Exercises to Applied Analysis, WS 2018

Ulisse Stefanelli

January 7, 2019

1. Let (X,d) be a metric space and $f,\ g : X \to \mathbb{R}\cup\{\infty\}$ be proper. Prove or disprove the following:
   (a) f has compact support $\Rightarrow$ f is l.s.c. (lower semicontinuous).
   (b) f and g have compact sublevels $\Rightarrow$ $f+g$ is l.s.c.
   (c) f and g have compact sublevels $\Rightarrow$ $f-g$ is l.s.c.
   (d) $f(x) \geq d(x,x_0)$ for some $x_0 \in X$ and all $x \in X$ $\Rightarrow$ the sublevels of f are bounded.
   (e) g^2 has compact sublevels $\Leftrightarrow$ g has compact sublevels.
   (f) $f+g$ has compact sublevels and $-g$ is l.s.c. $\Rightarrow$ f is l.s.c.
2. Let (X,d) be a metric space and $f : X \to \mathbb{R}\cup\{\infty\}$ be proper. Show that f is l.s.c. iff $\{x \in X \mid f(x) \leq \ell\}$ is closed for all $\ell \in \mathbb{R}$.
3. Show that $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$.
4. Check that $\{u \in L^2(0,1) \,:\, |u| = 1 \text{ a.e.}\}$ contains a weakly convergent sequence in $L^2(0,1)$.
5. Use Lax-Milgram-Lions in order to check that the elliptic problem $u + \Delta^2 u = 1$ on $B = \{x \in \mathbb{R}^d \,:\, |x| < 1\}$, $u = \Delta u = 0$ on ∂B has a unique solution in $H^2(B)$.
6. Find conditions on $b \in L^\infty(\Omega;\mathbb{R}^d)$ and $c \in L^\infty(\Omega)$ with $\Omega \subset \mathbb{R}^d$ nonempty, open, smooth, and bounded in order the bilinear form
$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega b(x)\cdot\nabla u(x)v(x)\,\mathrm{d}x + \int_\Omega c(x)u(x)v(x)\,\mathrm{d}x$$
to be coercive on $H^1(\Omega)$.
7. Prove that $-\Delta u + u^+ = 0$ admits a unique solution on $u \in H^1_0(B)$ with $B = \{x \in \mathbb{R}^d \,:\, |x| < 1\}$
8. Prove that $-\Delta u + \mathrm{e}^u = 0$ admits a unique solution $u \in H^1_0(B)$ with $B = \{x \in \mathbb{R}^d \,:\, |x| < 1\}$. Hint: truncate, solve, remove the truncation.
9. Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, smooth, and bounded. Prove that the integropartial differential equation $\partial_t u(x,t) - \Delta u(x,t) = 1 + \int_0^t \sin(u(x,s))\mathrm{d}s$ for $(x,t) \in \Omega\times(0,T)$ with $u(\cdot,0) = 0$ and $u = 0$ on $\partial\Omega\times(0,T)$ has a unique solution. Hint: try a fixed point.
10. Find the eigenvalues of the operator $-\Delta$ on $[0,\pi]^2$ with homogeneous Dirichlet boundary conditions. Hint: try to solve for $u(x,y) = \sin(nx)\sin(my)$.