WELL-POSEDNESS AND LONG-TIME BEHAVIOR FOR A CLASS OF DOUBLY NONLINEAR EQUATIONS

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Abstract. This paper addresses a doubly nonlinear parabolic inclusion of the form
\[ \mathcal{A}(u_t) + \mathcal{B}(u) \ni f. \]
Existence of a solution is proved under suitable monotonicity, coercivity, and structure assumptions on the operators \( \mathcal{A} \) and \( \mathcal{B} \), which in particular are both supposed to be subdifferentials of functionals on \( L^2(\Omega) \). Since unbounded operators \( \mathcal{A} \) are included in the analysis, this theory partly extends Colli & Visintin [24]. Moreover, under additional hypotheses on \( \mathcal{B} \), uniqueness of the solution is proved. Finally, a characterization of \( \omega \)-limit sets of solutions is given and we investigate the convergence of trajectories to limit points.

1. Introduction. The present analysis is concerned with the study of doubly nonlinear parabolic problems of the form
\[ \alpha(u_t) - \text{div}(b(x, \nabla u)) + W'(u) \ni f, \]
in a bounded and regular domain \( \Omega \subset \mathbb{R}^d \) \((d = 1, 2, 3)\). Here \( \alpha \subset \mathbb{R} \times \mathbb{R} \) and \( b(x, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( x \in \Omega \), are assumed to be maximal monotone while \( W \) is a \( \lambda \)-convex function (see the following assumption (H1) and [8] for the definition of these kind of perturbations of convex functionals) and \( W' \) is its derivative.

Equations of the type of (1.1) stem in connection with phase change phenomena [14, 17, 23, 32, 46, 47], gas flow through porous media [68], damage [15, 16, 30, 31, 54], and, in the specific case \( \alpha(\lambda r) = \alpha(r) \) for all \( \lambda > 0 \) and \( r \in \mathbb{R} \) (which is however not included in the present analysis), elastoplasticity [25, 50, 51, 52], brittle fractures [26], ferroelectricity [56], and general rate-independent systems [29, 48, 49, 53, 55].

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The applicative interest of (1.1) is related to the fact that it may arise in connection with the gradient flow of the energy functional

$$E(u) = \int_{\Omega} (\phi(x, \nabla u) + W(u) - fu) \quad (D\phi = b),$$

(1.2)

with respect to the metric induced by the dissipation functional

$$F(u_t) = \int_{\Omega} \hat{\alpha}(u_t) \quad (\partial \hat{\alpha} = \alpha),$$

(1.3)

where $\partial$ denotes Clarke’s gradient [21] in $L^2(\Omega)$. Namely, by taking variations in $L^2(\Omega)$, inclusion (1.1) may be derived from the kinetic relation

$$\partial F(u_t) + \partial E(u) \ni 0,$$

(1.4)

which represents indeed some balance between the variation in energy and the dissipation in the physical system under consideration.

The mathematical interest in (1.1) dates back at least to Barbu [11] (see also Arai [5] and Senba [62]) and Colli & Visintin [24] obtained several existence results in an abstract hilbertian framework. The reader is also referred to [22] for some extension to Banach spaces. In particular, the results in [24] address relation (1.1) in the situation of a linearly bounded function $\alpha$ and a convex potential $W$. More recently, Segatti extended some result of [24] to classes of $\lambda$-convex potentials $W$ [61]. In the same paper, the existence of a global attractor is investigated. As no uniqueness results are available for (1.1) in the genuine doubly nonlinear case, the analysis of [61] is tailored to Ball’s theory of generalized semiflows [9, 10].

The first issue of this paper is indeed to weaken the requirements of [24, 61] and prove novel existence and regularity results in the situation of unbounded graphs $\alpha$. This extension turns out to be crucial from the point of view of applications since it entails the possibility of choosing dissipation densities $\hat{\alpha}$ with bounded domain. In fact, a quite common choice is $\hat{\alpha}(r) \sim r^2$ for $r \geq 0$ and $\hat{\alpha}(r) = +\infty$ for $r < 0$ representing indeed the case of irreversible evolutions $u_t \geq 0$. Let us point out that a first result in this direction for the case of convex potentials $W$ and linear functions $b$ has been obtained in [17].

A second main focus of this paper is on uniqueness of solutions to (1.1). Let us stress that uniqueness is generally not expected in genuine doubly nonlinear situations. We however address here the case of locally smooth functions $W'$ which are singular at the bounded extrema of their domain. By requiring some suitable compatibility between the singular growth of $W'$ and the non-degeneracy of $b$, we prove that solutions to (1.1) stay uniformly away from the singular points of $W'$ (separation property). Hence, continuous dependence on data follows at least in the specific case of linear functions $b$. This is, to our knowledge, the first uniqueness result available for doubly nonlinear equations of the class (1.1).

The third part of the paper is devoted to the description of the long-time behavior of solutions to (1.1). First of all, we prove a characterization of the (non-empty) $\omega$-limit sets of trajectories as solutions to a suitable stationary problem. Secondly, as $b = \text{Id}$, $W'$ is analytic in the interior of its domain, and $\alpha$ fulfills suitable growth conditions, an application of the Lojasiewicz-Simon [44, 65] inequality entails that the $\omega$-limit reduces to a single point and the whole trajectory converges (this fact turns out to be particularly interesting since the above mentioned stationary limit problem may exhibit a continuum of solutions). Let us stress that both results hold independently of the uniqueness of a solution.
For the sake of completeness, we shall mention that other types of doubly nonlinear equations have attracted a good deal of interest in recent years. In particular, equations of the form
\[(Au)_t + B(u) \ni f, \tag{1.5}\]
where $A$ and $B$ are nonlinear maximal monotone operators (even nonlocal in time or with an explicit time dependence) have been addressed, and various existence, uniqueness and long time behaviour results are available. With no claim of completeness, we quote [3, 4, 6, 13, 27, 34, 36, 43, 57, 63, 66, 67], among many others.

Plan of the paper. Section 2 is devoted to the detail of our assumptions and the statement of the main results. Then, Section 3 brings to the proof of the existence result by means of a time-discretization procedure and passage to the limit technique. Section 4 is devoted to the proof of the separation property and uniqueness. Finally, Section 5 addresses the long-time behavior issues.

2. Statement of the problem and main results.

2.1. Assumptions and preliminary material. Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$. Set, for $t > 0$, $Q_t := \Omega \times (0, t)$. Set also $H := L^2(\Omega)$ and denote by $||\cdot||$ the norm both in $H$ and in $H^d$ and by $\|\cdot\|_X$ the norm in the generic Banach space $X$. Moreover, we indicate by $(\cdot, \cdot)$ the scalar product in $H$ and by $\langle \cdot, \cdot \rangle$ the duality between a Banach space $V$ and its topological dual $V'$. In the sequel, the same symbol $c$ will be used to indicate some positive constants, possibly different from each other, appearing in the various hypotheses and computations and depending only on data. When we need to fix the precise value of one constant, we shall use a notation like $c_i$, $i = 1, 2, \ldots$, instead.

In the sequel, we shall present the basic assumptions and notations which will be kept, with small variations, for all the rest of the paper.

: Assumption (H1). Let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph and define the domain of $\beta$ in $\mathbb{R}$ as the interval $I := \text{dom}_\mathbb{R} \beta = \{r \in \mathbb{R} : \beta(r) \neq \emptyset\}$. We assume, for simplicity, that $\beta$ is normalized in such a way that $0 \in I$ and $\beta(0) = 0$. We denote as $\beta^0$ the minimal section (cf., e.g., [18, p. 28]) of the graph $\beta$. We shall also indicate by $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ the convex and lower semicontinuous function such that $\hat{\beta}(0) = 0$ and $\beta = \partial \hat{\beta}$, $\partial$ denoting the subdifferential in the sense of Convex Analysis (here in $\mathbb{R}$). Note that [18, Prop. 2.11, p. 39] $D(\hat{\beta}) := \{r \in \mathbb{R} : \hat{\beta}(r) < +\infty\} \subset T$. We assume the potential $W$ to be $\lambda$-convex, namely of the form (since $\hat{\beta}$ is assumed to be convex)

$$W(r) := \hat{\beta}(r) - \frac{\lambda r^2}{2} + c_W \quad \text{for } r \in D(\hat{\beta}), \tag{2.1}$$

where $\lambda \geq 0$ and $c_W$ is an integration constant. Moreover, we suppose the coercivity property

$$\exists \eta > 0 : W'(r)r \geq \eta r^2 \quad \text{for } |r| \text{ sufficiently large in } \text{dom}_\mathbb{R} \beta \tag{2.2}$$

(where with some abuse of notation we are denoting by $W'(r)$ the (multi-)function $\beta(r) - \lambda r$, see Remark 2.2 below).

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1We choose $d \in \{1, 2, 3\}$ just for simplicity and in view of possible physical applications. However, we stress that all the following results hold in any space dimension.
Note that (2.2) entails no restriction if \( I \) is bounded. Possibly modifying the integration constant \( c_W \) in (2.1) it is not restrictive to suppose also that, for \( \eta \) as above,

\[
W(r) \geq \frac{\eta r^2}{2} \quad \forall r \in D(\widehat{\beta}).
\]  

(2.3)

In the sequel, \( \beta \) will be identified with a maximal monotone operator (still denoted as \( \beta \)) from \( H \) to \( H \), whose domain (in \( H \)) is the set

\[
\text{dom}_H \beta := \{ v \in H \text{ such that there exists } \eta \in H : \eta \in \beta(v) \text{ a.e. in } \Omega \}.\]

(2.4)

**Remark 2.1.** We point out that, even if \( I \) is open and bounded, given \( v \in C(\overline{\Omega}) \cap \text{dom}_H \beta \), it is not excluded that there exists a nonempty set (necessarily of zero Lebesgue measure) \( \Omega_{\infty} \) such that \( v(x) \in \partial I \) when \( x \in \Omega_{\infty} \).

\begin{itemize}
  \item **Assumption (H2).** Let \( \alpha \subset \mathbb{R} \times \mathbb{R} \) be a maximal monotone graph and suppose that \( 0 \in J := \text{dom}_R \alpha = \{ r \in \mathbb{R} : \alpha(r) \neq 0 \} \). Moreover, let, for some \( \sigma > 0 \) and \( S_- \leq 0 \leq S_+ \),
  \[
  \alpha'(r) \geq 2\sigma \quad \text{for a.e. } r \in \text{int } J \setminus [S_-, S_+],
  \]
  (2.5)
  Note that \( \alpha \) is single-valued apart (possibly) from a countable number of points in \( J \) and that any selection of \( \alpha \) is almost everywhere differentiable. As before, let \( \widehat{\alpha} : \mathbb{R} \rightarrow [0 + \infty] \) be the convex and lower semicontinuous function such that \( \widehat{\alpha}(0) = 0 \) and \( \alpha = \partial \widehat{\alpha} \). As above, the operator \( \alpha \) will be identified with its (possibly multivalued) realization on \( H \). Clearly, (2.5) implies that
  \[
  \exists c = c(\sigma, S_-, S_+, \Omega) \geq 0 : \int_{\Omega} (\alpha^0(v)v \, dx \geq \sigma|v|^2 - c \quad \forall v \in H.
  \]
  (2.6)

**Remark 2.2.** From this point on, in order to simplify the presentation, we will systematically refer to the multi-functions \( \beta \), \( W \), and \( \alpha \) by using the notation for (single-valued) functions. In particular the symbol \( \beta(u) \) will be used in order to denote some suitable selection \( \eta \in H \) such that \( \eta \in \beta(u) \) almost everywhere in \( \Omega \), and so on. We believe that this simplification will cause no confusion but rather help clarifying some statements and proofs. This notational convention is taken throughout the remainder of the paper with the exception of distinguished points where we add some extra comment.

\begin{itemize}
  \item **Assumption (H3).** Let \( \phi : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty) \) be such that:
    \[
    \phi(x, \cdot) \in C^1(\mathbb{R}^d) \quad \text{for a.e. } x \in \Omega,
    \]
    (2.7)
    \[
    \phi(x, \cdot) \text{ is convex and } \phi(x, 0) = 0 \text{ for a.e. } x \in \Omega,
    \]
    (2.8)
    \[
    \phi(\cdot, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^d.
    \]
    (2.9)
    Then, we can set
    \[
    b := \nabla_{\xi} \phi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d.
    \]
    (2.10)
    We assume that, for a given \( p > 1 \), \( \phi \) satisfies the growth conditions
    \[
    \exists \kappa_1, \kappa_2, \kappa_3 > 0 : \phi(x, \xi) \geq \kappa_1|\xi|^p - \kappa_2, \quad |b(x, \xi)| \leq \kappa_3(1 + |\xi|^{p-1})
    \]
    for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^d \).
    (2.11)
    Finally, we require that at least one of the following properties holds:
    \[
    \text{either } p > 6/5 \text{ in (2.11),}
    \]
    (2.12)
    \[
    \text{or } \exists \eta > 0, q > 2 : W(r) \geq \eta r^q - c \text{ for all } r \in D(\widehat{\beta}).
    \]
Let us now set either \( D(\Phi) := H \cap W^{1,p}_0(\Omega) \) or \( D(\Phi) := H \cap W^{1,p}(\Omega) \) depending whether homogeneous Dirichlet or homogeneous Neumann boundary conditions are taken into account. Both spaces \( W^{1,p}_0(\Omega) \) and \( W^{1,p}(\Omega) \) are intended to be endowed with the full \( W^{1,p} \)-norm in the sequel. Then, we introduce the functional
\[
\Phi : H \to [0, +\infty], \quad \Phi(u) := \int_\Omega \phi(x, \nabla u(x)) \, dx,
\]
which is defined as identically \(+\infty\) outside its domain \( D(\Phi) \).

By means of [33, Thm. 2.5, p. 22] one has that \( \Phi \) is convex and lower semicontinuous on \( H \). Thus, by [18, § 2.7], its subdifferential \( \partial \Phi \) in \( H \) is a maximal monotone operator in \( H \times H \). Moreover, we can introduce the operator \( B : \text{dom}_H B \to H \) by setting
\[
(Bv, z) := \int_\Omega b(x, \nabla v(x)) \cdot \nabla z(x) \, dx,
\]
for all \( z \in D(\Phi) \), defining
\[
\text{dom}_H B := \left\{ v \in D(\Phi) : \sup_{z \in D(\Phi) \setminus \{0\}} \left| \int_\Omega b(x, \nabla v(x)) \cdot \nabla z(x) \, dx \right|/|z| < +\infty \right\},
\]
and extending by continuity without changing symbols to \( z \in H \). In the case of Dirichlet conditions, the latter positions simply mean that \( Bv = -\text{div}(b'(\cdot, \nabla v(\cdot))) \), where the divergence is intended in the sense of distributions.

The inclusion \( B \subset \partial \Phi \) is almost immediate to check. It is however remarkable that also the converse inclusion can be proved (see [59, Ex. 2.4]) by showing that \( \Phi \) is indeed maximal. We shall not provide here a direct proof of this fact but rather sketch the main steps toward its check. First of all, one could consider some suitable approximation \( b_\varepsilon, \varepsilon > 0 \), (by a careful mollification, for instance) of the function \( b \) and solve the maximality problem for the regularized operator. Then, by estimating a priori and extracting suitable sequences of indices, we prove that some sequences of solutions to the regularized problem admit a limit as \( \varepsilon \to 0 \). Finally, by exploiting the maximality of \( \Phi(\cdot) \to b(\cdot, \xi(\cdot)) \) from \( L^p(\mathbb{R}^d) \) to \( L^{p'}(\mathbb{R}^d), \; p' = p/(p-1) \) (we used here (2.11)), we identify the limit and conclude.

Defining now the energy \( E : H \to [0, +\infty] \) as
\[
E(u) := \int_\Omega \left( \phi(x, \nabla u(x)) + W(u(x)) \right) \, dx,
\]
it is not difficult to see that \( E \) is a quadratic perturbation (due to the \( \lambda \)-term in (2.1)) of a convex functional. Moreover, we have

**Lemma 2.3.** \( E \) has compact sublevels in \( H \).

**Proof.** Let \( \{v_n\} \subset H \) a sequence of bounded energy. If \( p > 6/5 \), \( \{v_n\} \) is precompact in \( H \) thanks to (2.11) and compactness of the embedding \( W^{1,p}_0(\Omega) \subset H \). If, instead, the second of (2.12) holds, a priori \( \{v_n\} \) only admits a subsequence which is strongly converging in \( L^p(\Omega) \) and bounded in \( L^2(\Omega) \). However, since this implies, up to a further extraction, convergence almost everywhere, the conclusion follows.

The next simple Lemma describes the structure of \( \partial E \).

**Lemma 2.4.** The subdifferential of \( E \) coincides with the sum \( \mathcal{B} := B + \beta - \lambda \text{Id} \), with domain \( \text{dom}_H \mathcal{B} = \text{dom}_H B \cap \text{dom}_H \beta \).
Proof. It is enough to work on the nonlinear part of $\mathcal{B}$, i.e. to show that
\[ B + \beta = \partial \left( E + \lambda \frac{|\cdot|^2}{2} \right). \] (2.17)

The inclusion $\subset$ is clear. To get the converse one, it suffices to show that $B + \beta$ is maximal. With this aim, we proceed by regularization and replace $\beta$ with its (Lipschitz continuous) Yosida approximation $[18, \text{p. 28}] \beta_\varepsilon$, with $\varepsilon > 0$ intended to go to 0 in the limit. Then, given $g \in H$, by $[18, \text{Lemme 2.4, p. 34}]$ the elliptic problem
\[ v_\varepsilon \in H, \quad Bv_\varepsilon + \beta_\varepsilon(v_\varepsilon) + v_\varepsilon = g, \] (2.18)
has at least one solution $v_\varepsilon \in H$ which belongs, additionally, to $\text{dom}_H B$. In other words, the operator
\[ B + \beta_\varepsilon : \text{dom}_H B \to H \] (2.19)
is maximal monotone. As $\beta_\varepsilon$ is Lipschitz continuous and $\beta_\varepsilon(0) = 0$, it is possible to test (2.18) by $\beta_\varepsilon(v_\varepsilon)$, obtaining, for some $c > 0$ independent of $\varepsilon$,
\[ |\beta_\varepsilon(v_\varepsilon)| + |Bv_\varepsilon| \leq c. \] (2.20)
The boundedness of $\beta_\varepsilon(v_\varepsilon)$ in $H$ permits then to apply $[18, \text{Thm. 2.4, p. 34}]$ which concludes the proof.

Assumption (H4). Let us be given
\[ f \in W^{1,1}_{\text{loc}}([0, +\infty); H), \] (2.21)
\[ u_0 \in \text{dom}_H \mathcal{B}. \] (2.22)

2.2. Main Theorems. We shall state our abstract Cauchy problem as follows
\[ \alpha(u_t) + Bu + W'(u) = f, \quad \text{in } H, \] (2.23)
\[ u|_{t=0} = u_0, \quad \text{in } H. \] (2.24)

Our first result is concerned with the existence of at least one solution to the above problem. Its proof will be outlined in Section 3.

Theorem 2.5. [Existence] Let us assume (H1)–(H4) and take $T > 0$. Then, there exists a function $u : Q_T \to \mathbb{R}$ such that
\[ u \in W^{1,\infty}(0, T; H), \quad Bu, \beta(u) \in L^\infty(0, T; H), \] (2.25)
\[ E(u) \in L^\infty(0, T), \] (2.26)
which satisfies (2.23) a.e. in $(0, T)$ and the Cauchy condition (2.24). More precisely, there exists $c > 0$ depending on $u_0$, $f$, and $T$ such that
\[ \|u\|_{W^{1,\infty}(0, T; H)} + \|Bu\|_{L^\infty(0, T; H)} + \|\beta(u)\|_{L^\infty(0, T; H)} + \|E(u)\|_{L^\infty(0, T)} \leq c. \] (2.27)

Finally, if either
\[ S_- = S_+ = 0 \text{ in (2.5),} \] (S0)
\[ f \in L^2(0, +\infty; H), \quad f_t \in L^1(0, +\infty; H) \] (f1)
or
\[ f \in H \quad \text{is independent of time,} \] (f2)
then the constant $c$ in (2.27) is independent of $T$. 
Remark 2.6. Referring to the previous Remark 2.2, we note that, in case \( \alpha \) and \( \beta \) are multivalued, the statement above has to be intended in the following sense: we prove the existence of selections \( \xi, \eta \in L^\infty(0,T;H) \) such that \( \xi \in \alpha(u_t) \) and \( \eta \in \beta(u) \) almost everywhere in \( \Omega \times (0,T) \) and

\[
\xi + Bu + \eta - \lambda u = f \quad \text{in } H, \quad \text{a.e. in } (0,T). \quad (2.28)
\]

Let us make precise the comparison of our result with the corresponding Theorem proved in [24]. Indeed, in our notation, [24, Thm. 2.1] (which is, among the various results of [24], the closest to our theory) can be formulated as follows:

Theorem 2.7. [Colli & Visintin, 1990] Let (H1)–(H3) hold with \( \lambda = 0 \) in (2.1), let \( T > 0 \), and let \( \alpha \) be sublinear, namely:

\[
|\alpha(r)| \leq c(1 + |r|) \quad \text{for all } r \in \mathbb{R} \quad \text{and some } c > 0.
\]

Moreover, in place of (H4), assume \( E(u_0) < \infty \) and \( f \in L^2(0,T;H) \). Then, (2.23)–(2.24) admits at least one solution \( u \) satisfying (2.26) together with

\[
u \in H^1(0,T;H), \quad \alpha(u_t), Bu, \beta(u) \in L^2(0,T;H).
\]

By comparing Theorems 2.5 and 2.7 we readily check that our result refers to more general classes of nonlinearities \( \alpha \) and provides more regular solutions. On the other hand, in place of our assumption (2.22), in [24] the authors were able to consider less regular initial data, satisfying the finite energy condition \( E(u_0) < \infty \).

One of the main novelties of the present work is that for a significant class of potentials \( W \) we are able to show uniqueness and further regularity. In order this to hold we have to introduce the following.

: Assumption (H5). Suppose that, for some \( \nu > 0 \), one has

\[
dom_B \subset C^{0,\nu}(\overline{\Omega})
\]

and that there exists a nondecreasing function \( \gamma : [0, +\infty) \rightarrow [0, +\infty) \) such that

\[
\|v\|_{C^{0,\nu}(\overline{\Omega})} \leq \gamma(\|v\| + \|Bv\|) \quad \forall v \in \dom_B B.
\]

Moreover, let \( I = \dom_B \beta \) be an open set and, in case \( I \) is, e.g., right-bounded, set \( \overline{\tau} := \sup I \) and assume in addition that there exist \( c > 0, r_1 < \overline{\tau}, \) and \( \kappa > 0 \) such that

\[
\beta^0(r) \geq \frac{c}{(\overline{\tau} - r)^\kappa}, \quad \forall r \in (r_1, \overline{\tau}),
\]

where \( \kappa \) and \( \nu \) satisfy the compatibility condition

\[
2\kappa \nu \geq d.
\]

Analogously, if \( I \) is left-bounded, set \( \underline{\tau} := \inf I \) and assume the analogues of (2.33) in a right neighborhood \( (\underline{\tau}, r_0) \) of \( \underline{\tau} \), together with (2.34).

Remark 2.8. Condition (2.33) states that \( \beta \) explodes at \( \overline{\tau} \) at least as fast as a sufficiently high negative power of \( (\overline{\tau} - r) \), made precise by (2.34). Of course, the higher this power, the faster the system is forced by energy dissipation to keep the solution away from the barrier at \( r = \overline{\tau} \). Note that, unfortunately, even for \( d = 1 \), this excludes from our analysis the logarithmic potentials of the type

\[
W(r) = (1 - r) \log(1 - r) + (1 + r) \log(1 + r),
\]

which are relevant for applications (cf. e.g., [58]), since they are not enough coercive in proximity of the barriers (here given by \( \underline{\tau} = -1, \overline{\tau} = 1 \)). However (cf. the forthcoming Remark 2.9), in one
dimension \((d = 1)\) it is possible to consider some potentials \(W\) which are bounded on \([r, r]\) and thus qualitatively similar to \((2.35)\).

**Remark 2.9.** Property \((2.31)\) depends of course on the operator \(B\) and on the space dimension \(d\). For instance, if \(d = 3\) and \(B = -\Delta\), then \((2.31)\) holds for \(\nu \leq 1/2\). If \(B\) is a nonlinear operator coming from a functional \(\Phi\) of \(p\)-growth (cf. \((2.11)\)), \(p \geq 2\), satisfying the regularity conditions \([60, (22), (35)]\) (this is the case, e.g., of the \(p\)-Laplace operator), and moreover Dirichlet conditions are taken (i.e. \(D(\Phi)\) is chosen as \(W_0^{1,p}(\Omega)\)), then one can apply, e.g., \([60, \text{Thm. 2}]\), yielding \(\text{dom}_H B \subset W^{\zeta,p}(\Omega)\) for any \(\zeta < 1 + 1/p\), which in turn entails \((2.31)\), with a suitable \(\nu\), for \(p \geq 2\) if \(d = 2\) and for \(p > 2\) if \(d = 3\).

**Proposition 2.10.** [Separation property] Let us assume \((H1)\)--\((H5)\) and let \(u\) be any solution in the regularity setting of Theorem 2.5. Then, \(u(t) \in C^{0,r}(\Omega)\) for every \(t \in [0, T]\) and \(\exists c_0 > 0\) such that

\[
\|u(t)\|_{C^{0,r}(\Omega)} \leq c_0 \quad \forall t \in [0, T].
\]

Moreover, \(\exists r_* < 0, r^* > 0\), with \(r^* < r\) if \(I\) is right-bounded and \(r_* > r\) if \(I\) is left-bounded, such that

\[
\tau_* \leq u(x,t) \leq r^* \quad \forall (x,t) \in \Omega_T.
\]

Finally, \(c_0, \tau_*,\) and \(r^*\) are independent of \(T\) if either \((S0)\)--\((F1)\) or \((F2)\) hold.

Let us now come to uniqueness, which requires the linearity of \(B\), i.e. the following additional assumption.

: **Assumption (H6).** Let us consider a matrix field \(D\) such that

\[
D \in L^\infty(\Omega; \mathbb{R}^{d \times d}), \quad D(x) \text{ is symmetric for a.e. } x \in \Omega, \quad (2.38)
\]

\[
D(x)\xi \cdot \xi \geq a|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \Omega \text{ and some } a > 0. \quad (2.39)
\]

Then, we assume that \(\phi(x, \xi) := \frac{1}{2}(D(x)\xi)\cdot\xi\) (cf. \((H2)\)) and define \(\Phi\) and \(B\) correspondingly, with the choice of \(D(\Phi) = H^1(\Omega)\), or \(D(\Phi) = H_0^1(\Omega)\), depending on the desired boundary conditions.

Setting \(V := D(\Phi)\), which is now a Hilbert space, \(B\) turns out to be linear elliptic from \(V\) to \(V'\) and satisfies, for \(a\) as in \((2.39)\),

\[
\langle Bv, v \rangle \geq a|\nabla v|^2, \quad \langle Bv, z \rangle \leq \|D\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})}|\nabla v||\nabla z|, \quad \forall v, z \in V. \quad (2.40)
\]

**Theorem 2.11.** [Uniqueness] Let us assume \((H1)\)--\((H6)\) and \((S0)\). Moreover, suppose that

\[
\beta \text{ coincides with a locally Lipschitz continuous function in } I. \quad (2.41)
\]

Then, the solution \(u\) provided by Theorem 2.5 is unique.

**Remark 2.12.** Let us observe that \((2.41)\) excludes the presence of vertical segments in the graph \(\beta\), but does not give any further restriction on its behavior at the boundary of \(I\). As regards conditions \((2.31), (2.32)\) in \((H5)\), we note that, whenever \((H6)\) holds, these are related to the regularity properties of \(D\) and the space dimension. For instance, if \(D\) is Lipschitz continuous, then it is well known \([35, \text{Thm. 8.12, p. 186}]\) that \(\text{dom}_H B \subset H^2(\Omega)\) (recall that we are always supposing \(\Omega\) sufficiently smooth). Instead, if it is only \(D \in L^\infty(\Omega)\), then \(\text{dom}_H B \subset H^\zeta(\Omega)\) for any \(\zeta < 3/2\) \([60, \text{Rem. 4.4}]\).
The next results are related to existence and characterization of \( \omega \)-limit sets of trajectories of the system. We consider here global solutions \( u : [0, +\infty) \to H \) to (2.23)--(2.24) satisfying the regularity frame of Theorem 2.5. We remark once again that what follows holds for any such function \( u \) and does not require uniqueness.

**Theorem 2.13. [Characterization of the \( \omega \)-limit set]** Let (H1)--(H4), (S0), and either (f1) or (f2) hold. Then, for all functions \( u \) whose existence is stated in Theorem 2.5 and any sequence of times \( \{t_n\} \) with \( t_n \uparrow +\infty \), there exist a not relabeled subsequence and a function \( u_\infty \in H \) such that

\[
u(t_n) \to u_\infty \quad \text{strongly in } H.
\]  

Moreover, \( u_\infty \) is a solution of the stationary problem

\[
Bu_\infty + W'(u_\infty) = g \quad \text{in } H, 
\]

where \( g = 0 \) if (f1) holds and \( g = f \) if, instead, (f2) is satisfied.

**Remark 2.14.** In the theorem above we are implicitly assuming that the multifunction \( \alpha \) is such that

\[
\alpha(0) = \{0\}. 
\]

It will be clear from the proof that, if (2.44) does not hold, then we still have (2.42), but in place of (2.43) we have the weaker \( -Bu_\infty - W'(u_\infty) + g \in \alpha(0) \). Moreover, let us point out that our analysis could be easily extended in order to cover the case \( f = f_1 + f_2 \) where \( f_1 \) and \( f_2 \) fulfill (f1) and (f2), respectively.

It is well known that, since \( W \) is not convex, problem (2.43) may well admit infinite solutions [39]. Thus, the question of convergence of all the trajectory \( u(t) \) to one of these solutions is a nontrivial one and is not answered by the preceding Theorem. Here, we are able to show this property under more restrictive assumptions, and the basic tool in the proof is the so-called Lojasiewicz-Simon method. To detail this technique, originally devised in [44, 45, 65], let us state and comment the further assumptions we need.

: **Assumption (H7).** Assume that, if (f2) holds, then in addition

\[
f \in L^\infty(\Omega), \quad |f(x)| \leq M \quad \text{for a.e. } x \in \Omega 
\]

and some \( M > 0 \). Moreover, assume that

there exists an open interval \( I_0 = (\underline{r}, \bar{r}) \), with \( [\underline{r}, \bar{r}] \subset I \),

such that \( W'(r) + M < 0 \ \forall r \leq \underline{r}, \quad W'(r) - M > 0 \ \forall r \geq \bar{r} \),

where \( r \) is taken in \( I \). In case (f1) holds, assume the same with \( M = 0 \). Suppose also that

\[
W_{I_0} \quad \text{is real analytic.} 
\]

**Remark 2.15.** If (H7) holds, a simple maximum principle argument shows that there exists \( \epsilon > 0 \) such that any solution \( u_\infty \) to (2.43) fulfills

\[
\underline{r} + \epsilon \leq u_\infty(x) \leq \bar{r} - \epsilon \quad \forall x \in \Omega. 
\]

Note also that it is not excluded that \( [\underline{r}, \bar{r}] = I \). This means that \( \beta \), and hence \( W' \), may contain vertical half lines at the extrema. However, \( W' - g \) must have the right sign at least in a neighborhood of the barriers. A similar class of potentials has been considered in [37, Thm. 2.7].
Let us now choose $B = -\Delta$ along with either homogeneous Dirichlet or Neumann boundary conditions, and take $V$ correspondingly (see (H6)). Consider again a solution $u_\infty$ to (2.43). Then, the Łojasiewicz-Simon inequality (cf. [28, Prop. 6.1], see also [1]) states that there exist $c_\epsilon, \epsilon > 0$, $\theta \in (0, 1/2)$ such that
\begin{equation}
|E(v) - E(u_\infty)|^{1-\theta} \leq c_\epsilon \|Bv + W'(v)\|_V.
\end{equation}
for all $v \in V$ such that
\begin{equation}
\|v - u_\infty\|_V \leq \epsilon.
\end{equation}

**Remark 2.16.** Inequality (2.49) has been shown in [40, Thm. 2.1] in the Dirichlet case. The analogous statement for the Neumann case can be found in [1, Prop. 2.4].

**Remark 2.17.** It should be possible to prove the same result for more general (linear and symmetric) operators $B$ satisfying (H6). Namely, in order to extend inequality (2.49) (or the corresponding local version proved in [28, 2]), what seems to be needed is that the coefficients of $B$ are so regular that any solution $v$ to the equation $v + Bv = h$ for $h \in L^\infty(\Omega)$ lies in $W^{2,p}$ for $p > d$ (cf. also [41, Sec. 3]).

We can now state our characterization result for the $\omega$-limit.

**Theorem 2.18.** [Convergence to the stationary state] Assume (H1)–(H4), (S0), either (f1) or (f2), and (H7). Take also $B = -\Delta$. Moreover, if (f1) holds, assume also that there exist $c, \xi > 0$ such that
\begin{equation}
t^{1+\xi} \int_t^\infty |f(s)|^2 \, ds \leq c \quad \text{for all } t \geq 0.
\end{equation}
Finally, assume $J = \text{dom}_R \alpha = \mathbb{R}$ and that there exist $\sigma', \kappa_\infty, \ell_\infty > 0$ and $1 \leq p_\infty \leq q_\infty$ such that, for all $r \in \mathbb{R}$,
\begin{equation}
\sigma|r| + \kappa_\infty |r|^{p_\infty} \leq \alpha(r) \leq \sigma'|r| + \ell_\infty |r|^{q_\infty}.
\end{equation}
Finally, assume the constraint
\begin{equation}
\chi q_\infty \leq (p_\infty + 1), \quad \text{where}\end{equation}
\begin{equation}
\chi \in [1, 2] \quad \text{if } d = 1, \quad \chi \in (1, 2] \quad \text{if } d = 2, \quad \text{and} \quad \chi \in [6/5, 2] \quad \text{if } d = 3.
\end{equation}
Then, letting $u$ be a solution, the $\omega$-limit of $u$ consists of a unique function $u_\infty$ solving (2.43), where $g$ is as in Theorem 2.13. Furthermore, as $t \nearrow +\infty$,
\begin{equation}
u(t) \to u_\infty \quad \text{strongly in } V \cap C(\overline{\Omega}),
\end{equation}
i.e., we have convergence for the whole trajectory $u(t)$.

**Remark 2.19.** Let us point out that, as one proves convergence of the trajectory with respect to the norm of $H$, then (2.54) is immediate since, by (2.27) and being $B = -\Delta$, the trajectory associated to $u$ is precompact in $V \cap C(\overline{\Omega})$. The same might be extended to more general $B$ satisfying (H6) (cf. Remark 2.17).

**Remark 2.20.** Note that the assumptions entail $q_\infty \leq 5$ if $d = 3$. In general, the meaning of (2.53) is that functions $\alpha$ with linear growth at 0 and with behavior at $\infty$ ranging between two not too different powers $p_\infty \leq q_\infty$ are admissible.

**Remark 2.21.** The Łojasiewicz-Simon method used here is a powerful tool permitting to prove convergence of trajectories associated to various types of either parabolic or damped hyperbolic evolution equations (we quote, among the many recent works, [1, 2, 19, 28, 41]). In particular, relation (2.49) is a local version of the Łojasiewicz-Simon inequality, which has been proved to be useful to treat energy functionals of the form (2.16). Many other versions of the inequality are available in the recent literature.
Remark 2.22. In the Theorem above, we just considered operators \( B \) satisfying (H6). The main reason for this restriction is that, up to our knowledge, inequalities of Lojasiewicz-Simon type like (2.49) are known only for linear and symmetric elliptic operators. If one could prove (2.49), or a generalization of its, for other classes of elliptic operators, then Theorem 2.18 may be extended as well. In this regard, we refer to [20] where a first attempt to include nonlinear elliptic operators in the framework of the Lojasiewicz-Simon inequality is made.

3. Proof of existence. Let us show here Theorem 2.5 by means of a time-discretization and passage to the limit procedure.

3.1. Regularization and time discretization. Here we partly follow [47, Sec. 3]. First of all, we substitute the graphs \( \alpha \) and \( \beta \) in (2.23) by their Yosida regularizations \( \alpha \nu \) and \( \beta \nu \), with \( \nu, \varepsilon > 0 \) both intended to go to 0 in the limit. Correspondingly, we set \( \tau \)

By induction on \( \nu \), it is easy to show existence for this scheme. Actually, as \( \nu \) is fixed, the right hand side of (3.2) is a known function in \( H \). Moreover, the operator

is maximal monotone and Lipschitz continuous. In particular, its domain is the whole space \( H \). Since also \( B \) is maximal monotone, we can apply, e.g., [18, Cor. 2.7, p. 36], which yields that the

3.2. A priori estimates. Let us introduce some notation. For fixed \( N \) and given \( \nu^0, \ldots, \nu^N \) in \( H \), we define the piecewise linear and backward constant interpolants of a vector \( \{\nu^i\}_{i=0}^N \) respectively as follows:

where \( \alpha_i(t) := (t - (i - 1)\tau)/\tau \) for \( t \in ((i - 1)\tau, i\tau], \ i = 1, \ldots, N \). Moreover, let us introduce the translation operator \( \mathcal{T}_\tau \) related to the time step \( \tau \) by setting

Finally, for \( i = 1, \ldots, N \), we set \( \delta \nu^i := (\nu^i - \nu^i-1)\tau^{-1} \). It is clear that \( \delta \nu^i = \nu^i, t \) a.e. in \( [0, T] \).

With this notation (3.2) takes the form

Let us now prove a technical result, i.e., the extension of (2.5) to \( \alpha \nu \).
Lemma 3.1. There exists \( \nu > 0 \) such that
\[
\alpha^\nu_r(r) \geq \sigma \quad \text{for a.e. } r \in \mathbb{R} \setminus [2S_-, 2S_+], \nu \in (0, \nu].
\] (3.7)

Clearly, this also entails the extension of (2.6) to \( \alpha^\nu_r \), with \( \sigma / 2 \) in place of \( \sigma \).

Proof. Let us firstly assume for simplicity \( \alpha \) to be locally Lipschitz continuous. In this case, one clearly has
\[
(Id + \nu \alpha)([S_-, S_+]) \subset [2S_-, 2S_+] \quad \text{at least for sufficiently small } \nu.
\] (3.8)

Then, we have \((Id + \nu \alpha)' \geq 1 + 2\nu \sigma \) outside \([S_-, S_+], \) whence the resolvent \( j_\nu := (Id + \nu \alpha)^{-1} \) satisfies
\[
j_\nu'(r) = \frac{1}{(Id + \nu \alpha)^2(j_\nu(r))} \leq \frac{1}{1 + 2\nu \sigma} \quad \forall r \notin [2S_-, 2S_+].
\] (3.9)

Indeed, \( r \notin [2S_-, 2S_+] \Rightarrow j_\nu(r) \notin [S_-, S_+] \). From (3.9), we obtain
\[
\alpha^\nu_r(r) = 1 - \frac{j_\nu'(r)}{\nu} \geq \frac{2\sigma}{1 + 2\nu \sigma} \quad \forall r \notin [2S_-, 2S_+],
\] (3.10)

whence the thesis follows for \( \nu \) sufficiently small.

By suitably adapting the notations, the latter argument still holds for multi-valued graphs \( \alpha \) whenever the sets \( \alpha(S_-) \) and \( \alpha(S_+) \) are bounded. Indeed, in this case one can still establish \( r \notin [2S_-, 2S_+] \Rightarrow j_\nu(r) \notin [S_-, S_+] \) for small \( \nu \).

On the contrary, when \( \inf \alpha(S_-) = -\infty \) (or \( \sup \alpha(S_+) = +\infty \), or both) relation (3.7) is even easier to check since in this case \( \alpha^\nu_r(r) = 1/\nu \) for all \( r < 2S_- \) and sufficiently small \( \nu \). Hence (3.7) easily follows.

The same Lemma can be applied to the graph \( \beta_\nu \). More precisely, one can prove that, for \( \eta \) as in (2.2) and some \( \overline{r}, \overline{R} > 0 \),
\[
W_\varepsilon'(r) r \geq \frac{\eta r^2}{2} \quad \forall r \notin [\overline{R}, \overline{R}], \varepsilon \in (0, \overline{r}].
\] (3.11)

Since also the primitive \( W_\varepsilon \) is defined up to an integration constant, we can also extend (2.3), which takes the form
\[
W_\varepsilon(r) \geq \frac{\eta r^2}{4} \quad \forall r \in \mathbb{R}, \varepsilon \leq \overline{r}.
\] (3.12)

Let us now prove the a priori estimates. We point out that all the constants \( c \) in this section are independent of the approximation parameters \( \varepsilon, \nu, \) and \( \tau \) as well as of \( T \). Some specific constant will be noted by \( c_\alpha, j \geq 1 \).

First estimate. Let us test (3.2) by \( (u^i - u^{i-1}) \) in the scalar product of \( H \). Using the definition of subdifferential and the elementary equality
\[
\tau (v^i, \delta v^i) = \frac{1}{2} \left( |v^i|^2 + |v^i - u^{i-1}|^2 - |u^{i-1}|^2 \right),
\]
we get
\[
\tau (\alpha^\nu_r(\delta u^i), \delta u^i) \leq \Phi(u^i) - \Phi(u^{i-1}) + \tilde{\beta}_\varepsilon(u^i) - \tilde{\beta}_\varepsilon(u^{i-1}) + \frac{\varepsilon}{2} |u^i|^2 - \frac{\varepsilon}{2} |u^{i-1}|^2
\]
\[
\leq (f^i, u^i - u^{i-1}) + \frac{\lambda}{2} (|u^i|^2 - |u^i - u^{i-1}|^2 - |u^{i-1}|^2) + c,
\] (3.13)

where \( \tilde{\beta}_\varepsilon \) is the primitive of \( \beta_\varepsilon \) satisfying \( \tilde{\beta}_\varepsilon(0) = 0 \). Then (see Lemma 3.1), the first term gives
\[
\tau (\alpha^\nu_r(\delta u^i), \delta u^i) \geq \frac{\sigma \tau}{2} |\delta u^i|^2 - c_1 \tau,
\] (3.14)
with $c_1 = 0$ if $(S0)$ holds. Correspondingly, the first term on the right hand side of (3.13) can be estimated as

$$ (f^i, u^i - u^{i-1}) \leq \frac{\sigma \tau}{8} |\delta u^i|^2 + \frac{2 \tau}{\sigma} |f|^2. \quad (3.15) $$

Hence, let us collect $W_\varepsilon(u^i)$ from its summands in (3.13), split it and the $\alpha_\nu$-term into two halves, and sum over $i = 1, \ldots, m$, for arbitrary $m \leq N$. Noting also that $\Phi(u^0) < \infty$ (which is a consequence of (2.22) and of properties of subdifferentials), using Young’s inequality, and taking the maximum as $m$ ranges in $\{1, \ldots, N\}$, we obtain (for clarity in the $\tau$-notation)

$$ \frac{\sigma}{8} \|\tilde{u}_{\cdot\cdot\cdot i}\|^2_{L^2(0,T;H)} + \frac{1}{2} \int_0^T \int_\Omega \alpha_\nu(\tilde{u}_{\cdot\cdot\cdot i}) \tilde{u}_{\cdot\cdot\cdot i} + \|\Phi(\pi_{\cdot\cdot\cdot})\|_{L^\infty(0,T)} + \frac{1}{2} \|W_\varepsilon(\pi_{\cdot\cdot\cdot})\|_{L^\infty(0,T;L^1(\Omega))} $$

$$ + \frac{\eta}{8} \|\pi\|^2_{L^\infty(0,T;H)} \leq \frac{\sigma}{2} \|\tilde{f}\|^2_{L^2(0,T;H)} + c_1 T + c, \quad (3.16) $$

where $c_1$ is the same as in (3.14) (and is $0$ if $(S0)$ is fulfilled), while $c$ depends on the initial data and on the integration constant chosen such that $W_\varepsilon$ fulfills (3.12) (which also justifies taking its norm).

We also point out that, if $(F2)$ holds, then $f^i = f$ for all $i$. Hence, the first term in the right hand side of (3.13) can also be estimated in an alternative way. Namely, summing on $i$ we have

$$ \sum_{i=1}^m (f^i, u^i - u^{i-1}) = (f, u^m) - (f, u_0) \leq \frac{\eta}{16} |u^m|^2 + c, \quad (3.17) $$

where $c$ depends on $\eta, f, u_0$, but neither on $T$ nor on $m, N$. In this case, it is worth rewriting (3.16) in a different form, avoiding use of (3.14) and exploiting (3.17) in place of (3.15). Splitting again the $W_\varepsilon$-term into two halves, we then have

$$ \int_0^T \int_\Omega \alpha_\nu(\tilde{u}_{\cdot\cdot\cdot i}) \tilde{u}_{\cdot\cdot\cdot i} + \|\Phi(\pi_{\cdot\cdot\cdot})\|_{L^\infty(0,T)} + \frac{1}{2} \|W_\varepsilon(\pi_{\cdot\cdot\cdot})\|_{L^\infty(0,T;L^1(\Omega))} + \frac{\eta}{16} \|\pi\|^2_{L^\infty(0,T;H)} \leq c, \quad (3.18) $$

where the latter $c$ depends on the constant in (3.17), on the integration constant of $W_\varepsilon$, and on the initial data, but is independent of approximation parameters and of $T$.

Second estimate. Let us now test (3.2) in the scalar product of $H$ by

$$ \tau \delta(B + W_\varepsilon + \varepsilon) u^i = (B u^i + \beta_\varepsilon(u^i) + \varepsilon u^i - \lambda u^i) - (B u^{i-1} + \beta_\varepsilon(u^{i-1}) + \varepsilon u^{i-1} - \lambda u^{i-1}). $$

This gives

$$ \left( \alpha_\nu \left( \frac{u^i - u^{i-1}}{\tau} \right) , \tau \delta(B + W_\varepsilon + \varepsilon) u^i \right) + \left( (B + \beta_\varepsilon + \varepsilon) u^i - \lambda u^{i-1} , \tau \delta(B + W_\varepsilon + \varepsilon) u^i \right) = (f^i, \tau \delta(B + W_\varepsilon + \varepsilon) u^i). \quad (3.19) $$

Then, let us notice that, by monotonicity of $\alpha_\nu$ and $\beta_\varepsilon$ and $\alpha_\nu(0) = 0$, one has that

$$ \left( \alpha_\nu \left( \frac{u^i - u^{i-1}}{\tau} \right) , \beta_\varepsilon(u^i) - \beta_\varepsilon(u^{i-1}) + \varepsilon(u^i - u^{i-1}) \right) \geq 0. \quad (3.20) $$

Analogously, integrating by parts and using the monotonicity of $\alpha_\nu$ and $b$ (cf. (2.10)) and the Lipschitz continuity of $\alpha_\nu$, we get that

$$ \left( \alpha_\nu \left( \frac{u^i - u^{i-1}}{\tau} \right) , B u^i - B u^{i-1} \right) \geq 0. \quad (3.21) $$
Next, let us sum over $i = 1, \ldots, m, m \leq N$. Elementary calculations permit us to conclude that

\[
\sum_{i=1}^{m} \left( (B + \beta + \varepsilon)u^{i} - \lambda u^{i-1}, \delta (B + W^{\varepsilon}_x + \varepsilon)u^{i} \right)
= \frac{1}{2} \left| (B + \beta + \varepsilon - \lambda)u^{m} \right|^2 - \frac{1}{2} \left| (B + \beta + \varepsilon - \lambda)u^{0} \right|^2
+ \frac{1}{2} \sum_{i=1}^{m} \left| (B + \beta + \varepsilon)u^{i} - (B + \beta + \varepsilon)u^{i-1} \right|^2 - \frac{1}{2} \sum_{i=1}^{m} \left| \lambda u^{i} - \lambda u^{i-1} \right|^2. \tag{3.22}
\]

By collecting the above calculations we have proved that

\[
\frac{1}{2} \left| (B + W^{\varepsilon}_x + \varepsilon)u^{m} \right|^2 \leq \frac{1}{2} \left| (B + W^{\varepsilon}_x + \varepsilon)u^{0} \right|^2 + \lambda \sum_{i=1}^{m} (\alpha_{i} (\delta u^{i}), u^{i} - u^{i-1})
+ \frac{\lambda^2}{2} \sum_{i=1}^{m} |u^{i} - u^{i-1}|^2 + \sum_{i=1}^{m} (f_{i}, \tau \delta (B + W^{\varepsilon}_x + \varepsilon)u^{i}) \tag{3.23}
\]

Furthermore, let us observe that, in case (f2) holds, i.e. $f^i = f$ for all $i$, we get, for $c$ depending on initial data and on $f$,

\[
\sum_{i=1}^{m} (f_{i}, \tau \delta (B + W^{\varepsilon}_x + \varepsilon)u^{i}) = \left( f_{i}, (B + W^{\varepsilon}_x + \varepsilon)u^{m} - (B + W^{\varepsilon}_x + \varepsilon)u^{0} \right)
\leq c + \frac{1}{4} \left| (B + W^{\varepsilon}_x + \varepsilon)u^{m} \right|^2. \tag{3.24}
\]

Finally, by the monotonicity of $B$ and $\beta$, we have

\[
\frac{1}{4} \left| (B + \beta + \varepsilon - \lambda)u^{m} \right|^2 \geq \frac{1}{4} |Bu^{m}|^2 + \frac{1}{4} |\beta(u^{m})|^2 + \frac{(\varepsilon - \lambda)^2}{4} |u^{m}|^2
+ \frac{(\varepsilon - \lambda)}{2} (Bu^{m}, u^{m}) + \frac{(\varepsilon - \lambda)}{2} |\beta(u^{m}), u^{m}|,
\geq \frac{1}{8} |Bu^{m}|^2 + \frac{1}{8} |\beta(u^{m})|^2 - c_2(1 + \lambda^2)|u^{m}|^2, \tag{3.25}
\]

where $c_2$ can be chosen independently of $\varepsilon$ and $\lambda$.

Collecting (3.20)–(3.24) and taking the maximum for $m \in \{1, \ldots, N\}$, in the $\tau$-notation (3.19) becomes

\[
\frac{1}{8} \left\| B\pi_{\tau} \right\|^2_{L^\infty(0,T;H)} + \frac{1}{8} \left\| \beta_{\varepsilon}(\pi_{\tau}) \right\|^2_{L^\infty(0,T;H)}
\leq c + \lambda \int_{0}^{T} \int_{\Omega} \alpha(\tilde{u}_{\tau}, t) \tilde{u}_{\tau, t} + \frac{\tau \lambda^2}{2} \left\| \tilde{u}_{\tau, t} \right\|^2_{L^2(0,T;H)} + c_2(1 + \lambda^2) \left\| \pi_{\tau} \right\|^2_{L^\infty(0,T;H)}. \tag{3.26}
\]

As before, let us consider also the case where (f1) holds in place of (f2). Now, in place of (3.24), the term with $f$ has to be bounded by use of the following discrete Gronwall-like Lemma, whose proof can be easily deduced from [42, Prop. 2.2.1].

**Lemma 3.2.** Let $v^i, z^i \in H, i = 0, 1, \ldots, N$, be such that

\[
|v^m|^2 \leq c + c\tau \sum_{i=1}^{m} (z^i, (\delta v)^i) \quad \forall m = 1, \ldots, N, \tag{3.27}
\]

for some $c > 0$. Then, we have

\[
|v^m| \leq c' \quad \forall m = 1, \ldots, N. \tag{3.28}
\]
for some $c' > 0$ depending only on $c$, $v^0$, $z^0$, and $\sum_{i=1}^N \tau|\delta z^i|$.

By applying the latter Lemma to (3.23) with $v^i = (B + W_i^r + \varepsilon)u^i$ and $z^i = f^i$, we then arrive again at the estimate (3.26), of course with a different value of the constant $c$ which now depends in particular on the norm of $f$ in $L^1(0, +\infty; H)$.

Comprehensive estimate. We consider separately the general case and the specific one given by (f2). In the general case, we have to multiply (3.16) by $\max\{4\lambda, 8(c_2(1 + \lambda^2) + 1)\eta^{-1}\}$ and sum the result to (3.26), getting

$$\lambda \int_0^T \int_\Omega \alpha_v(\bar{\omega}_{\tau,t})\bar{\omega}_{\tau,t} + \frac{\lambda \sigma}{2}\|\bar{\omega}_{\tau,t}\|^2_{L^2(0,T;H)} + 4\lambda\|\Phi(\bar{\omega}_{\tau})\|_{L^\infty(0,T)} + 2\lambda\|W_c(\bar{\omega}_{\tau})\|_{L^\infty(0,T; L^1(\Omega))}
+ \frac{1}{8}\|B\bar{\omega}_{\tau}\|^2_{L^\infty(0,T;H)} + \frac{1}{8}\|\beta_c(\bar{\omega}_{\tau})\|^2_{L^\infty(0,T;H)} + \|\bar{\omega}_{\tau}\|^2_{L^\infty(0,T;H)}
\leq \frac{c_3^2}{\sigma} \|\bar{\omega}_{\tau}\|^2_{L^2(0,T;H)} + \frac{\tau \lambda^2}{2}\|\bar{\omega}_{\tau,t}\|^2_{L^2(0,T;H)} + c_3 T + c,$$

(3.29)

where $c_3$, which comes from $c_1$, is 0 if (S0) holds.

If (f2) holds, we can get something more precise if we multiply (3.18) by $\max\{2\alpha, 16(c_2(1 + \lambda^2) + 1)\eta^{-1}\}$ and sum the result to (3.26). This gives

$$\lambda \int_0^T \int_\Omega \alpha_v(\bar{\omega}_{\tau,t})\bar{\omega}_{\tau,t} + 2\lambda\|\Phi(\bar{\omega}_{\tau})\|_{L^\infty(0,T)} + \lambda\|W_c(\bar{\omega}_{\tau})\|_{L^\infty(0,T; L^1(\Omega))} + \|\bar{\omega}_{\tau}\|^2_{L^\infty(0,T;H)}
+ \frac{1}{8}\|B\bar{\omega}_{\tau}\|^2_{L^\infty(0,T;H)} + \frac{1}{8}\|\beta_c(\bar{\omega}_{\tau})\|^2_{L^\infty(0,T;H)} \leq \frac{\tau \lambda^2}{2}\|\bar{\omega}_{\tau,t}\|^2_{L^2(0,T;H)} + c.$$

(3.30)

Estimate (3.29) (or (3.30)) is the basic ingredient needed in order to pass to the limit within the approximations.

3.3. Passage to the limit. We remove here first the approximation in $\tau$ and then, simultaneously, those in $\varepsilon$ and $\nu$. In order to take the first limit, let us fix $T > 0$. All the convergence properties listed below are intended to hold up to successive extractions of subsequences of $\tau \downarrow 0$, not relabeled. Let us notice that the second term in the right hand side of (3.29) is estimated, for $\tau$ sufficiently small, by the second on the left hand side. Moreover, note that a straightforward comparison argument in equation (3.2) entails that $\alpha_v(\bar{\omega}_{\tau})$ is uniformly bounded with respect to $\tau$ in $L^\infty(0, T; H)$. Hence, for suitable limit functions $u, \xi, \eta$ we obtain

$$\bar{\omega}_{\tau,t} \to u_t \quad \text{weakly-* in } L^\infty(0, T; H),$$

(3.31)

$$\bar{\omega}_{\tau} \to u \quad \text{weakly-* in } L^\infty(0, T; H),$$

(3.32)

$$B\bar{\omega}_{\tau} \to \eta \quad \text{weakly-* in } L^\infty(0, T; H),$$

(3.33)

$$\alpha_v(\bar{\omega}_{\tau}) \to \xi \quad \text{weakly-* in } L^\infty(0, T; H),$$

(3.34)

where it is a consequence of standard argument the fact that the limits of $\bar{\omega}_{\tau}$ and $\bar{\omega}_{\tau}$ do coincide. Moreover, it is clear that (3.29) entails

$$\|E(\bar{\omega}_{\tau})\|_{L^\infty(0,T)} + \|E(\bar{\omega}_{\tau})\|_{L^\infty(0,T)} \leq c,$$

(3.35)

whence, by (3.32), Lemma 2.3, and [64, Thm. 1] it is not difficult to get

$$\bar{\omega}_{\tau} \to u \quad \text{strongly in } L^\infty(0,T; H).$$

(3.36)

More precisely, one has that

$$\bar{\omega}_{\tau}(t), \bar{\omega}_{\tau}(t) \to u(t) \quad \text{strongly in } H \quad \text{for every } t \in [0, T]$$

(3.37)
and c disappeared. Thus, for suitable limiting functions $\tau$, the take the $\tau$ in order to emphasize dependence on $n$ be taken in sequence, instead. Thus, let us take a sequence of $(\nu)$ parameters to 0 simultaneously, for brevity. However, it is clear that the two limits might also $	au$ from [12]. The passage to the limit in $a.e.$ in $Q$ corresponding limit $\beta$ two inclusions some care is needed since the operators themselves are approximations. Indeed, one

Now, we pass to the limit as $n \to +\infty$ (hence $\varepsilon_n, \nu_n \to 0$) in (3.38) written at level $n$. We obtain

$$\xi + \eta + \gamma - \lambda u = f,$$

and we have to identify $\xi, \eta$, and $\gamma$ in terms of $\alpha(u_t), Bu$, and $\beta(u)$, respectively.

The identification $\eta \in Bu$ follows immediately from (3.41) and (3.43). As for the remaining two inclusions some care is needed since the operators themselves are approximations. Indeed, one
has that the functionals
\[ L^2(0,T;H) \ni u \mapsto \int_{\Omega} \beta_{\varepsilon_n}(u(x)) \, dx \]
converges in the sense of Mosco [7] in \( L^2(0,T;H) \) to
\[ L^2(0,T;H) \ni u \mapsto \int_{\Omega} \beta(u(x)) \, dx \quad \text{if } \beta(u) \in L^1(\Omega) \quad \text{and } +\infty \text{ otherwise.} \]
The latter functional convergence, (3.41), and (3.45) immediately give the identification \( \gamma \in \beta(u) \) a.e. in \( QT \) via [7, Prop. 3.56.c, p. 354 and Prop. 3.59, p. 361]. Moreover, owing to the lower semicontinuity of \( \Phi \) and the convergence (3.41), we readily check that
\[ \liminf_{n \to +\infty} (\Phi(u_n(T)) + \int_{\Omega} \beta_{\varepsilon_n}(u_n(T))) \geq \Phi(u(T)) + \int_{\Omega} \beta(u(T)). \tag{3.47} \]
Arguing once again along the lines of (3.39), the latter inequality entails in particular that (see (3.40))
\[ \limsup_{n \to +\infty} \int_{0}^{t} (\alpha_{\varepsilon_n}(u_n), u_{n,t}) \leq \int_{0}^{t} (\xi, u_t), \tag{3.48} \]
and the inclusion \( \xi \in \alpha(u_t) \) a.e. in \( QT \) follows from the above-cited results from [7].

The proof of existence is thus concluded. Let us make, anyway, two final observations. Actually, a by-product of our procedure is that also the limit solution \( u \) satisfies estimates analogous to (3.29), (3.30) (but without the term with \( \tau \) on the right hand side). We report, for completeness, the limit version of (3.29), which will be used again in the sequel
\[ \lambda \int_{0}^{T} \int_{\Omega} \alpha(u(t)) u_t + \frac{\lambda \sigma}{2} \| u_t \|_{L^2(0,T;H)}^2 + 4\lambda \| \Phi(u) \|_{L^\infty(0,T)} + 2\lambda \| W(u) \|_{L^\infty(0,T;L^1(\Omega))} \]
\[ + \frac{1}{8} \| B\nabla u \|_{L^2(0,T;H)}^2 + \frac{1}{8} \| \beta(u) \|_{L^\infty(0,T;H)}^2 + \| u \|_{L^\infty(0,T;H)}^2 \]
\[ \leq c(1 + \| f \|_{L^2(0,T;H)}^2) + c_3 T, \tag{3.49} \]
where \( c_3 = 0 \) if \( (S0) \) holds and still \( c, c_3 \) are independent of \( T \).

The second observation is that, thanks to the convergence (3.43), we can prove that
\[ \Phi(u_n(t)) \to \Phi(u(t)) \quad \text{for any } t \in [0,T]. \tag{3.50} \]
Actually, the definition of subdifferential written for \( u_n \) gives that
\[ \Phi(u_n(t)) \leq (Bu_n(t), u_n(t) - u(t)) + \Phi(u(t)) \quad \text{for any } t \in [0,T] \tag{3.51} \]
(and not only almost everywhere, see the Proof of Proposition 2.10 just below). Thus, by taking the limsup in (3.51) and recalling (3.41), (3.43), and the lower semicontinuity of \( \Phi \), we readily get (3.50). In the specific case in which \( \Phi(v) = \frac{1}{p} \| \nabla v \|_{p}^p \) (hence \( Bv \) is the \( p \)-laplacian), the convergence (3.50) entails the convergence of \( u_n \) in \( W^{1,p}(\Omega) \) (recall that \( p > 1 \) and thus \( W^{1,p}(\Omega) \) is uniformly convex).

4. **Separation property and uniqueness.** Henceforth, let us denote by \( \text{solution} \) any function \( u \) satisfying (2.23)–(2.24) in the sense and with the regularity made precise in Theorem 2.5.
Proof of Proposition 2.10. Let us first prove that, for any solution $u$, one has $u(t) \in \text{dom}_H B$ for all, and not just a.e., $t \in [0, T]$. Indeed, by (2.25), $u$ lies in $C^0([0, T]; H)$. Thus, assuming by contradiction that, for some $t$, $u(t) \notin \text{dom}_H B$, we can approximate $t$ by a sequence $\{t_n\}$ such that $u(t_n) \in \text{dom}_H B$ for all $n$. Since (2.27) holds, we can assume $\{Bu(t_n)\}$ bounded in $H$. Hence, extracting a subsequence $\{n_k\}$ such that $u(t_{n_k}) \to u(t)$ strongly in $H$ and $Bu(t_{n_k}) \to \tilde{B}$ weakly in $H$, by maximal monotonicity of $B$ we have that $\tilde{B} = Bu(t)$, whence the assert follows.

The same argument can be applied in order to check that $\beta^0(u(t)) \in H$ for all $t \in [0, T]$. As a consequence, by semicontinuity of norms with respect to weak convergence, we have more precisely that
\[ |u(t)| + |Bu(t)| + |\beta^0(u(t))| \leq c \quad \forall t \in [0, T], \tag{4.1} \]
where $c$ is the same constant as in (2.27) and, in particular, does not depend on $T$ if either (S0)–(f1) or (f2) hold. Consequently, thanks to (2.31)-(2.32), there exists one constant $\delta > 0$, depending only on $c$ in (4.1) through the function $\gamma$, such that
\[ |u(x_1, t) - u(x_2, t)| \leq \delta|x_1 - x_2| \quad \forall x_1, x_2 \in \overline{\Omega}, \quad t \in [0, T], \tag{4.2} \]

We can now prove the right inequality in (2.37) in the case when $I$ is right-bounded (if $I$ is not right-bounded, the inequality is trivial). The proof of the left inequality is analogous, of course. Set
\[ \rho := \delta^{-1/\nu}|r_1 - r|^1/\nu \tag{4.3} \]
and assume, by contradiction, that there exist $\bar{t} > 0$, $\bar{x} \in \overline{\Omega}$ such that $u(\bar{x}, \bar{t}) = \tau$. By (4.2), it is clear that
\[ |u(x, \bar{t}) - \tau| = |u(x, \bar{t}) - u(\bar{x}, \bar{t})| \leq \delta|x - \bar{x}|^{\nu} \leq |r_1 - \tau| \quad \forall x \in \overline{\Omega} \cap B(\bar{x}, \rho). \tag{4.4} \]

Since the value of $u$ cannot exceed $\tau$, (4.4) entails that $u(x, \bar{t}) \geq r_1$ for all $x \in \overline{\Omega} \cap B(\bar{x}, \rho)$. Then, by (2.33),
\[ \beta^0(u(x, \bar{t})) \geq \frac{c}{(\tau - u(x, \bar{t}))^\nu} = \frac{c}{(\tau-u(x, \bar{t}))^\nu} \quad \forall x \in \overline{\Omega} \cap B(\bar{x}, \rho). \tag{4.5} \]
By taking squares, integrating in space, and using (4.1) and (4.4), we obtain
\[ c \geq \int_{\Omega} (\beta^0)^2(u(x, \bar{t})) \, dx \geq \int_{\Omega \cap B(\bar{x}, \rho)} (\beta^0)^2(u(x, \bar{t})) \, dx \]
\[ \geq \int_{\Omega \cap B(\bar{x}, \rho)} |u(x, \bar{t}) - u(\bar{x}, \bar{t})|^{2\nu} \, dx \geq \int_{\Omega \cap B(\bar{x}, \rho)} c\delta^{-2\kappa} \, dx. \tag{4.6} \]
Now, since $\Omega$ is a smooth set (here Lipschitz would be enough), there exists $c_\Omega > 0$ such that, for any sufficiently small $r > 0$, $\Omega \cap B(\bar{x}, r)$ measures at least $c_\Omega r^d > 0$. Recalling (2.34), this entails that the latter integral in (4.6) is $+\infty$, yielding a contradiction.

This means that no solution $u$ can ever attain the value $\tau$. However, in order to show (2.37), we have to be more precise. We actually claim that, if $\mathcal{F} \subset H$ is any set such that
\[ \exists \delta' > 0 : \mathcal{F} \subset \mathcal{G}(\delta'), \quad \mathcal{G}(\delta') := \left\{ v \in H : |v| + |Bv| + |\beta^0(v)| \leq \delta' \right\}, \tag{4.7} \]
then there exists $\tau^*$ such that $v(x) \leq \tau^*$ for all $v \in \mathcal{F}$ and $x \in \overline{\Omega}$. Applying this to the family $\mathcal{F} = \{u(t)\}_{t \in [0, T]}$, we clearly get the upper inequality in (2.37), as desired. In addition, it is a by-product of the argument that $\tau^*$ is independent of $T$ if such is $c$ in (2.27), i.e., if either (S0)–(f1) or (f2) hold.
To prove the claim, let us proceed by contradiction. Namely, suppose that there are \( \{v_n\} \subset \mathcal{F}, \{x_n\} \subset \overline{\Omega} \) such that \( v_n(x_n) \not\to \pi \). Since \( \mathcal{S}(\delta') \) is bounded in \( C^{0,\delta}(\overline{\Omega}) \) by (2.31)-(2.32), then we can extract a subsequence \( n_k \) such that \( x_{n_k} \to \pi \in \overline{\Omega} \) and \( v_{n_k} \to v \) uniformly. Thus, \( v(\pi) = \pi \). But this is impossible, since it can be easily seen that also \( v \in \mathcal{S}(\delta') \); hence, for the first part of the argument \( v \) can never attain the value \( \pi \).

**Proof of Theorem 2.11.** Assume by contradiction there exist two solutions \( u_i, i = 1, 2 \). Then, writing (2.23) for \( i = 1, 2 \), taking the difference, and testing it by \( (u_1 - u_2)_t \), we get, for a.e. \( t > 0 \),
\[
\int_{\Omega} (\alpha((u_1)_t) - \alpha((u_2)_t) + B(u_1 - u_2) + W'(u_1) - W'(u_2))(u_1 - u_2)_t = 0. \tag{4.8}
\]
Next, we note that
\[
\int_{\Omega} B(u_1 - u_2)(u_1 - u_2)_t = \frac{1}{2} \frac{d}{dt} \int_{\Omega} B(u_1 - u_2)(u_1 - u_2) \tag{4.9}
\]
and, by (H2) and (S0),
\[
\int_{\Omega} (\alpha((u_1)_t) - \alpha((u_2)_t))(u_1 - u_2)_t \geq 2\sigma \int_{\Omega} |(u_1)_t - (u_2)_t|^2. \tag{4.10}
\]
Finally, by (2.37) and (2.41), it is clear that
\[
\int_{\Omega} (W'(u_1) - W'(u_2))(u_1 - u_2)_t \leq \frac{\sigma}{2}|(u_1)_t - (u_2)_t|^2 + c|u_1 - u_2|^2, \tag{4.11}
\]
with the last \( c \) depending only on \( \sigma \), on the constant \( c \) in (2.27), and on the Lipschitz constant of \( W' \) in the interval \([\tilde{\tau}^-, \tilde{\tau}^+]\).

Taking the integral of (4.8) on \((0, t)\) and exploiting the relation
\[
|(u_1 - u_2)(t)|^2 \leq t \|(u_1)_t - (u_2)_t\|_{L^2(0, t; H)}^2 \tag{4.12}
\]
the conclusion follows immediately by taking (4.9)-(4.11) into account and applying Gronwall’s Lemma.

5. **Long-time behavior. Proof of Theorem 2.13.** Let \( \{t_n\} \) be fixed in such a way that \( t_n \to +\infty \). Then, property (2.42) for some suitable (not relabeled) subsequence is a direct consequence of bound (2.27) and the precompactness in \( H \) of the trajectory from Lemma 2.3.

In order to conclude, we have to show that the limit \( u_\infty \) solves the stationary problem (2.43). To see this, let us first note that, if (H1) holds, then \( f(t) \) tends to 0 strongly in \( H \) as \( t \to \infty \). Then, we can consider the sequence of Cauchy problems for \( f_n(\cdot) := f(\cdot + t_n) \)
\[
\begin{align*}
\alpha((u_n)_t) + Bu_n + W'(u_n) &= f_n \quad \text{in } H \\
u_n(0) &= u(t_n),
\end{align*}
\tag{5.1}
\]
and it is clear that \( u_n(t) := u(t + t_n) \) solves (5.1), e.g., for \( t \in (0, 1) \). Moreover, by (2.27), where \( c \) is now independent of \( t \), we have
\[
u_n \to \bar{u} \text{ strongly in } C^0([0, 1]; H), \tag{5.2}
\]
\[
Bu_n \to \bar{B} \text{ weakly-\ast in } L^\infty(0, 1; H), \tag{5.3}
\]
\[
W'(u_n) \to \bar{w} \text{ weakly-\ast in } L^\infty(0, 1; H), \tag{5.4}
\]

for suitable limit functions \( \tilde{u}, \tilde{B}, \tilde{w} \). Due to the standard monotonicity argument [12, Lemma 1.3, p. 42], this immediately yields

\[
\tilde{B} = B\tilde{u}, \quad \tilde{w} = \beta(\tilde{u}) - \lambda\tilde{u} \quad \text{a.e. in } \Omega \times (0,1). \tag{5.5}
\]

Furthermore, by (3.49), which now holds with \( c_3 = 0 \),

\[
(u_n)_t \to 0 \quad \text{strongly in } L^2(0,1; H), \tag{5.6}
\]

whence \( \tilde{u} \) is constant in time. Since \( \tilde{u}(0) = u_\infty \) by (2.42) and the Cauchy condition in (5.1), we readily conclude that \( \tilde{u} \equiv u_\infty \) for a.e. \( t \in (0,1) \). Finally, still from (2.27) we have

\[
\alpha((u_n)_t) \to \tilde{\alpha} \quad \text{weakly-\ast in } L^\infty(0,1; H), \tag{5.7}
\]

where actually \( \tilde{\alpha} \equiv 0 \) a.e. in \( (0,1) \) thanks to (5.6), [12, Lemma 1.3, p. 42], and (2.44). This completes the proof of relation (2.43) and of the Theorem.

**Proof of Theorem 2.18.** Let us consider the non-autonomous case when (\( \mathbf{f1} \)) holds, the situation where we have (\( \mathbf{f2} \)) being simpler. We argue along the lines of [19, Sec. 3]. However, due to the presence of the nonlinearity \( \alpha \), our proof presents further technical complications. Let \( u_\infty \) be an element of the \( \omega \)-limit and note first that, by precompactness (cf. Remark 2.19), \( u_\infty \) is the limit in \( C(\overline{\Omega}) \) of some sequence \( \{u(t_n)\} \). Thus, by (2.48), at least for \( n \) sufficiently large, we have that

\[
\xi < u_n(x) < \tau \quad \forall x \in \Omega. \tag{5.8}
\]

This justifies the application of the Lojasiewicz-Simon inequality, since \( u_n \) eventually ranges in the interval where \( W' \) is analytic. In particular, the barriers at the extrema of \( W \) are excluded even in case (2.37) does not hold.

Then, similarly with [19], we can set (but note that we use here the norm in \( H \) instead of that in \( V \) (cf. the regularity of \( u_t \) and (5.11) below), in agreement with version (2.49) of the Lojasiewicz-Simon inequality):

\[
\Sigma := \{ t > 0 : \|u(t) - u_\infty\|_V \leq \epsilon/3 \}. \tag{5.9}
\]

Clearly, \( \Sigma \) is unbounded. Next, for \( t \in \Sigma \), we put

\[
\tau(t) := \sup \{ t' \geq t : \sup_{s \in [t,t')} \|u(s) - u_\infty\|_V \leq \epsilon \}, \tag{5.10}
\]

where, by continuity, \( \tau(t) > t \) for all \( t \in \Sigma \). Let us fix \( t_0 \in \Sigma \) and divide \( \mathcal{J} := [t_0, \tau(t_0)] \) into two subsets:

\[
A_1 := \{ t \in \mathcal{J} : |u_t(t)| \geq \left( \int_t^\tau |f(s)|^2 \, ds \right)^{1-\theta} \}, \tag{5.11}
\]

\[
A_2 := \mathcal{J} \setminus A_1. \tag{5.12}
\]

Letting now

\[
\Phi_0(t) := E(u(t)) - E(u_\infty) + \frac{1}{\sigma} \int_t^{\tau(t_0)} |f(s)|^2 \, ds, \tag{5.13}
\]

exploiting assumption (2.52) and Hölder’s inequality and making a comparison in (2.23), it is not difficult to see that

\[
\Phi_0(t) \leq -\frac{\sigma}{2}|u_t(t)|^2 - \kappa_\infty \|u_t(t)\|_{L^{p_\infty+1}(\Omega)}^{p_\infty+1} - \frac{1}{2\sigma_2} |f(t)|^2. \tag{5.14}
\]

Note that \( \Phi_0 \) is absolutely continuous thanks to [18, Lemme 3.3, p. 73]. Thus, we have [41, (3.2)]

\[
\frac{d}{dt} \left( (|\Phi_0|^\theta \text{ sign } \Phi_0)(t) \right) \leq -\theta |\Phi_0(t)|^{\theta-1} \left( \frac{\sigma}{2} |u_t|^2 + \kappa_\infty \|u_t\|_{L^{p_\infty+1}(\Omega)}^{p_\infty+1} + \frac{1}{2\sigma_2} |f|^2 \right)(t). \tag{5.15}
\]
where we used the continuous embeddings $H \subset L^\chi(\Omega) \subset V^\ast$ and the last constant $c$ also depends on $\sigma'$ and $\ell_\infty$ in (2.52).

Thus, being $\chi q_\infty \leq p_\infty + 1$ by (2.53) and $q_\infty \geq (p_\infty + 1)/2$ since $q_\infty \geq p_\infty \geq 1$, from (2.27) and (5.16) we have that

$$|\Phi_0(t)|^{\theta - 1} \geq c \left( |u_t(t)| + \|u_t(t)\|_{L^{p_\infty + 1}((\Omega)} + |f(t)| \right)^{-1}.$$  \hspace{1cm} (5.17)

Collecting now (5.14), (5.16), and (5.17), (5.15) gives

$$\left( |u_t(t)| + \|u_t(t)\|_{L^{p_\infty + 1}((\Omega)} + |f(t)| \right) \leq -c \frac{d}{dt} \left( \Phi_0(t) \sigma \Phi_0(t) \right),$$ \hspace{1cm} (5.18)

whence, integrating over $A_1$ and exploiting that $\Phi_0$ is a decreasing function (cf. (5.14)), we get that $|u_t|$ is integrable over $A_1$.

From this point on, the argument proceeds exactly as in [19]. Namely, by definition of $A_2$ and (2.51) and possibly taking some smaller $\theta$, one immediately gets that $|u_t|$ is integrable over $A_2$ and hence on $J$. This permits to show by a simple contradiction argument that $\tau(t_0) = \infty$ as $t_0 \in \Sigma$ is sufficiently large. This entails $u_t \in L^1(t_0, +\infty; H)$, whence the convergence of the whole trajectory to $u_\infty$ follows, as desired.

Finally, let us just briefly outline the changes to be done when, instead, (f2) holds. The most significant difference is that now it is convenient to include $f$ into the energy, setting, for $v \in H$, $E_f(v) := E(v) - (f, v)_H$. Then, it is clear that the Łojasiewicz-Simon inequality (2.49) still holds in the form

$$|E_f(v) - E_f(u_\infty)|^{1-\theta} \leq c \|B v + W'(v) - f\|_{V^\ast}.$$ \hspace{1cm} (5.19)

At this point, the proof is performed similarly, provided that one defines $\Phi_0$ in (5.13) with $E_f$ in place of $E$ and without the integral term. Moreover, one directly gets the integrability of $|u_t(t)|$ on $J$ and no longer needs to split it into the two subsets $A_1$ and $A_2$. The details of the argument are left to the reader.

**Remark 5.1.** Let us note that, if (f1) and (2.51) hold, then it is also possible to estimate the decay rate of solutions as in [37, 38]. Namely, one can prove (cf., e.g., [38, (3.7)]) that

$$|u(t) - u_\infty| \leq c t^{-\mu} \quad \forall t > 0,$$ \hspace{1cm} (5.20)

where $\mu > 0$ depends on $\theta, \xi$ and $c$ depends only on data (and in particular not on time).

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