<u>Hilbert triplet</u> V, H Hilbert, $\langle \cdot, \cdot \rangle$ duality product between V' and V, $(\cdot, \cdot)$ scalar product in H. Assume $V \hookrightarrow H$ densely. Then one can identify H with a subspace of V' via the map $I : H \longrightarrow V'$ defined by

$$\langle Ih, v \rangle := (h, v) \quad \forall v \in V$$

Note that Ih is indeed linear and continuous as $|\langle Ih, v \rangle| = |(h, v)| \leq \|h\|_H \|v\|_H$
$$\leq \|h\|_H \, c_{V \hookrightarrow H} \|v\|_V$$

In particular $\|Ih\|_{V^*} \leq c_{V \hookrightarrow H} \|h\|_H$. This defines the Hilbert triplet

$$V \hookrightarrow H \hookrightarrow V'$$

Example : $H_0^1(D) \subset L^2(D) \subset (H_0^1(D))' = H^{-1}(D)$

The notation 'h' is often used instead of 'Ih'. Note that the density of $V \hookrightarrow H$ is needed in order to have I to be injective.

Two different formulations

Let $A: V \longrightarrow V'$ linear, continuous, and coercive and $l \in V'$ be given. We have checked that

$$Au = l \quad \text{in } V'$$

admits a unique solution $u \in V$. In case l is 'more regular', say $l \in H$, one has that the single-space problem

$$A_H u = l \quad \text{in } H$$

with $A_H v = Av$ for $v \in D(A_H) = \{w \in H : Aw \in H\}$ is solvable as well.

Example Dirichlet: $\begin{cases} -\Delta u = f & \text{in } \Omega, \; f \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega \end{cases}$

V-V' duality formulation:
find $u \in H^1_0(\Omega)$: $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H^1_0(\Omega)$

Single-space formulation:
find $u \in H^1_0(\Omega)$ with $-\Delta u \in L^2(\Omega)$:
$-\Delta u = f$ in $L^2(\Omega)$ (meaning a.e. = strong solution)

Remark for non-homogeneous Neumann no single-space formulation available, for $l \in V' \setminus H$.

Elliptic regularity

Let D be of class C^2 or convex with ∂D bounded (or $D = \mathbb{R}^d_+$) and $f \in L^2(D)$. Then, the unique sol. $u \in H^1_0(D)$ of $Au = f$ is actually in $H^2(D)$. If Ω is of class C^{m+2} and $f \in H^m(\ )$ for $m > d/2$ then $u \in C^2(D)$. If D is C^∞ and $f \in C^\infty(D)$ then $u \in C^\infty(D)$.

Let now A be nonlinear

$$A : D(A) \subset V \longrightarrow V'$$

and $l \in V'$. A condition for the solvability of $Au = l$ in V' is the following.

Thm Assume A is

1) monotone $\left(\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq 0 \ \forall v_1, v_2 \in V \right)$
2) maximal (the graph of A in $V \times V'$ is not strictly included in the graph of any other monotone operator from V to V')
3) coercive $\left(\liminf_{\|v\|_V \to +\infty} \dfrac{\langle Av, v \rangle}{\|v\|_V} = +\infty \right)$.

Then, A is onto and the equation $Au = l$ can be solved, for any $l \in V^*$. If A is strictly monotone ($\langle Av_1 - Av_2, v_1 - v_2 \rangle > 0 \ \forall v_1 \neq v_2$), such solution is unique.

<u>Application</u> $\beta : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous and nondecreasing. Then, for all $u \in L^2(D)$ one has that $\beta(u) \in L^2(D)$ as well, at least for D bounded. Define $A : H_0^1(D) \to H^{-1}(D)$ as

$$\langle Au, v \rangle = \int_D \nabla u \cdot \nabla v + \int_D \beta(u) v.$$

1) A is monotone

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle = \int_D |\nabla(u_1 - u_2)|^2 +$$
$$+ \int_D (\beta(u_1) - \beta(u_2))(u_1 - u_2) \geq 0$$

2) A is maximal (this we do not check ...)

3) A is coercive

$$\langle Au, u \rangle = \int_D |\nabla u|^2 + \int_D \beta(u) u =$$
$$= \int_D |\nabla u|^2 + \int_D (\beta(u) - \beta(0))(u - 0) + \int_D \beta(0) u$$
$$\geq \|\nabla u\|^2_{L^2} - |\beta(0)| \, \|u\|_{L^1} \geq$$
$$\geq \|\nabla u\|^2_{L^2} - |\beta(0)| \, |D|^{1/2} \|u\|_{L^2} \geq \|u\|_V^2 - C\|u\|_V$$
$$\geq \frac{1}{2}\|u\|_V^2 - \frac{C^2}{2},$$ where we have used

$$ab \leq \frac{1}{2\delta} a^2 + \frac{\delta}{2} b^2 \quad \forall \delta > 0, \ a, b \in \mathbb{R}$$