x \chi_0\|^2 + \int \int_{Q_t} (\theta - \theta_c) \chi = \\
= \int_0^t \langle \theta - \theta_c - \partial_t \chi, \chi \rangle &> - \int_0^t (k \partial_{xt} \chi + \partial_x \chi, \partial_x \chi).
\end{align*}

A comparison in (1.17) gives

\begin{equation}
\limsup_{\varepsilon \to 0} \int_0^t \langle \beta_\varepsilon(\chi_\varepsilon), \chi_\varepsilon \rangle \leq \int_0^t \langle \eta, \chi \rangle; \tag{3.34}
\end{equation}

in view of [2, Prop. 1.1, p. 42], (3.34) implies (1.18).
Our next goal is to pass to the limit in (2.7). We remark that from (3.27), (3.28), and (3.30), we achieve that
\[
\theta_{\varepsilon} \partial_t \chi_{\varepsilon} \rightarrow \theta \partial_t \chi \text{ weakly star in } L^\infty(0,T;H).
\]
The critical terms are \((\partial_t \chi_{\varepsilon})^2\) and \(k(\partial_{xt} \chi_{\varepsilon})^2\). In order to prove that
\[
\lim_{\varepsilon \rightarrow 0} \iint_{Q_T} (\partial_t \chi_{\varepsilon})^2 + k(\partial_{xt} \chi_{\varepsilon})^2 = \iint_{Q_T} (\partial_t \chi)^2 + k(\partial_{xt} \chi)^2, \tag{3.35} \]
(i.e. \(\partial_{xt} \chi_{\varepsilon}\) actually converges strongly in \(L^2(0,T;V)\)) thanks to (3.28), one has only to show that
\[
\limsup_{\varepsilon \rightarrow 0} \iint_{Q_T} (\partial_t \chi_{\varepsilon})^2 + k(\partial_{xt} \chi_{\varepsilon})^2 \leq \iint_{Q_T} (\partial_t \chi)^2 + k(\partial_{xt} \chi)^2. \tag{3.36} \]
The procedure is analogous to the one performed in [4, sec. 4] and for the sake of brevity we omit the details. We only want to outline that the key point is to multiply both sides of (2.8) by \(\partial_t \chi_{\varepsilon}\) and then exploit —besides the other convergences— (3.34). This completes the proof of Theorem 1.2.

### 3.8 Asymptotic analysis

We only sketch a possible asymptotic analysis when \(k \rightarrow 0\), relying on the previous estimates. Thanks to the previous bounds, we consider a triplet \((\theta^k, \chi^k, \eta^k)\) solution of Problem (P). We deduce the existence of a triplet of functions \((\theta^0, \chi^0, \eta^0)\) satisfying (1.25-1.32), so the proof of Thm. 1.4 is performed.

### References


