Exercises to Stochastic PDEs, WS 2018

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November 14, 2018

1. Let $x \in \mathbb{R} \mapsto y(x,\omega)$ solve $y' = \omega y$ on $\mathbb{R}$ and $y(0) = 1$ with $\omega \in \{0,1,2\}$ equiprobable. Compute
$$P(\{y(1) \geq 1\}),\ P(\{y(1) \leq 1\}),\ P(\{y(1) \leq 5\}),\ E(y(x)),\ E((y(x) - E(y(x)))^2.$$

2. Let $x \in \mathbb{R} \mapsto y(x,\omega)$ solve $y' = \omega y$ on $\mathbb{R}$ and $y(0) = 1$ with ω being a real random variable with probability distribution ρ given by $\rho(t) = 1$ for $t \in [0,1]$ and $\rho = 0$ otherwise. Compute
$$P(\{y(1) \geq 1\}),\ P(\{y(1) \leq 1\}),\ P(\{y(1) \geq 2\}),\ E(y(x)),\ E((y(x) - E(y(x)))^2.$$

3. Let X be a real random variable and $f : \mathbb{R} \to \mathbb{R}$ be smooth. Check that the variance fulfills $E((X - E(X))^2) = E(X^2) - E(X)^2$ and use a Taylor expansion of f at $E(X)$ in order to check that $E(f(X)) \sim f(E(X)) + f''(E(X))E((X - E(X))^2)/2$ for $E((X - E(X))^2)$ small (assume all integrals to be finite).

4. Let X be a real random variable on the space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = \{1,2,3,4\}$, $\mathcal{F}$ power set of Ω (set of all subsets of Ω), and $\mu(A) = \#A/4$, where $\#A$ is the cardinality of A. Let the σ-algebra $\mathcal{G} = \{\emptyset, \{1,2\}, \{3,4\}, \Omega\}$ be given. Find the conditional expectation $E(X|\mathcal{G})$.

5. Let the random variables X_n be independent and take the values ± 1 with equal probability (coin toss). Are the processes
$$A_n = \sum_{i=2}^{n} (X_i - X_{i-1}), \quad B_n = \sum_{i=1}^{n} \sum_{j=1}^{i} X_j, \quad C_n = \max_{j=1,\dots,n} \{X_j\}$$
markovian? Are they martingales?

6. Let $X_0 = 1$ and $X_n \in \mathbb{N}$ be the total result of the roll of X_{n-1} regular dice (6 faces, numbered 1 to 6, equiprobable). Is X_n markovian? Is it a martingale? Compute $E(X_n)$.

7. Write the variational formulation of the boundary value problem in one dimension: $-u'' + u = 1$ in $(0,1)$, $u(0) = 0$, and $u'(1) = 3$. Prove that it admits a weak solution.

8. Find conditions on $\eta > 0$, $\gamma \geq 0$, and $\beta \in \mathbb{R}$ such that the bilinear form
$$a(u,v) = \int_0^1 \eta\, u'v' + \beta\, u'v + \gamma\, uv \,\mathrm{d}x$$
is coercive on $H^1(0,1)$ (namely, such that there exists $\alpha > 0$ with $a(u,u) \geq \alpha \int_0^1 (u')^2 + u^2 \,\mathrm{d}x$ for all $u \in H^1(0,1)$).

The list will be progressively expanded along the course.