SCRIPT OF THE COURSE
TOPICS IN NONLINEAR EVOLUTION

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Abstract. I am collecting here some notes from the course on *Topics in Nonlinear Evolution*, SS16, Uni Wien. These are of course to be taken with no implicit understanding for completeness. I will be much indebted to anyone pointing me out mistakes.

1. Introduction: finite dimensions

Let us introduce some notation by starting from the finite dimensional case: the state of some system is described by $u \in \mathbb{R}^m$ and driven by the *energy* $E : \mathbb{R}^m \to \mathbb{R}$, here assumed to be sufficiently regular. Then, the *equilibria* of the system are the critical points of the energy

$$\nabla E(u) = 0.$$

By introducing a time-dependent linear *forcing* $t \mapsto f(t) \in \mathbb{R}^m$ one can consider the complementary energy $u \mapsto E(u) - f(t) \cdot u$ and search for a trajectory $t \mapsto u(t)$ sitting at critical points for all times, namely solving

$$\nabla E(u(t)) = f(t).$$

The latter is usually referred to as *metastable* (or, in some instances, *quasistatic*) evolution: it consists of a continuum of equilibria. Although relevant to many evolution problems, especially in continuum mechanics, metastable evolution fails to describe very common situations (as simple as that of a falling object).

A second very relevant evolution mode is that of a *gradient flow*

$$\dot{u}(t) + \nabla E(u(t)) = 0. \tag{1}$$

By multiplying (1) by $\dot{u}$ one gets

$$|\dot{u}|^2 + \frac{\mathrm{d}}{\mathrm{d}t} E(u) = 0$$

so that $t \mapsto E(u(t))$ is non-increasing and it constant iff u at an equilibrium.

More generally, we will look at *dissipative evolutions*

$$\nabla D(\dot{u}(t)) + \nabla E(u(t)) = 0 \tag{2}$$

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where the *dissipation* potential $D : \mathbb{R}^m \to [0,\infty)$ convex with $0 = D(0)$. Again by multiplying by $\dot u$ one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) = -\nabla D(\dot u)\cdot \dot u \leq 0$$

where the inequality comes from convexity. Gradient flows and dissipative evolutions can indeed describe (in some idealized case) the motion of a falling object in an extremely viscous fluid, where inertia effects are negligible.

Yet another example is *lagrangian flow*

$$\ddot u(t) + \nabla E(u(t)) = 0. \tag{3}$$

The usual test with $\dot u$ reveals that

$$\frac{1}{2}|\dot u|^2 + E(u) \ \text{ is constant along trajectories.}$$

Lagrangian flows do describe falling objects, but in void. By letting $H(q,p) = |p|^2/2 + E(q)$ and $u = (q,p)$, the system (3) corresponds to the *hamiltonian flow*

$$J\dot u(t) + \nabla H(u(t)) = 0. \tag{4}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the symplectic operator. Note that the latter is obviously not a dissipative evolution as J is not symmetric, hence not the gradient of a quadratic potential (if $D(u) = u\cdot Au/2$ then $\nabla D(u) = (A{+}A^t)u/2$).

Finally, the proper description of the falling object will require a combination of dissipative and lagrangian modes.

1.1. **Gradient flow solutions are global to the right.**

An early consequence of the gradient-flow structure is that solutions are global for $t\to\infty$. Indeed, assume $E > -c$ (bounded below) and ∇E continuous. Then, let u solve (1) on $[0,T)$ and compute, for all $t<T$

$$\int_0^t |\dot u|^2 = E(u_0) - E(u(t)) \leq E(u_0) - c.$$

By passing to the sup in t one has that u is in $H^1(0,T)$. Hence, u is Hölder since $H^1(0,T)\subset C^{1/2}[0,T]$. In particular, u is continuous and the limit $u(T-)$ exists and is finite. This is enough to find a solution on $[0,T+\varepsilon)$ by gluing u with a solution v of (1) along with Cauchy condition $v(T) = u(T-)$.

Here is some concrete example. Let $E(u) = u^4/4$. Then, $E\geq 0$ and $\nabla E(u) = u^3$ is continuous (but of course not globally Lipschitz continuous). The solution to

$$\dot u + u^3 = 0, \quad u(0) = -1$$

is $u(t) = -(1/(2t+1))^{1/2}$ which is well defined for $t \geq 0$. On the contrary, let us show the necessity of the lower bound on the energy by letting $E(u) = u^3/3$. Then, the solution to

$$\dot{u} + u^2 = 0, \quad u(0) = -1$$

is $u(t) = 1/(t-1)$ which cannot be extended for $t \geq 1$.

2. Functional analytic toolbox

We recalled some basics in functional analysi. In particular, generalities on

- Banach spaces, dual, separability, reflexivity, uniform convexity, representation, Lax-Milgram lemma, orthonormal bases,
- C^k, L^p, $W^{k,p}$ spaces and distributions,

2.1. **Hilbert spaces.** Let H be a real, separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We will use the following.

Lemma 2.1 (Orthonormal basis)**.** *There exists a countable set $\{u_i\}$ so that $(u_i, u_j) = \delta_{ij}$ and the closure of $\mathrm{span}(u_i) = H$. In particular, $u = \sum_i (u, u_i) u_i$ for all $u \in H$ and $(u, v) = \sum_i (u, u_i)(v, u_i)$ for all $u, v \in H$, where the sums are absolutely converging. The mapping*

$$u \mapsto \{(u, u_1), (u, u_2), \dots\}$$

is an isometry between H and $\ell^2 : \{a = \{a_1, a_2, \dots\} \ : \ \sum_i a_i^2 < \infty\}$ endowed with the scalar product $(a, b)_{\ell^2} := \sum_i a_i b_i$.

Lemma 2.2 (Projection on the closed convex set)**.** *Let $K \subset H$ be nonempty, convex, and closed. Then, for all $u \in H$ there exists a unique $u_0 \in K$, the* projection *of u on K, satisfying*

$$(u - u_0, k - u_0) \leq 0 \quad \forall k \in K.$$

The map $u \mapsto u_0$ is a contraction.

Lemma 2.3 (Lax-Milgram-Lions)**.** *Let $a : H \times H \to \mathbb{R}$ bilinear, continuous, and coercive. For all $\ell \in H^*$ there exists a unique $u \in H$ such that*

$$a(u, v) = \ell(v) \quad \forall v \in H. \tag{5}$$

We have that

$$\|u\| \leq \frac{M}{\alpha} \|\ell\|_{H^*}$$

where M and α are the continuity and coercivity constants of a, respectively. If a is symmetric then u solves (5) iff u minimizes the quadratic functional $J(u) = a(u, u)/2 - \ell(u)$.

Lemma 2.4 (Riesz representation)**.** *The map $u \in H \mapsto \ell_u \in H^*$, where $\ell_u(v) := (u, v)$ for all $v \in H$, is an isometry. In particular, H is isometric to H^*.*

2.2. **Variational formulation of a semilinear elliptic problem.** Let us briefly remind some basic variational theory. As a matter of example, we shall concentrate on the the semilinear elliptic problem

$$-\Delta u + f(u) = g \quad \text{in } \Omega, \tag{6a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{6b}$$

Here $\Omega \subset \mathbb{R}^d$ is a nonempty, open, connected, bounded subset of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$, $f = \nabla E : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and E is convex and has a minimum in 0 (so that f is monotone and $0 = f(0)$), and $g \in L^2(\Omega)$. Note that, for all $u \in L^2(\Omega)$ the term $f(u)$ belongs to $L^2(\Omega)$ by Lipschitz continuity as

$$|f(u)| \leq |f(u) - f(0)| + |f(0)| \leq L_f|u| + |f(0)|$$

where $L_f \geq 0$ stands for the Lipschitz constant of f.

A *classical solution* of (6) is a twice-differentiable function u solving (6) pointwise. This notion makes little sense here as g is actually not defined on negligible sets. By relaxing the solution requirement on negligible set, we shall be focusing on *strong solutions*, namely functions $u \in L^2(\Omega)$ with $\Delta u \in L^2(\Omega)$ so that (6a) holds as an equation in $L^2(\Omega)$ and the boundary condition (6b) holds as equation in $L^2(\partial\Omega)$. Equivalently, we say that u is a strong solution if the system (6) is solved almost everywhere. We shall term a *weak solution* to (6) a function $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega f(u) \cdot v = \int_\Omega g \cdot v \quad \forall v \in H^1_0(\Omega). \tag{7}$$

Here the matrix product $\nabla u \cdot \nabla v$ is defined as $\nabla u \cdot \nabla v = u_{i,j} v_{i,j}$ (summation convention). Clearly classical solutions are strong and strong solutions are weak. We shall here concentrate on weak solutions, for their analysis can be performed on variational grounds. We shall leave aside for the moment the existence problem (we will discuss it later) and simply assume that u solves (7). By choosing $v = u$ in (7) and using $f(u) \cdot u \geq 0$ a.e. (due to the convexity of E and the fact that $f(0) = 0$) one obtains the estimate

$$\begin{aligned}\|\nabla u\|^2_{L^2} \leq \int_\Omega g \cdot u \leq \|g\|_{L^2}\|u\|_{L^2} &\leq \frac{1}{2\delta}\|g\|^2_{L^2} + \frac{\delta}{2}\|u\|^2_{L^2} \\ &\leq \frac{1}{2\delta}\|g\|^2_{L^2} + \frac{\delta}{2}C_{\mathrm{P}}^2\|\nabla u\|^2_{L^2}\end{aligned}$$

where $C_{\mathrm{P}} = C_{\mathrm{P}}(\Omega, n)$ is the constant in the Poincaré inequality

$$\|v\|_{L^2} \leq C_{\mathrm{P}}\|\nabla v\|_{L^2} \quad \forall v \in H^1_0(\Omega),$$

δ is any positive number, and we have used the elementary inequality

$$ab \leq \frac{\delta}{2}a^p + \frac{1}{2\delta}b^{p'} \quad \forall \delta > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{p'} = 1.$$

In particular, by choosing δ so small that $\delta C_{\mathrm{P}}^2/2 \leq 1/2$ we have

$$\|\nabla u\|_{L^2}^2 \leq \frac{1}{\delta}\|g\|_{L^2}^2. \tag{8}$$

In case $\partial\Omega$ is smooth or Ω is polygonal and convex, We can obtain another, stronger estimate by taking the L^2-norm of the left- and the right-hand side of the equation (this is now formal, as Δu need not be a function, but it can be made precise, with the above provisions on the domain). Namely, we have

$$\begin{aligned}\|\Delta u\|_{L^2}^2 + \|f(u)\|_{L^2}^2 &= \|g\|_{L^2}^2 + 2\int_\Omega \Delta u \cdot f(u)\\ &= \|g\|_{L^2}^2 - 2\int_\Omega (\nabla u)^t \cdot \nabla f(u) \nabla u \leq \|g\|_{L^2}^2\end{aligned}$$

where the inequality follows from the fact that $\nabla f = \nabla^2 E$ is positive semi-definite. In particular, we have that $\|\Delta u\|_{L^2} \leq \|g\|_{L^2}$. From here, invoking classical elliptic regularity we can conclude that

$$\|u\|_{H^2} \leq C\|g\|_{L^2}. \tag{9}$$

In particular, the weak solution u is in $H^2(\Omega)$ and it is hence a strong solution.

Finally, one can easily check that weak solutions are unique. Let u_1 and u_2 be two weak solutions. Take the difference of the equations in (7) written for u_1 and u_2 and choose $v = u_1 - u_2$ in order to obtain

$$\|\nabla(u_1 - u_2)\|_{L^2}^2 \leq \|\nabla(u_1 - u_2)\|_{L^2}^2 + \int_\Omega (f(u_1) - f(u_2)) \cdot (u_1 - u_2) = 0$$

where the inequality follows from the fact that f is monotone. Hence, $\nabla(u_1 - u_2) = 0$ so that indeed $u_1 = u_2$.

2.3. **Compactness in Banach spaces.** Let B be a real Banach space with norm $\|\cdot\|$. Let us recall classical compactness results [5], starting from weak compactness.

Theorem 2.5 (Kakutani)**.** $\{\|x\| \leq 1\}$ *is weakly compact iff* B *is reflexive.*

We will use this tool, for instance, for $L^p(\Omega)$ and $W^{k,p}(\Omega)$ for $1 < p < \infty$ (all reflexive) in order to extract convergent subsequences from bounded sequences as in the following.

Corollary 2.6. *Let* u_n *be bounded in* $L^p(\Omega)$ *for some* $1 < p < \infty$*. Then, there exists a (not relabeled) subsequence such that*

$$\int_\Omega u_n v \to \int_\Omega uv \quad \forall v \in L^{p'}(\Omega)$$

where $1/p + 1/p' = 1$*, this being the definition of weak convergence in* $L^p(\Omega)$*.*

Here is the compactness theorem for the weak-star topology.

Theorem 2.7 (Banach-Alaoglu-Bourbaki)**.** *If B is the dual of a separable Banach space then $\{\|x\| \leq 1\}$ is weak-star compact.*

The latter applies to $L^\infty(\Omega)$ (non reflexive) which is the dual of $L^1(\Omega)$, which is separable.

Corollary 2.8. *Let u_n be bounded in $L^\infty(\Omega)$. Then, there exists a (not relabeled) subsequence such that*

$$\int_\Omega u_n v \to \int_\Omega uv \quad \forall v \in L^1(\Omega),$$

this being the definition of weak-star convergence in $L^\infty(\Omega)$.

We have no hope to get strong compactness form mere boundedness. In fact, we have the following.

Theorem 2.9. *$\{\|x\| \leq 1\}$ is strongly compact iff B is finite-dimensional.*

Still, one can obtain strong compactness by proving boundedness in a compact subset. In particular, we will be using the following embeddings.

Theorem 2.10 (Sobolev embeddings)**.**

$$W^{1,p}(\Omega) \subset\subset \begin{cases} L^q(\Omega) & \forall q < p_* = np/(n-p), \quad \textit{for } p < n \\ L^q(\Omega) & \forall q < \infty, \quad \textit{for } p = n \\ C(\overline{\Omega}) & \textit{for } p > n. \end{cases} \tag{10}$$

Note that the symbol $\subset\subset$ means here *compactly embedded into*. Being no proof, an homogeneity argument allow us to recover p_*. Assume some inequality

$$(|u(x)|^q \mathrm{d}x)^{1/q} \leq C\,(|\nabla_x u(x)|^p)^{1/p}$$

to hold for all given u, change variables as $x \mapsto \lambda y$ for $\lambda > 0$, and call $v(y) = u(\lambda y)$. One has that

$$\|u\|_{L^q(\mathbb{R}^d)} = \lambda^{n/q}\|v\|_{L^q(\mathbb{R}^d)} = C(\lambda^{n-p})\left(\int_{\mathbb{R}^d} |\nabla_y v(y)|^p \mathrm{d}y\right)^{1/p}$$

so that, necessarily $n/q = n-p$. Namely $q = p_*$. The case $p_* = \infty$ is special as $W^{1,p} \not\in L^\infty$: take $n = 2$ and $u(x) = |\ln|\ln|x|||$ for $|x| < 1/2$. Then $u \in L^p(B_{1/2}(0))$ for all $p < \infty$ as

$$\int_{B_{1/2}(0)} |u|^p \mathrm{d}x = 2\pi \int_0^{1/2} |\ln(p|\ln r|)|\mathrm{d}r < \infty$$

but $u \not{i}nL^\infty(B_{1/2}(0))$. On the other hand

$$\nabla u(x) = \frac{x}{|x^2|\ln|x|} \in L^2(B_{1/2}(0))$$

and $u \in H^1(B_{1/2}(0))$.

The embedding result can be iterated:

$$W^{k,p} \subset W^{k-1,p_*} \subset W^{k-2,(p_*)_*} \subset \cdots \subset L^{\hat{p}}$$

where

$$\frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}, \quad \frac{1}{(p_*)_*} = \frac{1}{p_*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}, \quad \dots, \quad \frac{1}{\hat{p}} = \frac{1}{p} - \frac{k}{n}.$$

As a matter of application of the latter embeddings consider the semilinear elliptic problem (6) where g replaced by g_n and denote by u_n the corresponding weak solution. Namely, we have

$$\int_\Omega \nabla u_n \cdot \nabla v + \int_\Omega f(u_n) v = \int_\Omega g_n v \quad \forall v \in H^1_0(\Omega). \tag{11}$$

Let us now assume that $g_n \to g$ in $L^2(\Omega)$. We would like to check that the solutions u_n converge to some limit u solving (7). By using (8) and (9) we know that u_n are bounded in $H^1(\Omega)$. In particular, we can extract a subsequence u_{n_k} such that $u_{n_k} \rightharpoonup u$ in $H^1(\Omega)$ (by Theorem 2.5 , since $H^1(\Omega)$ is reflexive) and strongly in $L^2(\Omega)$ (from the Sobolev embedding (10)). In particular

$$\int_\Omega \nabla u_n \cdot \nabla v \to \int_\Omega \nabla u \cdot \nabla v \quad \forall v \in H^1_0(\Omega)$$

and

$$\int_\Omega |f(u_{n_k}) - f(u)|^2 \leq L_f^2 \int_\Omega |u_{n_k} - u|^2 \to 0.$$

Hence, $f(u_{n_k}) \to f(u)$ in $L^2(\Omega)$ and we can pass to the limit in (11) getting (7).

Since the solution u to (6) is unique, the convergence $u_{n_k} \to u$ actually holds for the whole sequence u_n as well. Indeed, any subsequence of u_n admits a subsequence converging to a solution of (6), namely converging to u. This entails that the whole sequence converges to u as well.

Lemma 2.11 (Norm convergence implies strong convergence)**.** *let $x_n \rightharpoonup x$ in the Hilbert space H and $\limsup \|x_n\| \leq \|x\|$. Then, $x_n \to x$.*

Proof. Compute

$$\begin{aligned} \limsup \|x_n - x\|^2 &= \limsup(\|x_n\|^2 - 2(x_n, x) + \|x\|^2) \\ &\leq \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0. \end{aligned}$$

The lemma is indeed valid in uniformly convex spaces (more precisely, it entails a characterization of uniformly convex spaces). □

3. Direct method

Let X be a topological space fulfilling the first axiom of countability (so that the topology can be determined by sequencial convergence). We say that $I : X \to (-\infty, \infty]$ is *lower semicontinuous* (l.s.c.) if

$$I(x) \leq \liminf_{n\to\infty} I(x_n) \quad \forall x_n \to x.$$

Equivalently, one can check that I is l.s.c. iff its *epigraph* $\mathrm{epi}(I) = \{(x, r) \in X \times \mathbb{R} \mid I(x) \leq r\}$ is closed or its *sublevels* $\{I \leq r\}$ are closed for every $r \in \mathbb{R}$. Moreover, we say that I is *coercive* if

$$\inf_X I = \inf_K I$$

for some $K \subset\subset X$.

Theorem 3.1 (Direct Method). *Let $I : X \to (-\infty, \infty]$ be l.s.c. and coercive. Then, the problem*

$$\min_X I \tag{12}$$

admits a solution.

Proof. Take $x_n \in K$ so that $I(x_n) \leq \inf_X I + 1/n$ and extract (no relabeling) in order to have $x_n \to x$. Then,

$$I(x) \overset{\text{l.s.c}}{\leq} \liminf_{n\to\infty} I(x_n) \leq \inf_X I + \liminf_{n\to\infty} 1/n = \inf_X I.$$

Hence, $I(x) = \inf_X I$. Namely, x solves (12). □

Often spaces may allow different topologies and one could ask which one is better to find minima to functions. In order to apply the Direct method one needs some balanced choice: the finer the topology the easier to be l.s.c. but the fewer the compacts.

By dropping either lower semicontinuity or coercivity problem (12) may of course have no solutions (this was already the case for the Weierstraß Theorem in $\mathbb{R}^d$, which is nothing but a special case of Theorem 3.1).

As a first application of the Direct Method let $X = H^1_0(\Omega)$, $\Omega \subset \mathbb{R}^d$, and I be the Dirichlet integral

$$I(u) = \frac{1}{2}\int_\Omega \nabla u \cdot A\nabla u - \int_\Omega gu$$

where $g \in L^2(\Omega)$ and the matrix $A(x) = (a_{ij}(x))$ is symmetric, $x \mapsto a_{ij}(x)$ are bounded and measurable, and there exists $\alpha > 0$ such that $\alpha|\xi|^2 \leq \xi \cdot A(x)\xi$ for all $\xi \in \mathbb{R}^d$, a.e. $x \in \Omega$. Then, f is l.s.c. with respect to the weak topology of $H^1_0(\Omega)$ as it is the sum of the squared (semi)norm and a continuous perturbation. Fix any sublevel K of I. Then, K is bounded in

$H^1_0(\Omega)$ hence weakly compact. Eventually, the Direct Method applies and one has a minimizer u of I on $H^1_0(\Omega)$. This in particular solves

$$\frac{1}{2}\int_\Omega \nabla u\cdot A\nabla u - \int_\Omega gu = I(u) \leq I(v) = \frac{1}{2}\int_\Omega \nabla v\cdot A\nabla v - \int_\Omega gv \quad \forall v \in H^1_0(\Omega)$$

which can be easily proved to be equivalent to

$$\int_\Omega \nabla u \cdot A\nabla v = \int_\Omega gv \quad \forall v \in H^1_0(\Omega).$$

Namely, u is a weak solution to

$$-\mathrm{div}(A\nabla u) = g.$$

Thus, by arguing as in Subsection 2.2, it is unique. Uniqueness follows also from the strict convexity of f, see below.

Let's now apply the Direct Method to problem (7). First of all we reformulate (7) as a minimum problem by proving the following

Lemma 3.2. *u solves (7) iff $I(u) = \min\{I(v)\,|\, v \in H^1_0(\Omega)\}$.*

Proof. Let u a minimizer of I and $v \in H^1_0(\Omega)$. Then, $t \mapsto h(t) = I(u+tv)$ has a critical point at $t = 0$. Then

$$0 = h'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{t^2}{2}|\nabla v|^2 + t\nabla u : \nabla v + E(u+tv) - g\cdot(u+tv)\right)\Bigg|_{t=0}$$

$$= \int_\Omega (\nabla u : \nabla v + f(u)\cdot v - g\cdot v).$$

That is, u solves (7).

On the other hand, let u solve (7). Then is $t = 0$ is a critical point of $t \mapsto I(u+tv)$ for all $v \in H^1_0(\Omega)$. As I is convex (see later on), u is necessarily a minimum point. $\square$

We shall now check that I is lower semicontinuous and coercive with respect to the weak topology of $H^1_0(\Omega)$. As for the corcivity it is enought to remark that sublevels of I are bounded in H^1. In fact we have that

$$\begin{aligned} C > I(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega g\cdot u \geq C\|u\|^2_{H^1} - \|g\|_{L^2}\|u\|_{L^2} \\ &\geq C\|u\|^2_{H^1} - \frac{1}{2C}\|g\|^2_{L^2} - \frac{C}{2}\|u\|^2_{L^2} \geq \frac{C}{2}\|u\|^2_{H^1} - \frac{1}{2C}\|g\|^2_{L^2} \end{aligned}$$

so that

$$\begin{aligned} I(u) < C \ &\Rightarrow\ \frac{C}{2}\|u\|^2_{H^1} \leq C + \frac{1}{2C}\|g\|^2_{L^2} \\ &\Rightarrow\ \|u\|_{H^1} \leq \left(\frac{2}{C}\left(C + \frac{1}{2C}\|g\|^2_{L^2}\right)\right)^{1/2} \end{aligned}$$

and u is bounded.

Let us now check the lower semicontinuity. To this aim let $u_n \to u$ weakly in $H^1(\Omega)$ (hence strongly in $L^2(\Omega)$). Then, $\int_\Omega g \cdot u_n \to \int_\Omega g \cdot u$. Moreover, $\nabla u_n \to \nabla u$ weakly in $L^2(\Omega)$, and

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \leq \liminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla u_n|^2.$$

Eventually, $u_n \to u$ a.e. in Ω, so that $E(u_n) \to E(u)$ a.e. in Ω and

$$\int_\Omega E(u) \leq \liminf_{n\to\infty} \int_\Omega E(u_n)$$

by Fatou's lemma as $E \geq 0$. The functional I is hence lower semicontinous.

Given a sequence $I_n : X \to (-\infty; \infty]$ and $I : X \to (-\infty; \infty]$ we say that I_n Γ-converges to I iff

Γ-$\liminf$ inequality: $\quad I(x) \leq \liminf_{n\to\infty} I_n(x_n) \quad \forall x_n \to x$

Recovery sequence: $\quad \forall y \in X \ \exists y_n \to y$ such that $I_n(y_n) \to I(y)$.

Theorem 3.3 (Fundamental Theorem of Γ-convergence)**.** *Let I_n Γ-converge to f, x_n minimize I_n, and $x_n \to x$. Then, x minimizes I.*

Proof. Let y be such that $I(y) < I(x)$ and let y_n be the corresponding recovery sequence. Hence,

$$I(y) < I(x) \overset{\Gamma-\liminf}{\leq} \liminf_{n\to\infty} I_n(x_n) \leq \liminf_{n\to\infty} I_n(y_n) = I(y).$$

A contradiction. ☐

The Fundamental Theorem can be generalized by asking x_n to be just quasi-minimizers of I_n, namely

$$I_n(x_n) \leq \inf_X I_n + \mathrm{o}(1).$$

Γ-limits are l.s.c. In fact, the Γ-limit of a constant sequence $I_n = I$ it the l.s.c.-envelope of I (its relaxation).

The *viceversa* of the Fundamental Theorem does not hold: take $I_n(x) = |x|/n$ for $x \in \mathbb{R}$. Then, $I_n \xrightarrow{\Gamma} I \equiv 0$. Still, $x = 2$ is a minimizer of I and it is not the limit of minimizers of I_n

Γ-convergence is implied by uniform convergence for sequences of continuous functions. Indeed, uniform convergence implies that (1) the limit I is continuous and (2) the sequence converges pointwise. From (1), for any $x_n \to x$, we have

$$|I(x) - I_n(x_n)| \leq |I(x) - I(x_n)| + \|I - I_n\| \to 0.$$

That is that the $\Gamma - \liminf$ inequality holds. From (2) we have the the existence of a recovery sequence (just take it constant). On the other hand, Γ-convergence is stable via continuous perturbations. let G be continuous and $I_n \xrightarrow{\Gamma} I$. Then, one readily has that $I_n + G \xrightarrow{\Gamma} f + G$.

Γ-convergence is different from pointwise convergence: take $I_n(x) = |x|^n$ for $x \in \mathbb{R}$. Then, $I_n \to G$ pointwise, with $G(x) = 0$ for $|x| < 1$, $G(\pm 1) = 1$, and $G = \infty$ elsewhere. On the other hand, $I_n \xrightarrow{\Gamma} I$ with $I(x) = 0$ for $|x| \leq 1$ and $I = \infty$ elsewhere.

Γ-convergence is nonlinear: take $I_n(x) = \sin(nx)$ for $x \in \mathbb{R}$. Then, $I_n \xrightarrow{\Gamma} -1$. On the other hand also $-I_n \xrightarrow{\Gamma} -1$.

Let X be a Banach space. then, we say that $I_n \to I$ in the sense of *Mosco convergence* iff $I_n \xrightarrow{\Gamma} I$ both w.r.t. the weak and the strong topology of X. In particular, iff the two conditions hold

$$\Gamma\text{-}\liminf \text{ inequality:} \qquad I(x) \leq \liminf_{n\to\infty} I_n(x_n) \quad \forall x_n \rightharpoonup x,$$

$$\text{Recovery sequence:} \qquad \forall y \in X \ \exists y_n \to y \text{ such that } I_n(y_n) \to I(y).$$

4. Convex functions

Here is a minimal primer on convex functions on infinite dimensional spaces. Let B be a Banach space. We say that $\phi : B \to (-\infty, \infty]$ is *convex* iff $\phi(tu + (1-t)v) \leq t\phi(u) + (1-t)\phi(v)$ for all $u, v, \in B$ and $t \in [0, 1]$, and that it is *proper* if its *essential domain* $D(\phi) = \{u \in B \mid \phi(u) \neq \infty\}$ is not empty. Moreover we say that ϕ is *strictly convex* iff the convexity inequality holds with the strict sign $<$ whenever $u \neq v$ and $t \in (0, 1)$. and that it is λ-*convex* ($\lambda \in \mathbb{R}$) iff

$$\phi(tu + (1 - tv)) \leq t\phi(u) + (1-t)\phi(v) - \frac{\lambda}{2} t(1-t)\|u - v\|^2$$

for all $u, v, \in B$ and $t \in [0, 1]$. Finally, we wil say that ϕ is *uniformly convex* if it is λ-convex with $\lambda > 0$.

As ϕ is convex iff its *epigraph* $\mathrm{epi}(\phi)$ is convex. Hence, a convex function is weakly l.s.c. iff it is strongly l.s.c. (as $\mathrm{epi}(\phi)$ is weakly closed iff strongly closed).

Lemma 4.1 (Unique minimum)**.** *If ϕ is strictly convex than the minimization problem $\min_B \phi$ has at most one solution.*

As a matter of example let us take $f : \mathbb{R} \to \mathbb{R}$ Borel and define the functional $\phi : L^1(\Omega) \to (-\infty, \infty]$ by

$$\phi(u) = \begin{cases} \displaystyle\int_\Omega f(u(x)) & \text{if } \ f(u) \in L^1(\Omega) \\ +\infty & \text{elsewhere in } \ L^1(\Omega) \end{cases}$$

Lemma 4.2 (Weak l.s.c.)**.** *ϕ is weak l.s.c in $L^p(\Omega)$ iff f is convex.*

We can use this lemma to prove the following

Lemma 4.3. *Let $|\Omega| < \infty$, $f : \mathbb{R} \to (-\infty, \infty]$ convex with $|f(u)| \geq c|u|^p - 1/c$ for $c > 0$ and $p > 1$. Then, the problem $\min \phi$ has e unique solution in $L^p(\Omega)$.*

We say that $v^* \in B^*$ *belongs to the subdifferential of ϕ at u* and write $v^* \in \partial\phi(u)$ iff $u \in D(\phi)$ and

$$\langle v^*, w - u\rangle \leq \phi(w) - \phi(u) \quad \forall w \in B.$$

If ϕ is differentiable at u then $\nabla\phi(u) = \partial\phi(u)$. If ϕ is not differentiable at u then $\partial\phi(u)$ may be a (convex) set. In general $\partial\phi : B \to 2^{B^*}$ with $D(\partial\phi) = \{u \in D(\phi) \mid \partial\phi(u) \neq \emptyset\} \subset D(\phi)$ although they are the same in finite dimensions. Example $\phi(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x/2$ with $D(\phi) = H^1$ and $\partial\phi(u) = -\Delta u$ with $D(\partial\phi) = H^2$.

Lemma 4.4 (Fermat)**.** *Let ϕ be convex and proper. Then, $\phi(u) = \min \phi \iff 0 \in \partial\phi(u)$.*

In general, we have that $\partial\phi_1 + \partial\phi_2 \subset \partial(\phi_1 + \phi_2)$. The reverse inclusion does not hold. We however have $\partial\phi_1+\partial\phi_2 \equiv \partial(\phi_1+\phi_2)$ if a domain condition holds.

Lemma 4.5 (Sum of subdifferentials)**.** *Let ϕ_1, ϕ_2 be convex, proper, and l.s.c. with $D(\partial\phi_1) \cap \mathrm{int}(D(\partial\phi_2))$. Then, $\partial(\phi_1 + \phi_2) = \partial\phi_1 + \partial\phi_2$*

In particular $\partial(\phi_1+\phi_2) = \partial\phi_1+\partial\phi_2$ if one of the two functions is smooth.

Given $\phi : B \to (-\infty, \infty]$ proper, we define the *(Fenchel-Legendre) conjugate* $\phi^* : B^* \to (-\infty, \infty]$ as

$$\phi^*(u^*) := \sup_{u\in B}\{\langle u^*, u\rangle - \phi(u)\}.$$

The latter is convex and l.s.c. (as it is the sup of convex functions).

Theorem 4.6 (Fenchel-Moreau)**.** *Let ϕ be proper, convex, l.s.c. Then, ϕ^* be also proper, convex, l.s.c. and $\phi^{**} = \phi$.*

Lemma 4.7 (Fenchel inequalities)**.**

$$\phi(u) + \phi^*(u^*) \geq \langle u^*, u\rangle \quad \forall u \in B,\ u^* \in B^*$$
$$\phi(u) + \phi^*(u^*) = \langle u^*, u\rangle \iff u^* \in \partial\phi(u) \iff u \in \partial\phi^*(u^*)$$

Lemma 4.8 ([2, Thm. 3.18])**.** *Let $\phi : B \to (-\infty, \infty]$ with B being reflexive. Then,*

$$\phi_n \stackrel{Mosco}{\to} \phi \iff \phi_n^* \stackrel{Mosco}{\to} \phi^*.$$

Given $\phi : B \to (-\infty, \infty]$ proper, convex, and l.s.c., let us now introduce a canonical smoothing via the *Moreau-Yosida regularization* $\phi_\lambda : B \to (-\infty, \infty]$ defined as

$$\phi_\lambda(u) = \min\left(\frac{\|u - v\|^2}{2\lambda} + \phi(v)\right).$$

Note that the latter minimization always has a unique solution $J_\lambda u$ (which is called the *resolvent*) as the corresponding functional is uniformly convex and l.s.c.

From this point on let $B = H$ be a Hilbert space. One has that, $J_\lambda : H \to H$ is a contraction and that, for all $u \in H$,

$$\begin{aligned}
&\phi_\lambda(u) = \frac{\|u - J_\lambda u\|^2}{2\lambda} + \phi(J_\lambda u),\\
&J_\lambda u + \lambda\partial\phi(J_\lambda u) \ni u,\\
&J_\lambda u \to u.
\end{aligned}$$

Assume momentarily that ϕ is bounded below (this can be relaxed out). Then

$$\frac{\|u - J_\lambda u\|^2}{2\lambda} \leq \phi_\lambda(u) \leq \phi(u)$$

and the convergence $J_\lambda u \to u$ follows. The convergence $J_\lambda u \to u$ in particular proves that $D(\partial\phi)$ is dense in $D(\phi)$ as all $u \in D(\phi)$ are strong limits of $J_\lambda u \in D(\partial\phi)$. On the other hand we have that

$$\phi(u) \leq \liminf \phi(J_\lambda u) \leq \limsup \phi(J_\lambda u) \leq \phi(u)$$

so that $\phi(J_\lambda u) \to \phi(u)$. Eventually

$$\phi(J_\lambda u) \leq \phi_\lambda(u) \leq \phi(u)$$

so that $\phi_\lambda(u) \to \phi(u)$ as well. Moreover the convergence $\phi_\lambda(u) \to \phi(u)$ is monotone.

The Yosida approximation ϕ_λ can be proved to be $C^{1,1}$. Indeed, we have

$$\partial\phi_\lambda(u) = \frac{u - J_\lambda u}{\lambda} = \partial\phi(J_\lambda) \quad \forall u \in H.$$

More precisely, $\partial\phi_\lambda$ is Lipschitz continuous of constant $1/\lambda$. Recall that we have $\phi_\lambda(u) \nearrow \phi(u)$ for all $u \in D(\phi)$. We can hence prove that $\phi_\lambda \to \phi$ in the Mosco sense. Indeed, if $u_\lambda \to u$ we have $J_\lambda u_\lambda \to u$ and $\phi(J_\lambda u_\lambda) \to \phi(u)$ as before and

$$\phi(u) \leq \liminf \phi(J_\lambda u_\lambda) \leq \liminf \phi_\lambda(u_\lambda).$$

Finally, we remark that

$$\partial\phi_\lambda(u) = \partial\phi(J_\lambda u) \quad \forall u \in H.$$

5. Monotone operators

From [4]. We say that $A : D(A) \subset H \to 2^H$ is *monotone* if

$$(u^* - v^*, u - v) \geq 0 \quad \forall u^* \in Au,\ v^* \in Av$$

and that it is *maximal monotone* if its graph cannot be extended by set inclusion to the graph of a monotone operator.

Examples include: monotone real functions, constraints, nonnegative matrices in $\mathbb{R}^d$ (either symmetric or not), positive linear operators, monotone (hemi)continuous operators, some rotations, subdifferentials of convex, proper, l.s.c. functions, derivatives.

Lemma 5.1. *A monotone operator A is maximal iff $R(I+\lambda A) = H$ for all $\lambda > 0$. If A is maximal monotone then*

A is closed,

$(I+\lambda A)^{-1}$ is bijective between $D(A)$ and H and is a contraction.

Lemma 5.2 (Surjectivity)**.** *Let A be maximal monotone and coercive, namely there exists u_0 such that*

$$\lim_{u\in D(A), \|u\|\to\infty} \frac{((Au)^\circ, u-u_0)}{\|u\|} > \infty.$$

Then $R(A) = H$.

Lemma 5.3 (Sum)**.** *Let A_1, A_2 be maximal monotone and*

$$D(A_1) \cap \operatorname{int} D(A_2) \neq \emptyset.$$

Then $A_1 + A_2$ is maximal monotone.

In particular, if A_1 is maximal monotone and A_2 is monotone, everywhere defined, and continuous then $A_1 + A_2$ is maximal monotone.

Let $B = L^2(\Omega)$, $A_1 = -\Delta u$ for $D(A) = H^2(\Omega)\cap H^1_0(\Omega)$, and $A_2(u) = f(u)$ with f Lipschitz continuous and monotone. Then, $\operatorname{int}(D(A_2)) = D(A_2) = L^2(\Omega)$ and A_1+A_2 is onto $L^2(\Omega)$. In particular, the semilinar elliptic problem (6) has a strong solution.

On the other hand, define $B = H^1_0(\Omega)$, $A_1, A_2 : H^1_0(\Omega) \to H^{-1}(\Omega)$ as

$$\langle A_1 u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \quad \forall v \in H^1_0(\Omega),$$
$$\langle A_2 u, v\rangle := \int_\Omega f(u) \cdot v \quad \forall v \in H^1_0(\Omega).$$

Then, both A_i are continuous and monotone, hence maximal monotone. Moreover $D(A_1) = D(A_2) = H^1_0(\Omega)$, so that $A_1 + A_2$ is maximal monotone as well. As A_1 is coercive in $H^1_0(\Omega)$, we have that $A_1 + A_2$ is onto. In particular, the variational formulation (7) of the semilinar elliptic problem (6) has a weak solution also for $g \in H^{-1}(\Omega)$. This is particularly relevant in connection with nonhomogeneous Neumann boundary conditions.

Following the same regularization procedure as for convex functions (here applied directly to the operators) a maximal monotone operator A can be *Yosida regularized* to A_λ for $\lambda > 0$ by letting

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$$

where $J_\lambda : H \to D(A)$ is the *resolvent* associating to $u \in H$ the unique solution u_λ to

$$u_\lambda + \lambda A u_\lambda \ni u.$$

As in the subdifferential case, the Yosida regularization A_λ is single-valued and Lipschitz continuous of constant $1/\lambda$. Moreover, one has that

$$\|A_\lambda u\| \leq \|(Au)^\circ\| := \min\{\|u^*\| \mid u^* \in Au\}.$$

In addition, it is pointwise convergent $A_\lambda u \to Au$ for all $u \in D(A)$. In case of subdifferentials of convex, proper, and l.s.c. functions ϕ, the Yosida regularization $(\partial\phi)_\lambda$ corresponds to the subdifferential of the Moreau-Yosida approximation ϕ_λ.

Lemma 5.4 (Limsup tool)**.** *Let $u_n \rightharpoonup u$, $u_n^* \in Au_n$, $u_n^* \rightharpoonup u^*$, and $(u^*, u) \geq \limsup(u_n^*, u_n)$. Then, $u^* \in Au$ and $(u^*, u) \to (u_n^*, u_n)$.*

From [2]. We say that a sequence $A_n \to A$ in the *graph sense* if for all $u^* \in Au$ there exist sequences $u_n \to u$ and $u_n^* \to u^*$ so that $u_n^* \in Au_n$. An example of graph-convergent sequence is A_λ: given (u, Au) it suffices to choose $(u, A_\lambda u)$. Then, the limsup tool can be extended to the case of a graph-convergent sequence as follows.

Lemma 5.5 (Limsup tool, extended)**.** *Let $A_n \to A$ in the graph sense, $u_n \rightharpoonup u$, $u_n^* \in A_n u_n$, $u_n^* \rightharpoonup u^*$, and $(u^*, u) \geq \limsup(u_n^*, u_n)$. Then, $u^* \in Au$.*

Lemma 5.6 ([2, Th. 3.66])**.** *Let $\phi_n, \phi : H \to [0, \infty]$ convex, proper, and l.s.c., all minimized at 0. Then,*

$$\partial\phi_n \to \partial\phi \text{ in the graph sense } \iff \phi_n \to \phi \text{ in the Mosco sense.}$$

The minimality in 0 in the last statement is assumed for simplicity only and could be replaced by a more general normalization condition [2].

6. Young measures

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^d$ be of finite measure and $u_n : \Omega \to \mathbb{R}^m$ be measurable and bounded. Then, there exists a not relabeled subsequence and a weak-star measurable function $\nu : \Omega \to M(\mathbb{R}^m)$ (finite Radon measures) such that*

$$\begin{aligned} &(1) \quad \nu_x \geq 0, \ \|\nu_x\| = 1 \quad \text{for a.e. } x \in \Omega, \\ &(2) \quad \forall f \in C(\mathbb{R}^m) \quad f(u_n) \overset{*}{\rightharpoonup} f_\infty \quad \text{in } L^\infty(\Omega) \\ &\qquad \text{where} \quad f_\infty(x) = \langle f, \nu_x \rangle = \int_{\mathbb{R}^m} f(\xi)\, \mathrm{d}\nu_x(\xi). \end{aligned}$$

We say that ν is the *Young measure* generated by the sequence u_n. An analogous theorem holds if we replace $\mathbb{R}^d$ with a compact metric space. A classical application is the following convergence theorem in L^p spaces.

Corollary 6.2. *Let u_n be bounded in $L^p(\Omega, \mathbb{R}^m)$. Then, there exists a not relabeled subsequence and a Young measure ν_x such that*

- *ν_x is weakly-star measurable, namely $x \mapsto \int_{\mathbb{R}^m} f(\xi) \mathrm{d}\nu_x(\xi)$ is Lebesgue measurable for all f continuous,*
- *if $p < \infty$ one has*
$$\int_\Omega \int_{\mathbb{R}^m} |\xi|^p \mathrm{d}\nu_x(\xi) \mathrm{d}x < \infty.$$
If $p = \infty$ there exists $K \subset\subset \mathbb{R}^m$ such that
$$\operatorname{supp} \nu_x \subset K \quad \text{for a.e. } x \in \Omega.$$
- *For all $f : \mathbb{R}^m \to \mathbb{R}$ lowe semicontinuous with $f^-(u_n) = \max\{0, f(u_n)\}$ equiintegrable we have that*
$$\int_\Omega \left(\int_{\mathbb{R}^m} f(x, \xi) \mathrm{d}\nu_x(\xi) \right) \mathrm{d}x \leq \liminf_{n \to \infty} \int_\Omega f(u_n) \, \mathrm{d}x.$$
- *For all $f : \mathbb{R}^m \to \mathbb{R}$ continuous with $f(u_n)$ equiintegrable we have that*
$$f(u_n) \rightharpoonup f_\infty \quad \text{in} \quad L^1 \quad \text{with} \quad f_\infty(x) = \int_{\mathbb{R}^m} f(x, \xi) \mathrm{d}\nu_x(\xi).$$
Note that if $|f(x)| \leq c(1 + |x|^q)$ for $q < p$ then are $|f(u_n)|$ equiintegrable.

7. *BV*

Let $M(\Omega, \mathbb{R}^m)$ denote the space of finite Radon vector measures (*Radon* = positive, finite on compacts), The function $u \in L^1(\Omega)$ is of *bounded variation* if its distributional gradient ∇u is in $M(\Omega, \mathbb{R}^m)$. We let
$$BV(\Omega) = \{u \in L^1(\Omega) \mid \nabla u \in M(\Omega, \mathbb{R}^m)\}$$
which is a Banach spaces when endowed with the norm
$$\|u\|_{BV} = \|u\|_{L^1} + |\nabla u|(\Omega).$$

We have that $W^{1,1}(\Omega) \subset BV(\Omega)$. By letting
$$\mathrm{Var}(u) = \sum \left\{ \int_\Omega u \operatorname{div} \varphi \mid \varphi \in C^1_{\mathrm{c}}(\Omega; \mathbb{R}^m), \ \|\varphi\|_{L^\infty} \leq 1 \right\},$$
we have that
$$u \in BV(\Omega) \Longrightarrow \mathrm{Var}(u) < \infty \ \text{ and } \ |\nabla u|(\Omega) = \mathrm{Var}(u)$$
but also that
$$\mathrm{Var}(u) < \infty \ \text{ and } \ u \in L^1(\Omega) \Longrightarrow u \in BV(\Omega).$$

Bounded sets in $BV(\Omega)$ are strongly precompact in $L^1(\Omega)$. More precisely, as $M(\Omega, \mathbb{R}^d) = (C_c(\Omega;\mathbb{R}^m))^*$, then any BV-bounded sequence u_n admits a not relabeled subsequence such that

$$u_n \to u \ \text{ in } \ L^1(\Omega) \ \text{ so that } \ \nabla u_n \overset{*}{\rightharpoonup} \nabla u \ \text{ in } \ M(\Omega, \mathbb{R}^m).$$

In particular,

$$\int_\Omega f(x)\mathrm{d}(\nabla u_n)(x) \to \int_\Omega f(x)\mathrm{d}(\nabla u)(x) \quad \forall f \in C_c(\Omega;\mathbb{R}^m).$$

8. Vector-valued functions

From [15]. We shall be viewing $u : (x,t) \mapsto \mathbb{R}$, $u \in L^1(\Omega \times (0,T))$ as a function $t \mapsto u(\cdot,t) \in L^1(\Omega)$. In particular, we would like to handle functions $u : [0,T] \to B$ with values in a Banach space.

We say that a function is *simple* iff

$$u(t) = \sum_{i\in F} u_i \chi_{B_i}(t)$$

where the index set F is finite and B_i are measurable and disjoint. We say that u is *strongly measurable* if it is the a.e. limit of a sequence of simple functions.

Theorem 8.1 (Pettis' characterization)**.** *u is strongly measurable iff*

$t \mapsto \langle f, u(t)\rangle$ *is measurable* $\forall f \in B^*$ *(weakly measurable) and*

$\exists N$ *negligible such that* $\{u(t) \mid t \notin N\}$ *is separable (quasi-separable range).*

On separable spaces we have: strong measurability = weak measurability.

We say that u is *(Bochner) integrable* iff there exists a sequence u_n of simple functions so that

$$t \mapsto \|u(t) - u_n(t)\|_B \in L^1(0,T) \ \text{ and } \ \lim_{n\to\infty} \int_0^T \|u - u_n\|_B \mathrm{d}t = 0.$$

Under the latter, u_n is Cauchy, so that the integral of u is uniquely defined.

The space of (a.e. equivalence classes of) (Bochner) integrable functions $L^1(0,T;B)$ is a Banach space when endowed with the norm

$$\|u\|_{L^1(0,T;B)} = \int_0^T \|u\|_B$$

and classical proporties of the integral hold.

If $B = \mathbb{R}$ then Bochner = Lebesgue.

Theorem 8.2 (Bochner)**.**

$$u \in L^1(0,T;B) \iff \begin{cases} u \text{ is strongly measurable} \\ t \mapsto \|u(t)\| \in L^1(0,T) \end{cases}$$

Corollary 8.3.

$$\left.\begin{array}{r} u \textit{ is strongly measurable} \\ \|u(t)\| \leq g(t) \in L^1(0,T) \end{array}\right\} \iff u \in L^1(0,T;B).$$

We can define $L^p(0,T;B)$ and check that $(L^p(0,T;B))^* = L^{p'}(0,T;B^*)$ at least for B reflexive and $p \in (1,\infty)$. Note that $L^\infty = (L^1)^*$ but $L^1 \neq (L^\infty)^*$.

Lemma 8.4 (Lebesgue points)**.**

$$u \in L^1(0,T;B) \implies \frac{1}{h}\int_t^{t+h} u \to u(t) \textit{ for a.e. } t \in (0,T).$$

Note that $L^p(\Omega \times (0,T)) = L^p(0,T;L^p(\Omega))$ for all $p \in [1,\infty)$ but

$$L^\infty(0,T;L^\infty(\Omega)) \subset L^\infty(\Omega \times (0,T))$$

the inclusions being strict.

One can define the spaces of continuous functions $C^k([0,T];B)$, the Sobolev spaces $W^{k,p}(0,T;B)$, and the BV space $BV(0,T;B)$ as well.

9. Gradient flows

Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Assume to be given $\phi : H \to (-\infty,+\infty]$ convex, proper, and l.s.c., an initial datum $u^0 \in D(\phi)$, and $f \in L^2(\mathbb{R}_+;H)$. We are interested in finding $u \in H^1(\mathbb{R}_+;H)$ solving the *gradient flow*

$$u' + \partial\phi(u) \ni f \quad \text{for a.e. } t > 0, \tag{13}$$

$$u(0) = u^0. \tag{14}$$

We call trajectories solving the latter *gradient flow solutions.* Note that the inclusion in (13) is intended in H. In particular, given the asserted regularity of f and u', we can equivalently rewrite it as

$$\begin{aligned} &\exists \xi \in L^2(\mathbb{R}_+;H) \ \text{ such that} \\ &\xi \in \partial\phi(u) \text{ a.e. in } \mathbb{R}_+ \text{ and} \\ &u' + \xi = f \text{ a.e. in } \mathbb{R}_+. \end{aligned}$$

We call such ξ a *selection* in $\partial\phi(u)$. In the following, we shall equivalently use the latter or (13). The initial condition (14) makes sense as $H^1(\mathbb{R}_+;H) \subset C([0,\infty);H)$.

The existence theorem reads as follows.

Theorem 9.1 (Existence)**.** *Let $\phi : H \to (-\infty,\infty]$ be proper, convex, and l.s.c., $u^0 \in D(\phi)$, and $f \in L^2(\mathbb{R}_+;H)$. Then, there exists a (unique) gradient-flow solution.*

We shall give two proof of the latter in the following Subsections 9.3-9.4

In the following we will always assume ϕ to be bounded from below. This assumption is often not restrictive with respect to applications and simplifies the discussion.

9.1. **Some examples.** Let us collect here a number of examples of gradient-flow evolutions. In the following, $\Omega \subset \mathbb{R}^d$ is nonempty open and smoothly bounded and $g \in L^2(\Omega \times (0,T))$.

9.1.1. *ODEs.* Let $f \in C(\mathbb{R})$. Then, the autonomous ODE $\dot{u} + f(u) = 0$ corresponds to the gradient flow of the function $F(u) = \int_0^u f(s)\mathrm{d}s$. Let $f \in C(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \times f = 0$ so that $f = \nabla F$ for some F. Then, the system $\dot{u} + f(u) = 0$ is the gradient flow of F.

Consider now the constrained problem

$$(\dot{u} + f(u) - g)(u - \psi) = 0, \quad u \geq \psi, \quad \dot{u} + f(u) - g \geq 0$$

where $f \in C(\mathbb{R})$, $f = F'$, $g \in C[0,T]$, and $\phi \in \mathbb{R}$. The latter is the gradient flow in $\mathbb{R}$ of the function $\phi(u) = F(u)$ if $u \geq \psi$ and $\phi(u) = \infty$ otherwise.

9.1.2. *Linear parabolic PDEs.* Let $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ symmetric so that $\xi \cdot A\xi \geq \alpha|\xi|^2$ for some $\alpha > 0$ and all $\xi \in \mathbb{R}^d$. Then, the linear equation

$$u_t - \nabla \cdot (A(x)\nabla u) = g,$$

along with homogeneous Dirichlet boundary conditions, corresponds to the gradient flow of the functional

$$\phi(u) = \frac{1}{2}\int_\Omega \nabla u \cdot A\nabla u - \int_\Omega gu \quad D(\phi) = H^1_0(\Omega)$$

in $H = L^2(\Omega)$.

9.1.3. *Parabolic variational inequalities.* The problem

$$(u_t - \Delta u)(u - \psi) = 0, \quad u \geq \psi, \quad u_t - \Delta - g \geq 0$$

along with homogeneous Dirichlet boundary conditions is the the gradient flow of the functional

$$\phi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 - \int_\Omega gu & \text{for } \ u \in H^1_0(\Omega), \ u \geq \psi \text{ a.e.} \\ \infty & \text{elsewhere in } H = L^2(\Omega). \end{cases}$$

9.1.4. *Semilinear parabolic PDEs.* Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone operator and let $j : \mathbb{R} \to (-\infty, \infty]$ be a convex function so that $\beta = \partial j$. Then, the relation

$$u_t - \Delta u + \beta(u) \ni g$$

along with homogeneous Dirichlet boundary conditions, is the gradient flow in $H = L^2(\Omega)$ of the functional

$$\phi(u) = \begin{cases} \displaystyle \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega j(u) - \int_\Omega gu & \text{for } \ u \in H^1_0(\Omega), \ j(u) \in L^1(\Omega) \\ \infty & \text{elsewhere in } H = L^2(\Omega). \end{cases}$$

9.1.5. *Quasilinear parabolic PDEs.* Let $F : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be such that:

$$F(x, \cdot) \in C^1(\mathbb{R}^d) \quad \text{for a.e. } \ x \in \Omega, \tag{15}$$

$$F(x, \cdot) \quad \text{is convex and } \ F(x, 0) = 0 \ \text{ for a.e. } \ x \in \Omega, \tag{16}$$

$$F(\cdot, \xi) \quad \text{is measurable for all } \ \xi \in \mathbb{R}^d. \tag{17}$$

Then, we can set $b = \nabla_\xi F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. We assume that, for a given $p > 1$, F satisfies the growth conditions

$$\begin{aligned} &\exists c > 0 \quad \text{such that } \ F(x, \xi) \geq c|\xi|^p - (1/c), \\ &|b(x, \xi)| \leq (1/c)(1 + |\xi|^{p-1}) \quad \text{for a.e. } \ x \in \Omega \ \text{ and all } \ \xi \in \mathbb{R}^d. \end{aligned} \tag{18}$$

Then, the quasilinear equation

$$u_t - \nabla \cdot (b(x, \nabla u)) \ni g$$

along with homogeneous Dirichlet boundary conditions, is the gradient flow in $H = L^2(\Omega)$ of the functional

$$\phi(u) = \begin{cases} \displaystyle \int_\Omega F(x, \nabla u) - \int_\Omega gu & \text{for } \ u \in W^{1,p}_0(\Omega), \\ \infty & \text{elsewhere in } H = L^2(\Omega). \end{cases}$$

In particular, the choice $F(x, \xi) = |\xi|^p/p$ gives rise to the so-called *p-Laplacian equation*, whereas the choice $F(x, \xi) = (1 + |\xi|^2)^{1/2}$ corresponds to the *mean curvature flow for Cartesian surfaces* (note however that the latter does not directly fit into this theory because of a lack of lower semicontinuity).

9.1.6. *Degenerate parabolic PDEs.* Assume we are given $\beta : \mathbb{R} \to \mathbb{R}$ monotone and continuous with $\beta(0) = 0$ and superlinear growth at infinity [4]. Define j to be the only convex function such that $\beta = j'$ and $j(0) = 0$. Then, the equation

$$u_t - \Delta\beta(u) = 0$$

along with homogeneous Dirichlet boundary conditions, is the gradient flow in $H = H^{-1}(\Omega)$ of the functional

$$\phi(u) = \begin{cases} \displaystyle\int_\Omega j(u) & \text{for } \ u \in H^1_0(\Omega), \ j(u) \in L^1(\Omega) \\ \infty & \text{elsewhere in } H = H^{-1}(\Omega). \end{cases}$$

In particular, the choice $\beta(u) = (u-1)^+ - u^-$ corresponds to the classical *two-phase Stefan problem*, $\beta(u) = |u|^{m-2}u$ for $m > 2$ leads to the *porous medium equation*. The multivalued case $\beta(u) = \partial I_{[0,1]}$ (subdifferential of the indicator function of the interval $[0,1]$), related to the *Hele-Shaw cell equation*, can be handled as well.

9.2. **Basic properties.**

Lemma 9.2 (Chain rule)**.** *Let $u \in H^1(\mathbb{R}_+; H)$ and $\xi \in L^2(\mathbb{R}_+; H)$ with $\xi \in \partial\phi(u)$ a.e. in $\mathbb{R}_+$. Then, $t \mapsto \phi(u(t))$ is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = (\xi(t), u'(t)) \quad \textit{for a.a. } t > 0.$$

By testing (13) on u, integrating on $(0,t)$ and use the chain rule, we obtain the following.

Proposition 9.3 (Energy conservation)**.** *Let u be a gradient-flow solution. Then,*

$$\phi(u(t)) + \int_0^t |u'|^2 = \phi(u^0) + \int_0^t (g, u) \quad \forall t > 0. \tag{19}$$

Proposition 9.4 (Continuous dependence)**.** *Assume ϕ to be λ-convex for (possibly with $\lambda = 0$). Let u_i be gradient-flow solutions corresponding to data (u_i^0, g_i), for $i = 1, 2$. We have*

$$|u_1(t) - u_2(t)| \le e^{-\lambda t}|u_1^0 - u_2^0| + e^{-\lambda t}\int_0^t e^{\lambda s}|g_1(s) - g_2(s)|\mathrm{d}s \quad \forall t > 0. \tag{20}$$

In particular,

(i) *Gradient-flow solutions are unique;*
(ii) *The gradient flow is contractive for $g = 0$, exponentially contractive for $\lambda > 0$;*
(iii) *For $\lambda > 0$ and $g = 0$ solutions converge exponentially fast to the unique equilibrium.*

Proof. Take the difference of (13) written for $i = 1$ and the same relation for $i = 2$, test the resulting relation on $w(t) := (u_1 - u_2)(t)$ and integrate on $(0,t)$ getting

$$|w(t)|^2 + 2\lambda\int_0^t |w|^2 \le |w(0)|^2 + 2\int_0^t |g_1 - g_2|\,|w|.$$

Then, relation (20) results from an application of the Gronwall lemma, see [13, Lemma 3.7]. By taking $u_1^0 = u_2^0$ and $g_1 = g_2$ in (20) we obtain (i). The choice $g_1 = g_2 = 0$ in (20) gives (ii). Assume that ϕ is λ-convex for $\lambda > 0$. Then, ϕ admits a unique minimizer u_∞. By setting $u_2^0 = u_\infty$ we obtain $u_2(t) = u_\infty$ for all $t \in [0,T)$. Then, letting $u = u_1$ with $u_1^0 = u^0$, again by (20)

$$\lim_{t\to\infty} |u(t) - u_\infty| \leq \lim_{t\to\infty} \mathrm{e}^{-\lambda t}|u^0 - u_\infty| = 0.$$

This proves (iii). $\square$

Proposition 9.5 (Minimal element)**.** *The gradient-flow solution fulfills*

$$g - \dot{u} = (\partial\phi(u))^\circ := \{\|\xi\| \mid \xi \in \partial\phi(u)\} \quad a.e.$$

The introduced notion of gradient-flow solution makes sense for $u \in D(\phi)$ only, see (19). On the other hand, by letting $g = 0$ and exploiting contractivity (ii) of the flow one can define some weaker notion of solution for $u^0 \in \overline{D(\phi)}$. In particular, let $D(\phi) \ni u_n^0 \to u^0$. Then, as from (ii) one has

$$|u_m(t) - u_n(t)| \leq |u_m^0 - u_n^0| \to 0$$

we conclude that u_n is a Cauchy sequence in $C(0,T;H)$. As such, it has a limit, which can seen as a *weak solution* to the gradient flow.

9.3. **Existence by time-discretization.**

Theorem 9.6. *Let $\phi : H \to (-\infty,\infty]$ proper, convex, and l.s.c., $u^0 \in D(\phi)$, and $g \in L^2(0,T;H)$. Let, $N \in \mathbb{N}$ be given and define $\tau = T/N$, $g_i = (1/\tau)\int_{(i-1)\tau}^{\tau} g(s)\mathrm{d}s$ and $u^i = J_\tau(u^{i-1})$ for $i = 1,\ldots,N$. Namely, u^i is the unique solution of the minimization problem*

$$\min\left(\frac{|u - u^{i-1}|^2}{2\tau} + \phi(u) - (g^i, u)\right).$$

Let $u_\tau : [0,T] \to H$ be the piecewise affine interpolant of the values $\{u^i\}$ on the time partition $\{i\tau\}$. Then $u_\tau \to u$ in $H^1(0,T;H)$ where u solves (13)-(14).

Proof. We have that u^i solves $(u^i - u^{i-1})/\tau + \xi^i = g^i$ for $i = 1,\ldots,N$, where $\xi^i \in \partial\phi(u^i)$. Let $\overline{\eta}_\tau$ denote the right-continuous, piecewise interpolant on the time partition $\{i\tau\}$ of the vector $\{\eta^i\}_{i=1}^m$. We can rewrite the latter system as

$$\dot{u}_\tau + \overline{\xi}_\tau = \overline{g}_\tau \quad \text{a.e. in } (0,T). \tag{21}$$

From the minimality property of $u^i = J_\tau(u^{i-1})$ one has that

$$\phi(u^i) + \frac{1}{2\tau}|u^i - u^{i-1}|^2 - (g^i, u^i) \leq \phi(u^{i-1}) - (g^i, u^{i-1}).$$

Taking the sum for $i = 1, \dots, m$ for $m \leq N$ one has

$$\phi(u^m) + \frac{1}{2}\sum_{i=1}^{m} \tau \left|\frac{u^i - u^{i-1}}{\tau}\right|^2 \leq \phi(u^0) + \sum_{i=1}^{m} \tau |g^i| \left|\frac{u^i - u^{i-1}}{\tau}\right|.$$

The latter estimate and a comparison in (21) ensure that

$$\|u_\tau\|_{H^1(0,T;H)} \text{ and } \|\overline{\xi}_\tau\|_{L^2(0,T;H)} \text{ are bounded indep. of } \tau.$$

By noting indeed that $\|u_\tau - \overline{u}_\tau\|^2_{L^2(0,T;H)} \leq \tau^3 \|\dot{u}_\tau\|^2_{L^2(0,T;H)}/\sqrt{3}$, one can hence extract (not relabeled) subsequences so that

$$u_\tau, \ , \overline{u}_\tau \rightharpoonup u, \quad \dot{u}_\tau \rightharpoonup \dot{u}, \quad \overline{\xi}_\tau \rightharpoonup \xi \ \text{ in } L^2(0,T;H).$$

As $\overline{g}_\tau \to g$ in $L^2(0,T;H)$, we may pass to the limit in (21) and obtain

$$\dot{u} + \xi = g \ \text{ a.e. in } (0,T). \tag{22}$$

Note that $u_\tau \rightharpoonup u$ in $H^1(0,T;H)$ implies that $u_\tau(t) \rightharpoonup u(t)$ for all $t \in [0,T]$. In particular, $u^0 = u_\tau(0) \rightharpoonup u(0)$ so that the initial condition (14) is fulfilled.

In order to check for the a.e. inclusion $\xi \in \partial\phi(u)$ we test (21) on $\overline{u}_\tau$ and integrate in time in order to get

$$\int_0^T (\overline{\xi}_\tau, \overline{u}_\tau) \stackrel{(21)}{=} \int_0^T (\overline{g}_\tau - \dot{u}_\tau, \overline{u}_\tau) \leq \int_0^T (\overline{g}_\tau, \overline{u}_\tau) - \phi(u_\tau(T)) + \phi(u^0).$$

Apply the $\limsup$ as $\tau \searrow 0$ to both sides and exploit the strong convergence of $\overline{g}_\tau$ in order to get that

$$\limsup_{\tau \searrow 0} \int_0^T (\overline{\xi}_\tau, \overline{u}_\tau) \leq (g, u) - \frac{1}{2}|u(T)|^2 + \frac{1}{2}|u^0|^2 \stackrel{(22)}{=} \int_0^T (\xi, u).$$

Hence, the inclusion $\xi \in \partial\phi(u)$ follows by applying Lemma 5.4 and u solves the gradient flow (13)-(14). As gradient-flow solutions are unique, not just a subsequence but the whole sequence u_τ converges to u.

Test now (21) on $\dot{u}_\tau$ and integrate in time getting

$$\int_0^T |\dot{u}_\tau|^2 \stackrel{(21)}{\leq} -\phi(u_\tau(T)) + \phi(u^0) + \int_0^T (\overline{g}_\tau, \dot{u}_\tau).$$

Pass to the $\limsup$ as $\tau \searrow 0$ and use the lower semicontinuity of ϕ and, again, the strong convergence of $\overline{g}_\tau$ in order to get that

$$\limsup_{\tau \searrow 0} \int_0^T |\dot{u}_\tau|^2 \leq -\phi(u(T)) + \phi(u^0) + \int_0^T (g, \dot{u}) \stackrel{(22)}{=} \int_0^T |\dot{u}|^2.$$

Hence, the strong convergence $u_\tau \to u$ in $H^1(0,T;H)$ follows from Lemma 2.11. $\square$

9.4. **Existence by Moreau-Yosida regularization.**

Theorem 9.7. *Let $\phi : H \to (-\infty, \infty]$ proper, convex, and l.s.c., $u^0 \in D(\phi)$, and $g \in L^2(0,T;H)$. Let u_λ be a gradient-flow solution to*

$$u' + \partial\phi_\lambda(u) \ni g \quad \text{for a.e. } t > 0, \tag{23}$$

$$u(0) = u^0 \tag{24}$$

where ϕ_λ is the Moreau-Yosida regularization of ϕ at level $\lambda > 0$. Then, $u_\lambda \to u$ in $H^1(0,T;H)$ where u solves (13)-(14).

Proof. For all λ, the gradient-flow solution u_λ exists uniquely as $\partial\phi_\lambda$ is Lipschitz continuous. By testing (23) by $\dot{u}_\lambda$ we readily get that

$$\|u_\lambda\|_{H^1(0,T;H)} \text{ and } \|\xi_\lambda\|_{L^2(0,T;H)} \text{ are bounded indep. of } \lambda$$

where $\xi_\lambda = \partial\phi_\lambda(u_\lambda)$ a.e. By extracting some not relabeled subsequence we can proceed as in the proof of Theorem 9.6 and conclude □

Given any $T > 0$, we have established the existence of a gradient-flow solution u_T on $(0,T)$. By a standard argument we find a gradient flow solution in the whole of $\mathbb{R}_+$. In particular, let $u : \mathbb{R}_+ \to H$ be defined by

$$u(t) = u_T(t) \text{ for some } T \geq t.$$

This definition makes sense due to the uniqueness of gradient-flow solutions.

9.5. **Approximation.** Assume to be given $\phi_n : H \to [0, \infty]$ convex, proper, and l.s.c., $u_n^0 \in D(\phi_n)$, and $g_n \in L^2(\mathbb{R}_+;H)$. Then, for all n one has a unique $u_n \in H^1(0,T;H)$ solving

$$u_n' + \partial\phi_n(u_n) \ni g_n \quad \text{for a.e. } t > 0, \tag{25}$$

$$u_n(0) = u_n^0. \tag{26}$$

We now present an assumption frame under which u_n converge to a solution of the limiting gradient flow.

Theorem 9.8. *Assume that $\phi_n \to \phi$ in Mosco sense, $u_n^0 \to u^0$, $\sup \phi_n(u_n^0) < \infty$, and $g_n \to g$ in $L^2(\mathbb{R}_+;H)$. Then, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}_+;H)$ where u is the gradient-flow solution of* (13)-(14). *If additionally the initial data are* well-prepared, *i.e. $\phi_n(u_n^0) \to \phi(u^0)$, then $u_n \to u$ in $H^1(\mathbb{R}_+;H)$ and $\phi_n(u_n(t)) \to \phi(u(t))$ for all $t \geq 0$.*

9.6. **The Brezis-Elekand-Nayroles principle.** Assume $f = 0$ in (13) and let the functional J be defined over whole trajectories $u \in K = \{v \in H^1(0,T;H) \mid v(0) = u^0\}$ as

$$J(v) = \int_0^T \big(\phi(u) + \phi^*(-\dot{u})\big)\mathrm{d}t + \frac{1}{2}|u(T)|^2 - \frac{1}{2}|u^0|^2.$$

Our interest in J is revealed by the following.

Theorem 9.9 (Brezis-Elekand-Nayroles principle)**.** *u is a gradient-flow solution iff $J(u) = \min_K J = 0$.*

Proof. Let u be a gradient-flow solution. Then, by Lemma 4.7,

$$\phi(u) + \phi^*(-\dot{u}) - (-\dot{u}, u) = 0 \quad \text{a.e. in} \ \ (0,T). \tag{27}$$

By integrating the latter and imposing (14) we get $J(u) = 0$. On the other hand the above right-hand side of (27) is always nonnegative. In particular, $J(v) \geq 0$ for all $v \in K$, so that u is a minimizer of J.

Let now $J(u) = 0$ for some $u \in K$. Then, (14) and (27) hold. From Lemma 4.7 we deduce that u solves (13) as well. □

The idea behind the reformulation of gradient flows as minimization problems is that of exploiting the methods of the Calculus of Variations (Direct Method, relaxation, etc.). In this regard, one should remark that J is convex and l.s.c. with respect to the weak topology of $H^1(0,T;H)$. On the other hand, it is not immediate that J is coercive with respect to this topology. Indeed, this would follow for some quadratically growing ϕ^*. This would however imply a quadratically bounded ϕ, a pretty restrictive assumption w.r.t. applications, especially to PDEs. Coercivity can however be enforced by replacing J by

$$\tilde{J}(u) = \left(\int_0^T |\dot{u}|^2 + \phi(u(T)) - \phi(u^0) \right)^+ + J(u).$$

Theorem 9.9 holds for $\tilde{J}$ as well and $\tilde{J}$ is convex, l.s.c., and coercive w.r.t. the weak topology of $H^1(0,T;H)$.

The major drawback of the Brezis-Ekeland-Nayroles variational characterization is that it does not consists in a pure minimization but rather in a so-called *null-minimization* problem: One is additionally required to check that the minimum value is actually 0. This delicate point has been tackled from different directions.

For the sake of illustration, let us spell out the form of the functional J in the case of the heat equation as

$$J(u) = \frac{1}{2} \int_0^T \int_\Omega \left(|\nabla u|^2 + |\nabla(-\Delta)^{-1} u_t|^2 \right) \mathrm{d}t + \frac{1}{2} \int_\Omega |u(T)|^2 - \frac{1}{2} \int_\Omega |u^0|^2.$$

9.7. **The WED principle.** Fix $\varepsilon > 0$ and let $u^\varepsilon \in K$ minimize the Weighted Energy-Dissipation (WED) functional

$$I^\varepsilon(u) = \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |\dot{u}|^2 + \phi(u) \right) \mathrm{d}t.$$

The corresponding Euler-Lagrange system reads

$$-\varepsilon \ddot{u}^\varepsilon + \dot{u}^\varepsilon + \partial\phi(u^\varepsilon) \ni 0, \quad \varepsilon \dot{u}(T) = 0 \tag{28}$$

revealing indeed the fact that minimizing I^ε corresponds to solving an elliptic-in-time regularization of the gradient-flow, of course with an extra *final* condition. For all $\varepsilon > 0$, this corresponds to a *non-causal* approximation of the gradient-flow.

With respect to the Brezis-Ekeland-nayroles principle, the WED approach has the advantage of being a true minimization (instead of a *null-minimization*). At the same time, the functional I^ε is l.s.c. and coercive w.r.t. the weak topology of $H^1(0,T;H)$ and uniformly convex. In particular, the minimizer u^ε exists uniquely. The point is to check that in the limit $\varepsilon \searrow 0$ one has $u^\varepsilon \to u$ where u is a gradient-flow solution. This has been accomplished in [12]. The core of the argument is a maximal regularity estimate on (28). In particular, by taking the square of the left-hand side of the first equation in (28) one has

$$
\begin{aligned}
0 &= \int_0^T \big(|\varepsilon \ddot{u}^\varepsilon|^2 + |\dot{u}^\varepsilon|^2 + |\partial\phi(u)^\varepsilon|^2\big) \\
&\quad - 2\varepsilon \int_0^T (\ddot{u}^\varepsilon, \dot{u}^\varepsilon) + 2\int_0^T (\dot{u}^\varepsilon, \partial\phi(u^\varepsilon)) - 2\varepsilon \int_0^T (\ddot{u}^\varepsilon, \partial\phi(u^\varepsilon)) \\
&= \int_0^T \big(|\varepsilon \ddot{u}^\varepsilon|^2 + |\dot{u}^\varepsilon|^2 + |\partial\phi(u)^\varepsilon|^2\big) - \varepsilon|\dot{u}^\varepsilon(T)|^2 + \varepsilon|\dot{u}^\varepsilon(0)|^2 \\
&\quad + 2\phi(u^\varepsilon(T)) - 2\phi(u^0) + 2\varepsilon(\dot{u}^\varepsilon, \partial\phi(u^\varepsilon))(0) + \int_0^T (\dot{u}^\varepsilon, \mathrm{D}^2\phi(u^\varepsilon)\dot{u}^\varepsilon).
\end{aligned}
$$

At this level, the above computation is just formal as we are missing the necessary regularity. The argument can be however made precise at some approximation level (by time-discretizing, for instance). By assuming that $u^0 \in D(\partial\phi)$ (this can be relaxed) we obtain that

$$
\varepsilon\ddot{u}^\varepsilon, \ \dot{u}^\varepsilon, \ \partial\phi(u^\varepsilon) \ \text{ are bounded in } L^2(0,T;H) \text{ independently of } \varepsilon.
$$

Via the interpolation $\|\dot{u}^\varepsilon\|_{L^\infty} \leq \|\ddot{u}^\varepsilon\|_{L^2}^{1/2}\|\dot{u}^\varepsilon\|_{L^2}^{1/2}$ we deduce that

$$
\sqrt{\varepsilon}\dot{u}^\varepsilon \ \text{ is bounded in } L^\infty(0,T;H) \text{ independently of } \varepsilon.
$$

We can hence take the difference between (13) and (28), test it on $w^\varepsilon := u - u^\varepsilon$, and integrate in time in order to get

$$
\begin{aligned}
\frac{1}{2}|w^\varepsilon(t)|^2 &= -\varepsilon\int_0^t (\ddot{u}^\varepsilon, w^\varepsilon) = -\varepsilon(\dot{u}^\varepsilon, w^\varepsilon)|_0^t + \varepsilon\int_0^t (\dot{u}^\varepsilon, \dot{w}^\varepsilon) \\
&\leq \varepsilon^2|\dot{u}^\varepsilon(t)|^2 + \frac{1}{4}|w^\varepsilon(t)|^2 + \varepsilon\int_0^t (\dot{u}^\varepsilon, \dot{w}^\varepsilon) \leq \frac{1}{4}|w^\varepsilon(t)|^2 + C\varepsilon.
\end{aligned}
$$

In particular, $w^\varepsilon = u - u^\varepsilon \to 0$ in $C([0,T];H)$. As $\partial\phi(u^\varepsilon) \rightharpoonup \xi$ in $L^2(0,T;H)$, the latter implies that $\xi \in \partial\phi(u)$. In particular, u is a gradient-flow solution. The argument can be refined in order not to rely on the existence of a solution to (13)-(14).

9.8. **Lipschitz perturbations of gradient flows.** Let $f : H \to H$ be Lipschitz continuous and, as before, $\phi : H \to (-\infty, \infty]$ be proper, convex, and l.s.c., and $u^0 \in D(\phi)$. We are here interested in the following Lipschitz perturbation

$$u' + \partial\phi(u) \ni f(u) \quad \text{for a.e. } t > 0, \tag{29}$$

$$u(0) = u^0. \tag{30}$$

Note tat it makes sense to look for solutions $u \in H^1(0,T;H)$ to the latter as, correspondingly, the nonlinear term $f(u)$ belongs to $H^1(0,T;H) \subset L^2(0,T;H)$. We have the following.

Theorem 9.10. *There exists a unique solution $u \in H^1(0,T;H)$ to* (29)-(30).

Proof. Let us devise an iterative procedure. For all $\tilde{u} \in C([0,T];H)$ we define $S(u)$ to be the solution of the gradient flow

$$u' + \partial\phi(u) \ni f(\tilde{u}) \quad \text{for a.e. } t > 0,$$
$$u(0) = u^0$$

where the nonlinearity is fixed. This defines a map in $C([0,T];H)$. Let now $\tilde{u}_i$ for $i = 1, 2$ be given. By taking the difference between the respective equations we readily check that

$$|(S(u_1) - S(u_2))(t)|^2 \leq C \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|^2_{C([0,s];H)} \mathrm{d}s$$

where C depends on the Lipschitz constant of f as well. Then, one can prove by induction that

$$\|S^{(k)}(\tilde{u}_1) - S^{(k)}(\tilde{u}_2)\|^2_{C([0,t];H)} \leq \frac{C^k t^k}{k!} \|\tilde{u}_1 - \tilde{u}_2\|^2_{C([0,t];H)} \quad \forall t \in [0,T].$$

By choosing k large enough, the mapping $S^{(k)}$ is a strict contraction in $C([0,T];H)$. Hence, $S^{(k)}$ admits a (unique) fixed point $u \in C([0,T];H)$. This is also a fixed point of S. Indeed, from $u = S^{(k)}(u)$ we deduce that $S(u) = S(S^{(k)}(u)) = S^{(k)}(S(u))$ so that $S(u)$ is a fixed point of $S^{(k)}$ as well. Since $S^{(k)}$ has a unique fixed point, we have $u = S(u)$. In particular, u belongs to $H^1(0,T;H)$. $\square$

The latter theorem ensures that the existence theory for gradient flows of convex functions can be extended to the λ-convex case (with $\lambda < 0$) as one can write

$$\partial\phi = \partial(\phi - \lambda|\cdot|^2/2) + \lambda\mathrm{id}$$

where $\partial(\phi - \lambda|\cdot|^2/2)$ is maximal monotone and λid is nonmonotone but Lipschitz continuous. This particularly allows to solve gradient flows driven by $C^{1,1}$ functionals.

The perturbation can be generalized to $f(t,u)$ or even a nonlocal $F : C([0,T];H) \to C([0,T];H)$ so that

$$\|F(u_1)-F(u_2)\|_{C([0,t];H)} \leq C\|u_1-u_2\|_{C([0,t];H)}$$

for some $C > 0$ and all $t \in (0,T)$. This particularly includes the case of Volterra operators

$$F(u)(t) = \int_0^t k(t,s)f(u(s))\mathrm{d}s$$

for f Lipschitz and $k \in L^1$.

An application of this Lipschitz perturbation theory allows to cover the case of the linear equation on $u : \Omega \times (0,T) \to \mathbb{R}$

$$u_t - \Delta u + \beta \cdot \nabla u = g$$

for some $\beta \in \mathbb{R}^d$, together with initial and, say, homogeneous Dirichlet boundary conditions. Note that the latter is not a gradient flow as the operator $u \mapsto -\Delta u + \beta \cdot \nabla u$ is not symmetric.

10. Curves of maximal slope in metric spaces

We are now interested in extending the existence theory for gradient flows to metric spaces, namely out of the linear setting. In order to to this, we follow the original idea by De Giorgi [8]. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth. Then, one has the following chain of equivalences

$$\begin{aligned}
&\dot{u} + \nabla\phi(u) = 0 \quad \Longleftrightarrow \quad \frac{1}{2}|\dot{u} + \nabla\phi(u)|^2 = 0\\
&\Longleftrightarrow \quad \frac{1}{2}|\dot{u}|^2 + \nabla\phi(u)\cdot\dot{u} + \frac{1}{2}|\nabla\phi(u)|^2 = 0\\
&\Longleftrightarrow \quad \frac{1}{2}|\dot{u}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\phi(u) + \frac{1}{2}|\nabla\phi(u)|^2 = 0\\
&\Longleftrightarrow \quad \phi(u(t)) + \frac{1}{2}\int_0^t |\dot{u}|^2 + \frac{1}{2}\int_0^t |\nabla\phi(u)|^2 = \phi(u(0)) \quad \forall t > 0. \qquad (31)
\end{aligned}$$

The first relation in this chain is system of equations in $\mathbb{R}^d$ whereas all the others are scalar equations. In order to give sense to the first equation in $\mathbb{R}^d$ one needs to compute time derivative and gradients. This requires the possibility of taking differences of u. In particular, u should belong to a linear space. On the contrary, the last relation in the chain requires only that *the norm* of the time derivative and *the norm* of the gradients be defined. These are notions which can make sense also in a complete metric space (X,d), in relation to a proper and l.s.c. functional $\phi : X \to [0,\infty]$ (once again, the lower bound is inessential, but it greatly simplifies the arguments).

The strategy hence runs as follow: (1) define a surrogate for the norm of the time derivative (Subsection 10.1) and (2) a surrogate for the norm of the gradient (Subsection 10.2), (3) make precise the notion of solution to the latter in (31) (Subsection 10.3), and (4) prove that such solution exists (Subsection 10.6). In addition, we would like to comment on how this metric solution notion is indeed an extension of the Hilbert one. This is done in Subsection 10.5.

The material of this section corresponds to a simplified/condensed version of [1], to which we occasionally refer for some proofs and for both full generality and applications.

10.1. **Metric derivative.** We say that a curve $u : [0,T] \to X$ belongs to $AC^p([0,T];X)$, $p \in [1,\infty]$, if there exists $m \in L^p(0,T)$ such that

$$d(u(s),u(t)) \leq \int_s^t m(r)\mathrm{d}r \quad \text{for all } 0 < s \leq t < T. \tag{32}$$

For $p = 1$, we simply write $AC([0,T];X)$ and speak of *absolutely continuous curves.* For all $u \in AC^p([0,T];X)$, the limit

$$|u'|(t) = \lim_{s\to t} \frac{d(u(s),u(t))}{|t-s|}$$

exists for a.a. $t \in (0,T)$ [1, Thm. 1.1.2]. We will refer to it as the *metric derivative* of u at t. We have that the map $t \mapsto |u'|(t)$ belongs to $L^p(0,T)$ and it is minimal within the class of functions $m \in L^p(0,T)$ fulfilling (32).

If $X = H$ is a Hilbert space, then one clearly has that $AC^p(0,T;X) = W^{1,p}(0,T;H)$ and $|u'| = \|u'\|$ a.e. (here $\|\cdot\|$ is the norm in H).

10.2. **Local slope.** Let $\phi : X \to [0,+\infty]$ be lower semicontinuous and proper and let $D(\phi) := \{u \in X \ : \ \phi(u) < +\infty\}$ denote the effective domain of ϕ. We define the *local slope* of ϕ at $u \in D(\phi)$ as

$$|\partial\phi|(u) := \limsup_{v\to u} \frac{(\phi(u)-\phi(v))^+}{d(u,v)}.$$

We say that a function $g : X \to [0,+\infty]$ is a *strong upper gradient* for the functional ϕ if, for every curve $u \in AC(0,T;X)$, the function $g \circ u$ is Borel and

$$|\phi(u(t)) - \phi(u(s))| \leq \int_s^t g(u(r))|u'|(r)\mathrm{d}r \quad \text{for all } 0 < s \leq t < T$$

We explicitly observe that, whenever $(g \circ u)|u'| \in L^1(0,T)$, then $\phi \circ u \in W^{1,1}(0,T)$ and

$$|(\phi \circ u)'(t)| \leq g(u(t))|u'|(t) \quad \text{for a.e. } t > 0.$$

The notion of strong upper gradient will be relevant in the definition of metric solution to the gradient flow below. Let us however anticipate that, under the assumptions which will be spelled out for the existence theorem, the local slope $|\partial\phi|$ will turn out to be a strong upper gradient for ϕ.

10.3. **Curves of Maximal Slope.** Let $g : X \to [0, +\infty]$ be a strong upper gradient for ϕ and $u^0 \in D(\phi)$. We say that $u \in AC^2(0, T; X)$ is a *curve of maximal slope for the functional ϕ with respect to the upper gradient g starting from u^0* if $u(0) = u^0$ and

$$\phi(u(t)) + \frac{1}{2}\int_0^t |u'|^2(r)\mathrm{d}r + \frac{1}{2}\int_0^t g^2(u(r))\mathrm{d}r = \phi(u^0) \quad \forall 0 < t < T.$$

10.4. **Geodesically convex functionals.** We shall be concerned with a metric notion of convexity. We say that the functional ϕ is λ-geodesically convex for some $\lambda \in \mathbb{R}$ if

$$\begin{aligned}&\text{for all } v_0,\, v_1 \in D(\phi) \text{ there exists a } \textit{constant-speed geodesic } v : [0,1] \to X\\ &\text{(i.e. satisfying } d(v(s), v(t)) = (t-s)d(v_0, v_1) \text{ for all } 0 \le s \le t \le 1)\\ &\text{such that } v(0) = v_0, \quad v(1) = v_1, \quad \text{and } \phi \text{ is } \lambda\text{-convex on } v, \text{ i.e.}\\ &\phi(v(t)) \le (1-t)\phi(v_0) + t\phi(v_1) - (\lambda/2)t(1-t)d^2(v_0, v_1) \quad \text{for all } 0 \le t \le 1.\end{aligned} \tag{33}$$

Note that the term *geodesic* is suggestive of the flat case $X = \mathbb{R}^d$. Here, constant-speed geodesic are straight segments. The same holds true in Hilbert spaces. The existence of a constant-speed geodesic connecting two points can be always ensured in so-called *length spaces*. We shall however not discuss this point here and rather include the existence requirement of the constant-speed geodesic directly in the definition of geodesic convexity. The following result stems from a combination of [1, Cor. 2.4.10, Lem. 2.4.13, Thm. 2.4.9].

Proposition 10.1. *Let $\phi : X \to (-\infty, +\infty]$ be l.s.c. and geodesically convex ($\lambda \ge 0$). Then, the local slope $|\partial\phi|$ is l.s.c. and admits the representation*

$$|\partial\phi|(u) = \sup_{v \neq u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)} \quad \textit{for all } u \in D(\phi). \tag{34}$$

Furthermore, $|\partial\phi|$ is an upper gradient.

Proof. We just prove the representation formula here. Clearly

$$|\partial\phi|(u) = \limsup_{v \to u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)} \le \sup_{v \neq u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)}.$$

On the other hand, given $v \in D(\phi)$ let $t \mapsto v(t)$ be a constant-speed geodesic connecting u to v. Then,

$$\begin{aligned}|\partial\phi|(u) &= \limsup_{v \to u} \frac{(\phi(u) - \phi(v))^+}{d(u, v)} \ge \limsup_{t \to 0} \frac{(\phi(u) - \phi(v(t)))^+}{td(u, v(t))}\\ &\ge \limsup_{t \to 0} \frac{(\phi(u) - (1-t)\phi(u) - t\phi(v(t)))^+}{td(u, v)} = \frac{(\phi(u) - \phi(v))^+}{d(u, v)}\end{aligned}$$

so that the formula holds.

Let's check now that $|\partial\phi|$ is l.s.c. Let $u_n \to u$ and $v \in D(\phi)$ with $v \neq u$ be given. Then, $1/d(u_n, v) \to 1/d(u, v)$ and

$$\frac{(\phi(u) - \phi(v))^+}{d(u,v)} \leq \liminf_{n\to\infty} \frac{(\phi(u_n) - \phi(v))^+}{d(u_n, v)} \overset{(34)}{\leq} \liminf_{n\to\infty} |\partial\phi|(u_n)$$

and the lower semicontinuity follows by taking the sup on v.

□

10.5. **The Hilbert-space case.** In case $X = H$ is Hilbert and ϕ is convex, proper, and l.s.c. gradient-flow solutions and curves of maximal slope coincide (and are unique). Indeed, given a gradient-flow solution one exploits Lemma 9.2 in order to give sense to the chain of equivalences (31). This indeed gives rise to a curve of maximal slope as one can check that (recall also Proposition 9.5)

$$|\partial\phi|(u) = \|(\partial\phi(u))^\circ\| = \min\{\|\xi\| \mid \xi \in \partial\phi(u)\} \quad \forall u \in D(\partial\phi).$$

Indeed, for all $\xi \in \partial\phi(u)$ one has

$$|\partial\phi|(u) = \sup_{w\neq u} \frac{(\phi(u) - \phi(w))^+}{\|u - w\|} \leq \sup_{w\neq u} \frac{|(\xi, u - w)|}{\|u - w\|} \leq \|\xi\|.$$

On the other hand, for all $\eta \in H$ one has that

$$-|\partial\phi|(u)\|\eta\| \leq \phi(u + \eta) - \phi(u).$$

as $\eta \mapsto -|\partial\phi|(u)\|\eta\|$ is concave and $\eta \mapsto \phi(u + \eta) - \phi(u)$ is convex, by the Hahn-Banach Theorem there exists $\xi \in H$ such that

$$-|\partial\phi|(u)\|\eta\| \overset{(34)}{\leq} (\xi, \eta) \leq \phi(u + \eta) - \phi(u) \quad \forall \eta \in H.$$

The first inequality entails that $\|\xi\| \leq |\partial\phi|(u)$ while the second says that $\xi \in \partial\phi(u)$.

10.6. **Existence.** Let us now turn to the existence result. This will be stated and proved under very strong assumptions which have the benefit of simplifying the argument but also the drawback of being little suited for applications. In particular, the compactness assumption below is very restrictive and could be avoided, reproducing indeed the existence theory in the Hilbert-space case. We refer to [1] for such generalizations.

Theorem 10.2 (Existence of curves of maximal slope)**.** *Let (X, d) be complete, $\phi : X \to [0, \infty]$ be proper, l.s.c., geodesically convex, and have compact sublevels, and $u^0 \in D(\phi)$. Then, there exists a curve of maximal slope for ϕ w.r.t. the strong upper gradient $|\partial\phi|$ starting from u^0. That is, a curve $u \in AC^2(0, T; X)$ with $u(0) = u^0$ so that*

$$\phi(u(t)) + \frac{1}{2}\int_0^t |u'|(s)\mathrm{d}s + \frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\mathrm{d}s = \phi(u^0) \quad \forall 0 < t < T.$$

Before proving the theorem by time-discretization, let us introduce some notation and facts. Let $\tau > 0$ and define

$$J_\tau(u) = \mathrm{Arg\,min}_v \left(\frac{d^2(u,v)}{2\tau} + \phi(v) \right).$$

The latter is nonempty as the functional in the parenthesis is coercive and l.s.c. Then, define

$$\phi_\tau(u) = \frac{d^2(u,u_\tau)}{2\tau} + \phi(u_\tau) \quad \text{for some (hence all) } u_\tau \in J_\tau(u)$$

and $\widehat{d}^+_\tau(u) = \sup\{d(u,u_\tau) \mid u_\tau \in J_\tau(u)\}$ and $\widehat{d}^-_\tau(u) = \inf\{d(u,u_\tau) \mid u_\tau \in J_\tau(u)\}$. We can check the following

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau(u) = -\frac{1}{2\tau^2}(\widehat{d}^\pm_\tau(u))^2 \quad \text{for a.e. } \tau > 0. \tag{35}$$

In particular $\tau \mapsto \phi_\tau(u)$ is monotone decreasing. Indeed, take $0 < \tau_0, \tau_1$ and $u_{\tau_1} \in J_{\tau_1}(u)$ and compute

$$\phi_{\tau_0}(u) - \phi_{\tau_1}(u) \leq \frac{1}{2\tau_0}d^2(u,u_{\tau_1}) + \phi(u_{\tau_1}) - \frac{1}{2\tau_1}d^2(u,u_{\tau_1}) - \phi(u_{\tau_1})$$
$$\leq \frac{1}{2\tau_0}d^2(u,u_{\tau_1}) - \frac{1}{2\tau_1}d^2(u,u_{\tau_1}) = \frac{\tau_0 - \tau_1}{2\tau_0\tau_1}d^2(u,u_{\tau_1}).$$

By changing signs and exchanging the role of τ_0 and τ_1 we also have

$$\frac{\tau_0 - \tau_1}{2\tau_0\tau_0}d^2(u,u_{\tau_1}) \leq \phi_{\tau_0}(u) - \phi_{\tau_1}(u).$$

As $\tau \mapsto \phi_\tau(u)$ is monotone [1, (3.1.5)], by letting $\tau_0 < \tau_1$ we conclude that

$$0 \leq \frac{1}{2\tau_0\tau_1}(\widehat{d}^+_{\tau_0}(u))^2 \leq \frac{\phi_{\tau_0}(u) - \phi_{\tau_1}(u)}{\tau_1 - \tau_0} \leq \frac{1}{2\tau_0\tau_1}(\widehat{d}^-_{\tau_1}(u))^2$$

so that (35) follows.

A second relation will turn out useful in the following

$$|\partial\phi|(u_r) \leq \frac{1}{r}d(u,u_r) \quad \forall u_r \in J_r(u). \tag{36}$$

Indeed, we have that

$$\phi(u_r) - \phi(v) \leq \frac{1}{2r}(d^2(u,v) - d^2(u,u_r))$$
$$= \frac{1}{2r}(d(u,v) - d(u,u_r))(d(u,v) + d(u,u_r)) \leq \frac{1}{2r}d(v,u_r)(d(u,v) + d(u,u_r)).$$

By dividing by $d(v,u_r)$ for $v \neq u_r$ and passing to the lim sup as $v \to u_r$ we get (36).

Proof of Theorem 10.2. Let $\tau = T/N$ for some $N \in \mathbb{N}$ and define $u^i \in J_\tau(u^{i-1})$ for $i = 1, \ldots, N$. From (35) we have

$$\phi_\tau(u) + \int_{\tau_0}^{\tau} \frac{1}{2r^2}(\widehat{d}^+_r(u))^2 \mathrm{d}r = \phi_{\tau_0}(u).$$

For all $u \in D(\phi)$ we can take the limit $\tau_0 \to 0$ [1, (3.1.6)] and get

$$\phi_\tau(u) + \int_0^\tau \frac{1}{2r^2}(\widehat{d}_r^+(u))^2 \mathrm{d}r = \phi(u).$$

Let now, $u = u^{i-1}$ and use $u^i \in J_\tau(u^{i-1})$ in order to check that

$$\phi(u^i) + \frac{1}{2\tau}d^2(u^{i-1}, u^i) + \int_0^\tau \frac{1}{2r^2}(\widehat{d}_r^+(u^{i-1}))^2 \mathrm{d}r \leq \phi(u^{i-1}). \tag{37}$$

By summing up on i the latter entails in particular that

$$\phi(\bar{u}_\tau) \ \text{ and } \ \int_0^T |\overline{u}_\tau'|^2(t)\mathrm{d}t \ \text{ are bounded independently of } \ \tau.$$

We can now use a refined variant of the Ascoli-Arzelá Theorem [1, Prop. 3.3.1] ensuring that some not relabeled subsequence $\overline{u}_\tau$ converges pointwise to some limit $u : [0,T] \to X$ and $|\overline{u}_\tau'| \rightharpoonup m$ in $L^2(0,T)$. In particular, for all $0 < s \leq t < T$ we have

$$d(u(s), u(t)) \leq \liminf_{\tau\to 0} d(\overline{u}_\tau(s), \overline{u}_\tau(t)) \leq \liminf_{\tau \to 0} \int_0^s |\overline{u}_\tau'|(r)\mathrm{d}r = \int_0^s m(r)\mathrm{d}r.$$

This entails that $u \in AC^2(0,T;X)$ and, by the minimality of $|u'|$,

$$\int_0^t |u'|^2(r)\mathrm{d}r \leq \int_0^s m(r)\mathrm{d}r \leq \liminf_{\tau\to 0} \int_0^t |\overline{u}_\tau'|^2(r)\mathrm{d}r \quad \forall t > 0.$$

Define now the *De Giorgi interpolant* as

$$\tilde{u}_\tau(t) = \tilde{u}_\tau(t^{i-1}+r) \in J_r(u^{i-1}) \quad \text{for } t = t^{i-1}+r \in [(i-1)\tau, i\tau), \ i = 1, \dots, N.$$

Then, one has that $\tilde{u}_\tau(t^i) = \overline{u}_\tau(t^i) = u^i$ and one can check that $\tilde{u}_\tau(t) \to u(t)$ for all t as well [1, (3.2.7)]. Moreover, one has

$$d(\tilde{u}(t^{i-1}+r), u^{i-1}) \leq \widehat{d}_r^+(u^{i-1}).$$

In particular, by using (36) we get that the third term in the left-hand side of estimate (37) is bounded from below by $(1/2)\int_0^\tau |\partial\phi|^2(\tilde{u}_\tau(t_{i-1}+r))\mathrm{d}r$. By passing to the lim inf in (37) (suitably summed over i) and exploiting the l.s.c. of $|\partial\phi|$ (see Proposition 10.1) we obtain

$$\frac{1}{2}\int_0^t |u'|^2(r)\mathrm{d}r + \frac{1}{2}\int_0^t |\partial\phi|^2(u(r))\mathrm{d}r \leq \phi(u^0) - \phi(u(t)).$$

We now use the fact that $u \mapsto |\partial\phi|(u)$ is a strong upper gradient (again Proposition 10.1) in order to prove the converse inequality, namely

$$\begin{aligned}\phi(u^0) - \phi(u(t)) &\leq \int_0^t |\partial\phi|(u(r))|u'|(r)\mathrm{d}r \\ &\leq \frac{1}{2}\int_0^t |u'|^2(r)\mathrm{d}r + \frac{1}{2}\int_0^t |\partial\phi|^2(u(r))\mathrm{d}r.\end{aligned}$$

Hence, u is a curve of maximal slope. □

11. Doubly nonlinear flows

We shall now turn our attention to the *doubly nonlinear* equation

$$\partial\psi(\dot u) + \partial\phi(u) \ni f \tag{38}$$

where the *dissipation* potential $\psi : H \to [0,\infty]$ is convex, l.s.c, and $\phi(0) = 0$, corresponding to a generalized version of (2). We shall prove that under compactness assumptions on the sublevels of ϕ and growth assumptions on $\partial\psi$ equation (38) admits strong solutions. Note that (38) covers the case of gradient flows for $\phi(\dot u) = |\dot u|^2/2$ ($|\cdot|$ is the norm in H). Still, we shall be here assuming compactness, which was not needed on the gradient flow case. In this regard, the present nonlinear theory is *not* an extension of the gradient flow results (apart in finite dimensions).

Let us enlist here our assumptions:

$$H \ \text{ is a Hilbert space}, \tag{39}$$

$$\phi : H \to [0,\infty] \ \text{ is convex, proper and l.s.c.}, \tag{40}$$

$$\exists c > 0 \ : \ \phi(u) \geq c\|u\|_V - 1/c \ \ \forall u \in H, \ \ V \text{ Banach with } \ V \subset\subset H, \tag{41}$$

$$\psi : H \to [0,\infty] \ \text{ is convex and l.s.c. with } \ \psi(0) = 0, \tag{42}$$

$$\exists c > 0 \ : \ \psi(\dot u) \geq c|\dot u|^2 - 1/c \ \ \forall \dot u \in H, \tag{43}$$

$$\exists c > 0 \ : \ |v| \leq (1/c)(1 + |u|) \ \ \forall v \in \partial\psi(u), \tag{44}$$

$$f \in L^2(0,T;H), \ u^0 \in D(\phi). \tag{45}$$

Assumption (41) entails compactness of the sublevels of ϕ. On the other hand (43)-(44) correspond to the *linear growth* of $\partial\psi$ far from $0 \in \partial\psi(0)$.

We shall use the following useful compactness result.

Theorem 11.1 (Aubin-Lions Lemma)**.** *Let $A \subset\subset B \subset C$ where A, B, C are Banach spaces. Let $\{u_n\}$ be bounded in $W^{1,p}(0,T;C) \cap L^r(0,T;A)$ for $1 \leq r < \infty$ and $p = 1$ or $r = \infty$ and $p > 1$. Then, $\{u_n\}$ is relatively precompact in $L^r(0,T;B)$ ($C([0,T];B)$ if $r = \infty$).*

We follow [7] for an existence result.

Theorem 11.2 (Existence)**.** *Assume* (39)-(45)*. Then, there exists a solution to* (38)*. More precisely, there exist $u \in H^1(0,T;H)$, and $v, w \in L^2(0,T;H)$ such that*

$$v + w = f \quad \text{a.e. in } (0,T), \tag{46}$$

$$v \in \partial\psi(\dot u) \quad \text{a.e. in } (0,T), \tag{47}$$

$$w \in \partial\phi(u) \quad \text{a.e. in } (0,T), \tag{48}$$

$$u(0) = u^0. \tag{49}$$

Proof. We proceed by approximation. Let $\varepsilon > 0$ and ϕ_ε be the Yosida approximation of ϕ at level ε. Then, the problem

$$\varepsilon \dot{u}_\varepsilon + \partial\psi(\dot{u}_\varepsilon) + \partial\phi_\varepsilon(u_\varepsilon) \ni f, \quad u_\varepsilon(0) = u^0$$

admits a unique solution $u_\varepsilon \in H^1(0,T;H)$. Indeed, the equation can be equivalently rewritten as

$$\dot{u}_\varepsilon = (\varepsilon \mathrm{id} + \partial\psi)^{-1}(f - \partial\phi_\varepsilon(u_\varepsilon))$$

where the right-hand side is Lipschitz continuous with respect to u_ε.

Call now $w_\varepsilon = \partial\phi_\varepsilon$ and define $v_\varepsilon = f - w_\varepsilon - \varepsilon\dot{u}_\varepsilon$ so that $v_\varepsilon \in \partial\phi(\dot{u}_\varepsilon)$ a.e. Testing the equation by $\dot{u}_\varepsilon$ and integrating one gets

$$\varepsilon \int_0^t |\dot{u}_\varepsilon|^2 + \int_0^t (v_\varepsilon, \dot{u}_\varepsilon) + \phi_\varepsilon(u_\varepsilon(t)) = \phi_\varepsilon(u^0) + \int_0^t (f, \dot{u}_\varepsilon).$$

By using (44) and the fact that $\phi_\varepsilon(u^0) \leq \phi(u^0) < \infty$ from (45) one concludes that

$$\|u_\varepsilon\|_{H^1(0,T;H)} + \|\phi_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T)} < C \tag{50}$$

independently of ε. As J_ε is 1-Lipschitz and $\phi_\varepsilon(u_\varepsilon) = \phi(J_\varepsilon u_\varepsilon) + |u_\varepsilon - J_\varepsilon u_\varepsilon|^2/(2\varepsilon) \geq \phi(J_\varepsilon u_\varepsilon)$, by using the compactness (41) we get that

$$\|J_\varepsilon u_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} < C \tag{51}$$

independently of ε. The linear boundedness (44) entails that

$$\|v_\varepsilon\|_{L^2(0,T;H)} < C \tag{52}$$

and a comparison in the equation reveals that

$$\|w_\varepsilon\|_{L^2(0,T;H)} < C \tag{53}$$

independently of ε.

The estimates (50)-(53) and Theorem 2.5 and 11.1 yield

$$u_\varepsilon \rightharpoonup u \ \text{ in } H^1(0,T;H), \tag{54}$$

$$v_\varepsilon \rightharpoonup v \ \text{ in } L^2(0,T;H), \tag{55}$$

$$w_\varepsilon \rightharpoonup w \ \text{ in } L^2(0,T;H), \tag{56}$$

$$J_\varepsilon u_\varepsilon \to j \ \text{ in } C([0,T];H), \tag{57}$$

By observing that

$$u_\varepsilon - J_\varepsilon u_\varepsilon \in \varepsilon\partial\phi(J_\varepsilon u_\varepsilon) = \varepsilon\partial\phi_\varepsilon(u_\varepsilon) = \varepsilon w_\varepsilon \to 0 \ \text{ in } L^2(0,T;H)$$

we conclude that $u \equiv j$.

The convergences (55)-(56) yield (46). From (54) we deduce that

$$u_\varepsilon(t) \rightharpoonup u(t) \ \ \forall t \in [0,T]. \tag{58}$$

In particular, the initial condition (49) follows.

The convergences (56)-(57) and Lemma 5.4 entail (48). On the other hand, we can compute

$$\limsup_{\varepsilon\to 0}\int_0^T (v_\varepsilon,\dot u_\varepsilon) \overset{(46)}{=} \limsup_{\varepsilon\to 0}\int_0^T (f-w_\varepsilon,\dot u_\varepsilon)$$
$$= \limsup_{\varepsilon\to 0}\left(\int_0^T (f,\dot u_\varepsilon) - \phi_\varepsilon(u_\varepsilon(T)) + \phi_\varepsilon(u^0)\right)$$
$$\leq \int_0^T (f,\dot u) - \liminf_{\varepsilon\to 0}\phi_\varepsilon(u_\varepsilon(T)) + \phi(u^0)$$
$$\leq \int_0^T (f,\dot u) - \phi(u(T)) + \phi(u^0) \overset{(48)}{=} \int_0^T (f-w,\dot u) \overset{(46)}{=} \int_0^T (v,\dot u)$$

where we used $\phi_\varepsilon \to \phi$, both pointwise and in the Mosco sense. Hence, the inclusion (47) follows from Lemma 5.4. □

By inspecting the proof of Theorem 11.2 one realizes that the operator $\partial\psi$ can be replaced by a more general maximal monotone operator A, given that the growth assumptions (43)-(44) still hold. The result can be further extended to Banach spaces and polynomially growing operators $\partial\psi$ [6]. In particular, if (43)-(44) are replaced by

$$\exists c > 0 \ : \ \psi(\dot u) \geq c|\dot u|^p - 1/c \quad \forall \dot u \in H$$
$$\exists c > 0 \ : \ |v|^{p'} \leq (1/c)(1 + |u|^p) \quad \forall v \in \partial\psi(u)$$

for some $p > 1$, equation (38) has solutions $u \in W^{1,p}(0,T;H)$. However, the existence argument cannot be directly extended to the limiting case $p = 1$, as the weak relative compactness of bounded subsets of $W^{1,p}(0,T;H)$ is used. The case $p = 1$ is the focus of the next section.

Uniqueness for (38) is not to be expected. Take $H = \mathbb{R}$, $\phi = I_{[-1,1]}$, $\phi(x) = (|x|-1)^+$,$f = 0$, $u^0 = 0$. Then, $u(t) = \sin(\alpha t)$ solves(38) for all $\alpha \in \mathbb{R}$. Uniqueness can be ensured under additional assumptions. Let $\partial\psi$ be linear and $\partial\phi$ be strictly convex, let u_1 and u_2 be two solutions. Then, one computes that

$$0 \leq \int_0^T (\partial\phi(u_1)-\partial\phi(u_2), u_1-u_2) = -\int_0^T (\partial\psi(\dot u_1)-\partial\psi(\dot u_2), u_1-u_2)$$
$$= -\int_0^T (\partial\psi(\dot u_1-\dot u_2), u_1-u_2) = -\frac{1}{2}\psi(u_1(T)-u_2(T)) \leq 0.$$

In particular, we have

$$\int_0^T (\partial\phi(u_1)-\partial\phi(u_2), u_1-u_2) = 0$$

and $u_1 \equiv u_2$ follows by strict convexity.

12. Rate-independent flows

We shall now turn our attention to the *rate-independent* case of (38). This corresponds to a positively 1-homogeneous dissipation ψ, namely $\psi(\lambda\dot{u}) = \lambda\psi(\dot{u})$ for all $\lambda > 0$ and $\dot{u} \in H$. *Rate-independence* refers to the fact that, by letting u be a solution to (38) and $\alpha : [0, \widehat{T}] \to [0,T]$ and increasing diffeomorphism, the trajectory $t \mapsto (u \circ \alpha)(t)$ solves (38) with the datum f replaced by $f \circ \alpha$.

Being positively 1-homogeneous, the function ψ coincides with the *support function* of the convex set $K = \partial\psi(0)$, namely $\psi(\dot{u}) = \sup_{v\in K}(v, \dot{u})$.

We shall assume that ψ fulfills the *triangle inequality*

$$\psi(a{+}b) \leq \psi(a) + \psi(b) \quad \forall a,\, b \in H \tag{59}$$

which, together with homogeneity, entails convexity. Moreover, we assume that ψ controls the H-norm and is continuous (see (62)-(63) below).

We shall turn to an equivalent weak formulation of (38), in the spirit of (31). Indeed, by *assuming* to have a solution $u \in W^{1,1}(0,T;H)$ one can write

$$\begin{aligned}
&\partial\psi(\dot{u}) + \partial\phi(u) \ni f \quad \text{a.e.}\\
&\iff \quad (f{-}\partial\phi(u), \dot{u}) + \psi(\dot{u}) + \psi^*(f{-}\partial\phi(u)) = 0 \quad \text{a.e.}\\
&\iff \quad \int_0^t (f{-}\partial\phi(u), \dot{u}) + \int_0^t \psi(\dot{u}) + \int_0^t \psi^*(f{-}\partial\phi(u)) = 0 \quad \forall t \in [0,T].
\end{aligned}$$

By recalling that $\psi^* = I_K$ with $K = \partial\psi(0)$ one can rewrite the latter as

$$\begin{cases} f{-}\partial\phi(u) \in K \quad \text{a.e.}\\ E(t,u(t)) + \displaystyle\int_0^t \psi(\dot{u}) = E(0,u(0)) - \int_0^t (\dot{f}, u) \quad \forall t \in [0,T] \end{cases}$$

where we use the notation $E(t,u) = \phi(u) - (f(t), u)$ for the complementary energy. This inspires a derivative-free weak formulation of (38).

Definition 12.1 (Energetic solution)**.** *We say that $u : [0,T] \to H$ (everywhere defined) is an energetic solution of* (38) *if, for all $t \in [0,T]$ one has*

$$\text{Global stability:} \quad E(t,u(t)) \leq E(t,\widehat{u}) + \psi(u(t){-}\widehat{u}) \quad \forall \widehat{u} \in H, \tag{60}$$

$$\text{Energy conservation:} \quad E(t,u(t)) + \mathrm{Diss}(u,[0,t]) = E(0,u(0)) - \int_0^t (\dot{f}, u) \tag{61}$$

where $\mathrm{Diss}(u,[0,t]) = \sup \sum_i \psi(u(t_i){-}u(t_{i-1}))$*, the supremum being taken over all partitions* $\{0 = t_0 < \cdots < t_N = t\}$ *of the interval* $[0,t]$.

Note that (60) is just a reformulation of the inclusion $f - \partial\phi(u) \in K$ and translates the fact that the solution state $u(t)$ is *stable* at t: a transition to any other state $\widehat{u}$ would not be favorable in terms of energy change and dissipation cost. For the sake of later convenience, let us indicate the set of states fulfilling (60), anmely the set of *globally stable states* at time t, as

$$S(t) = \{u \in H \mid E(t, u(t)) \leq E(t, \widehat{u}) + \psi(u(t) - \widehat{u}) \ \ \forall \widehat{u} \in H\}.$$

The advantage of dealing with energetic solutions consists in reducing to a system of a *static* variational inequality (60) coupled with a *scalar* equation (61) instead of the *evolutionary* variational inequality (38). A remarkable advantage of the energetic formulation is that of being gradient-free, both for the functionals and the solution. This is particularly convenient for nonsmooth situations.

Our aim here is to prove a first existence result for energetic solutions [11]. In addition to (39)-(42) we shall assume

$$\psi \ \text{ is positively 1-homogeneous, continuous, and fulfills (59),} \tag{62}$$

$$\exists c > 0 \ : \ \psi(\dot{u}) \geq c|\dot{u}| \ \ \forall \dot{u} \in H, \tag{63}$$

$$f \in C^1([0,T]; H), \ \ u^0 \in S(0). \tag{64}$$

Theorem 12.2 (Existence of energetic solutions)**.** *Assume* (39)-(42)*,* (62)-(64)*. Then, there exists an energetic solution u with $u(0) = u^0$.*

In proving the existence theorem, we use a compactness tool based on the *Helly selection* principle [11, Thm. 5.1].

Lemma 12.3 (Extended Helly principle)**.** *Assume* (59)*,* (62)-(63)*. Let the sequence $u_n : [0,T] \to K \subset\subset H$ with* $\mathrm{Diss}(u_n, [0,T]) < C$ *independently of n. Then, there exists a not relabeled subsequence and a function $u : [0,T] \to$ such that $u_n(t) \to u(t)$ for all $t \in [0,T]$. Moreover, there exists a function $D : [0,T] \to \mathbb{R}$ such that* $\mathrm{Diss}(u_n, [0,t]) \to D(t) \geq \mathrm{Diss}(u, [0,t])$ *for all $t \in [0,T]$.*

Proof of Theorem 12.2. We proceed by time discretization. Let $N \in \mathbb{N}$ and $\tau = T/N$ be the time step and define the time partition $t^i = i\tau$, $i = 1, \ldots, N$. Starting from u^0, define inductively u^i by

$$u^i \in \mathrm{Arg\,min}_u \left(\psi(u - u^{i-1}) + E(t^i, u)\right) \quad \text{for } i = 1, \ldots, N.$$

Note that these minimization problems have solutions as $u \mapsto \psi(u - u^{i-1}) + \phi(u) - (f(t^i), u)$ is l.s.c. and coercive.

By minimality we have that

$$\begin{aligned}
&E(t^i, u^i) + \psi(u^i - u^{i-1}) \leq E(t^i, u^{i-1}) \\
&= E(t^{i-1}, u^{i-1}) - (f(t^i) - f(t^{i-1}), u^{i-1}) \\
&= E(t^{i-1}, u^{i-1}) - \int_{t^{i-1}}^{t^i} (\dot{f}, u^{i-1}).
\end{aligned}$$

Let now u_τ be defined as $u_\tau(t) = u(t_i)$ for $t \in [t^i, t^{i+1})$, and analogously for t_τ. By taking the sum for $i = 1, \ldots, m$ we have

$$E(t^m, u^m) + \mathrm{Diss}(u_\tau, [0, t^m]) = E(t^m, u^m) + \sum_{i=1}^m \psi(u^i - u^{i-1})$$
$$\leq E(0, u^0) - \int_0^{t^m} (\dot{f}, u_\tau(\cdot - \tau)). \tag{65}$$

We now use the assumptions on data (64) and apply a suitable discrete versin of Gronwall's lemma in order to check that

$$\|\phi(u_\tau)\|_{L^\infty(0,T)} + \mathrm{Diss}(u_\tau, [0, T]) < C$$

independently of τ. As (41) ensures that ϕ has compact sublevels we can apply the Helly Lemma 12.3 and deduce the existence of a function $u : [0, T] \to$ such that $u_\tau(t) \to u(t)$ for all $t \in [0, T]$. Moreover, we also have that that $\liminf_{\tau\to0} \mathrm{Diss}(u_\tau, [0, t]) \geq \mathrm{Diss}(u, [0, t])$ for all $t \in [0, T]$. The continuity of $\dot{f}$ (64) then suffices in order to pass to the lim inf as $\tau \to 0$ in (65) and obtain the *upper energy estimate*

$$E(t, u(t)) + \mathrm{Diss}(u, [0, t]) \leq E(0, u^0) - \int_0^t (\dot{f}, u). \tag{66}$$

Before proving the converse inequality (which will then correspond to (61)) let us check the global stability (60). Indeed, again by minimality and the triangle inequality (59) we have that, for all $\widehat{u} \in H$,

$$E(t^i, u^i) + \psi(u^i - u^{i-1}) \leq E(t^i, \widehat{u}) + \psi(\widehat{u} - u^{i-1})$$
$$\stackrel{(59)}{\leq} E(t^i, \widehat{u}) + \psi(\widehat{u} - u^i) + \psi(u^i - u^{i-1})$$

so that

$$E(t_\tau(t), u_\tau(t)) \leq E(t_\tau(t), \widehat{u}) + \psi(\widehat{u} - u_\tau(t)) \quad \forall \widehat{u} \in H.$$

Owing to the above-mentioned convergences, as $t_\tau(t) \to t$ and $u_\tau(t) \to u(t)$ and ψ is continuous (62) we can pass to the lim inf in the latter and obtain global stability (60).

Finally, let $\{0 = s^0 < s^1 < \cdots < s^M = t\}$ be any partition of $[0, t]$. From the already proved $u(s^{j-1}) \in S(s^{j-1})$ we have

$$E(s^{j-1}, u(s^{j-1})) \leq E(s^{j-1}, u(s^j)) + \psi(u(s^j) - u(s^{j-1}))$$
$$= E(s^j, u(s^j)) + \psi(u(s^j) - u(s^{j-1})) + \int_{s^{j-1}}^{s^j} (\dot{f}, u).$$

By taking the sum on j we get

$$E(0, u^0) \leq E(t, u(t)) + \sum_{j=0}^M \psi(u(s^j) - u(s^{j-1})) + \int_0^t (\dot{f}, u)$$

so that the *lower energy estimate*, namely the converse inequality w.r.t. (66), follows by taking the supremum over all partitions. In particular, the energy conservation (61) holds.

Note that the above argument provides the improved convergences

$$E(t, u_\tau(t)) \to E(t, u(t)) \quad \forall t \in [0, T]$$
$$\mathrm{Diss}(u_\tau, [0, t]) \to \mathrm{Diss}(u, [0, t]) \quad \forall t \in [0, T]$$

as well. □

13. Semilinear waves

Let $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary $\partial\Omega$ and $p > 1$. We consider the *semilinear wave* problem [10]

$$u_{tt} - \Delta u + |u|^{p-2}u = g \quad \text{in } \Omega \times (0, T), \tag{67}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{68}$$
$$u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \Omega \tag{69}$$

where the source $g : \Omega \times (0, T) \to \mathbb{R}$ and the initial data u_0, $u_1 : \Omega \to \mathbb{R}$ are given.

We reformulate the problem variationally (weak solutions) by looking for a trajectory $u = u(t)$ with values in the Banach space

$$V = H^1_0(\Omega) \cap L^p(\Omega).$$

Note that $V^* = H^{-1}(\Omega) + L^{p'}(\Omega)$. If $p \geq 2^* = 2n/(n-2)^+$ one has that $V = H^1_0(\Omega)$. The laplacian $-\Delta$ in (67) is then replaced by $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ and the equation is specified as

$$u'' + Au + |u|^{p-2}u = g \quad \text{in } V^*, \text{ a.e. in } (0, T). \tag{70}$$

We have the following.

Theorem 13.1 (Existence)**.** *Let $g \in L^2(0, T; L^2(\Omega))$, $u_0 \in V$, and $u_1 \in L^2(\Omega)$. Then, there exists $u \in H^2(0, T; V^*) \cap W^{1,\infty}(0, T; V) \cap L^\infty(0, T; L^2(\Omega))$ solving* (70) *along with $u(0) = u_0$ and $u'(0) = u_1$.*

Note that the initial conditions in the above statement make sense. Indeed, u is continuous from $[0, T]$ to V, so that $u(0) = u_0$ is well defined. On the other hand,

$$u'' = g - Au - |u|^{p-2}u \in L^2(0, T; L^2(\Omega)) + L^\infty(0, T; V^*) \subset L^2(0, T; V^*)$$

and u' is continuous from $[0, T]$ to V^*. In particular, all terms in equation (70) are $L^2(0, T; V^*)$.

Proof. We argue by Faedo-Galerkin approximations. Indicate with $(\cdot, \cdot)$ both scalar products in $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^d)$. Start by assuming g to be smooth in time. Let $v_i \in V$, $i \in \mathbb{N}$ be linearly independent and such that their linear combinations are dense in V. Such a set of functions exists as V is separable. For all $m \in \mathbb{N}$ we find a solution u_m of the form

$$u_m(t) = \sum_{i=1}^{m} u_{im}(t) v_i$$

of equation. This is determined by finding the time-dependent coefficients u_{im}, which come from

$$(u_m''(t), v_i) + (\nabla u_m(t), \nabla v_i) + (|u_m(t)|^{p-2} u_m(t), v_i) = (g(t), v_i) \qquad (71)$$

for all $i = 1, \ldots, m$, together with the initial conditions

$$u_m(0) = u_{0m} := \sum_{i=1}^{m} u_{0im} v_i \stackrel{V}{\to} u_0, \quad u_m'(0) = u_{1m} := \sum_{i=1}^{m} u_{1im} v_i \stackrel{L^2(\Omega)}{\to} u_1.$$

This is a system of ODEs defined by a locally Lipschitz vector field. As v_i are linearly independent, it has a a solution in a small interval $[0, t_0]$. The a priori estimates will show that indeed $t_0 = T$. In order to get such estimates multiply relation (71) by u_{im}' and add up on i getting, for all times,

$$(u_m'', u_m') + (\nabla u_m, \nabla u_m) + (|u_m|^{p-2} u_m, u_m') = (g, u_m').$$

This entails that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|u_m'\|_{L^2}^2 + \frac{1}{2} \|\nabla u_m\|_{L^2}^2 + \frac{1}{p} \|u_m\|_{L^p}^p \right) = \int_\Omega g u_m'$$

so that it follows from Gronwall that

$$\frac{1}{2} \|u_m'\|_{L^2}^2 + \frac{1}{2} \|\nabla u_m\|_{L^2}^2 + \frac{1}{p} \|u_m\|_{L^p}^p \leq C$$

for all $t \in [0, t_0]$, independently of m. Hence the solution can be extended up to T and, by extracting some not relabeled subsequence

$$u_m \to u \quad \text{weakly star in} \;\; W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; V).$$

We now check that u indeed solves the problem. First of all note that $u_m \to u$ almost everywhere by the Aubin-Lions Lemma 11.1. This implies that

$$|u_m|^{p-2} u_m \to |u|^{p-2} u \quad \text{weakly star in } L^\infty(0, T; L^p(\Omega)).$$

Indeed, $|u_m|^{p-2} u_m$ clearly has a weak-star limit in $L^\infty(0, T; L^p(\Omega))$ but it pointwise almost everywhere converging to $|u|^{p-2} u$, as the latter is bounded in $L^\infty(0, T; L^{p/(p-1)}(\Omega))$.

We can now pass to the limit in (71) the following sense

$$\begin{aligned} (\nabla u_m, \nabla v_i) \to (\nabla u, \nabla v_i) &\quad \text{weakly star in } L^\infty(0, T), \\ (|u_m|^{p-2} u_m, v_i) \to (|u|^{p-2} u, v_i) &\quad \text{weakly star in } L^\infty(0, T), \\ (u_m'', v_i) = \frac{\mathrm{d}}{\mathrm{d}t}(u_m', v_i) \to \frac{\mathrm{d}}{\mathrm{d}t}(u', v_i) = (u'', v_i) &\quad \text{in } \mathcal{D}'(0, T), \end{aligned}$$

so that

$$(u'', v_i) + (\nabla u, \nabla v_i) + (|u|^{p-2}u, v_i) = (g, v_i) \quad \text{in } \mathcal{D}'(0,T), \quad \forall i.$$

By using the density of linear combinations of the v_i we get that

$$(u'', v_i) + (\nabla u, \nabla v_i) + (|u|^{p-2}u, v_i) = (g, v_i) \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in V,$$

entailing indeed that $u'' = -Au - |u|^{p-2}u + g$ in $\mathcal{D}'(0,T;V^*)$. Hence $u'' \in H^2(0,T;V^*)$ and (70) holds. The initial conditions follow from $u_m(0) \to u_0$ and $u_m(0) \to u(0)$ on one side, and from

$$\int_\Omega u_1 v_i = \lim_{m\to\infty} \int_\Omega u_m'(0) v_i = \int_\Omega u'(0) v_i \quad \forall i$$

from the other side.

The case of nonsmooth g can then be obtained by approximation upon noticing that the estimate depends solely on the $L^2(0,T;L^2(\Omega))$ of g. □

Note that the existence proof works (even without changing notation) if we replace $A = -\Delta$ by a linear second order elliptic operator in divergence form $Au = -\partial_j(a_{ij}(x)\partial_i u)$ with bouneded measurable coefficients.

Theorem 13.2 (Uniqueness)**.** *Let $p \le 1 + d/(d-2)$ (any p if $d = 2$). Then, the solution from Theorem* 13.1 *is unique.*

Proof. Let u, v be two solutions and $w = u - v$. Take the difference of equations and test on w'. One gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w'(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) = \int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w'.$$

We estimate the right-hand side as

$$\begin{aligned}
&\left|\int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w\right| \le \int_\Omega \left||u|^{p-2}u - |v|^{p-2}v\right| |w'| \\
&= \int_\Omega \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\left(|\theta u + (1-\theta)v|^{p-2}(\theta u + (1-\theta)v)\right) \mathrm{d}\theta\right| |w| \\
&= \int_\Omega \left|\int_0^1 \left((p-2)|\theta u + (1-\theta)v|^{p-3} w(\theta u + (1-\theta)v) + |\theta u + (1-\theta)v|^{p-2} w\right) \mathrm{d}\theta\right| |w'| \\
&\le (p-1)\int_\Omega \sup\{|u|^{p-2}, |v|^{p-2}\}|w|\,|w'| \le (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right)|w|\,|w'| \\
&\overset{(a)}{\le} (p-1)\left(\||u|^{p-2}\|_{L^d} + \||v|^{p-2}\|_{L^d}\right)\|w\|_{L^{2^*}}\|w'\|_{L^2} \\
&\overset{(b)}{\le} (p-1)\left(\|u\|_{H^1}^{p-2} + \|v\|_{H^1}^{p-2}\right)\|w\|_{L^{2^*}}\|w'\|_{L^2} \\
&\overset{(c)}{\le} c\|w\|_{H^1}\|w'\|_{L^2}
\end{aligned}$$

where inequality (a) comes from

$$\frac{1}{d} + \frac{1}{2^*} + \frac{1}{2} = 1,$$

inequality (b) follows as

$$p \leq 1 + d/(d-2) \Rightarrow (p-2)d \leq 2^*,$$

and (c) from the fact that u, v are bounded in H^1. We conclude by Gronwall that $w = 0$. $\square$

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