INTERPOLATION OF OPERATORS ON $L^p$-SPACES

NATHALIE TASSOTTI AND ARNO MAYRHOFER

Abstract. We prove the Riesz-Thorin theorem for interpolation of operators on $L^p$-spaces and discuss some applications. We follow in large the presentation in [D. Werner, Funktionalanalysis (Springer, 2005), p. 72-79, II.4].

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1. The theorem of Riesz-Thorin

Motivation 1.1. Let $0 < p_0 < p_1$, $\Omega \subseteq \mathbb{R}^n$ open, $m(\Omega) < \infty$ and $T \in L(L^{p_0}(\Omega), L^{q_0}(\Omega))$. Then $T f$ is also defined for $f \in L^{p_1}(\Omega)$ since $L^{p_0}(\Omega) \subseteq L^{p_1}(\Omega)$. We want to suppose that $f \in L^{p_1}(\Omega)$ implies $T f \in L^{q_1}(\Omega)$ so that we have $T \in L(L^{p_1}(\Omega), L^{q_1}(\Omega))$.

We now want to study the operator $T$ on the $L^p(\Omega)$ spaces between $L^{p_0}(\Omega)$ and $L^{p_1}(\Omega)$. Additionally we want to get norm estimates from the norms $\|T\|_{p_0 \rightarrow q_0}$ and $\|T\|_{p_1 \rightarrow q_1}$.

Strictly speaking, if we talk about an operator $T : L^{p_0}(\Omega) \to L^{q_0}(\Omega)$ and $T : L^{p_1}(\Omega) \to L^{q_1}(\Omega)$ we mean that we have an operator $T_0 \in L(L^{p_0}(\Omega), L^{q_0}(\Omega))$ and an operator $T_1 \in L(L^{p_1}(\Omega), L^{q_1}(\Omega))$ for which

$$T_0 \mid_{L^{p_0}(\Omega) \cap L^{p_1}(\Omega)} = T_1 \mid_{L^{p_0}(\Omega) \cap L^{p_1}(\Omega)}$$

holds.

We will carry out our investigations for arbitrary open subsets $\Omega$ of $\mathbb{R}^n$ but it would be possible to deal with more general measure spaces. From now on we abbreviate $L^p(\Omega)$ with $L^p$.

Lemma 1.2 (Lyapunov inequality). Let $1 \leq p_0, p_1 \leq \infty$ and $0 \leq \theta \leq 1$. Define $p$ by $\frac{1}{k} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then $L^{p_0} \cap L^{p_1} \subset L^p$ and we have

$$\|f\|_p \leq \|f\|_p^{1-\theta} \|f\|_p^\theta$$

for all $f \in L^{p_0} \cap L^{p_1}$.

Proof. We will use Hölder’s inequality to prove inequality (1). Indeed we have $\|f g\|_q \leq \|f\|_p \|g\|_q$ whenever $1 = \frac{1}{p} + \frac{1}{q}$.

Let $x := (1-\theta)p$, $y := \theta p$, $\frac{1}{z_0} := \frac{1-\theta}{p_0}$, $\frac{1}{z_1} := \frac{\theta}{p_1}$. With these definitions we have
\[ x + y = p, \frac{1}{x_0} + \frac{1}{z_1} = 1, xz_0 = p_0 \text{ and } yz_1 = p_1. \]

Now using Hölder’s inequality we obtain
\[
\|f\|_p^p = \|f^x f^y\|_1 \leq \|f^x\|_{z_0} \|f^y\|_{z_1} = \left( \int |f|^{x z_0} \right)^{\frac{1}{z_0}} \left( \int |f|^{y z_1} \right)^{\frac{1}{z_1}}
\]
\[
= \left( \int |f|^{p_0} \right)^{\frac{1}{p_0}} \left( \int |f|^{p_1} \right)^{\frac{1}{p_1}} = \left( \int |f|^{1-\theta} \right)^{\frac{1}{\theta}} \left( \int |f|^{\theta} \right)^{\frac{1}{\theta}}.
\]

□

In the proof of the Riesz-Thorin theorem we will need the following result from complex analysis.

**Proposition 1.3** (Three line Lemma). Let \( F : S \to \mathbb{C} \) be bounded and continuous, where \( S := \{ z \in \mathbb{C} : 0 \leq \Re z \leq 1 \} \). Additionally let \( F \) be analytic on \( S^\circ \). For \( 0 \leq \theta \leq 1 \) let \( M_\theta := \sup_{y \in \mathbb{R}} |F(\theta + iy)| \). Then we have
\[
M_\theta \leq M_0^{1-\theta} M_1^\theta.
\]

To visualize the meaning of the proposition we take a look at the figure below.

**Proof.**

**Step 1:** First of all we investigate the case \( M_0, M_1 \leq 1 \). So we have to prove that \( M_\theta \leq 1 \).

Let \( z_0 = x_0 + iy_0 \in S^\circ, \epsilon > 0 \) and define \( F_\epsilon(x) := \frac{F(x)}{1+\epsilon^2} \).

This function is also bounded, continuous and analytic on \( S^\circ \). Moreover \( \lim_{|y| \to \infty} |F_\epsilon(x + iy)| = 0 \) uniformly for \( x \in [0, 1] \) since \( |F_\epsilon(x + iy)| \leq \frac{|F(x+iy)|}{\epsilon|y|} \) and \( F \) is bounded.

Let \( r > |y_0| \) such that \( |F_\epsilon(x + iy)| \leq 1 \) for \( 0 \leq x \leq 1 \) and \( |y| = r \). Furthermore let \( R \) be the compact rectangle \( [0, 1] \times \{ i[-r,r] \} \) This implies that \( |F_\epsilon(z)| \leq 1 \) on \( \partial R \). The maximum principle for analytic functions [K. Jähnich, Funktionentheorie (Springer, 2004), p. 30, Satz 13] now tells us that \( |F_\epsilon(z)| \leq 1 \forall z \in R \), in particular \( |F_\epsilon(z_0)| \leq 1 \) and thus \( |F(z_0)| = \lim_{\epsilon \to 0} |F_\epsilon(z_0)| \leq 1 \).

**Step 2:** Let \( M_0, M_1 \) be arbitrarily and \( G(z) = \frac{F(z)}{\alpha z^\beta} \) where \( \alpha > M_0 \) and \( \beta > M_1 \). Then \( G \) is continuous, bounded and analytic on \( S^\circ \) and \( |G(z)| \leq 1 \) on \( \partial S \) and by step 1 \( |G(z)| \leq 1 \) on \( S \) so \( M_\theta \leq \alpha^{1-\theta} \beta^\theta \) and \( M_\theta \leq M_0^{1-\theta} M_1^\theta \). □

Now that we have all the tools we need, we can formulate and prove the main result of this talk.
**Theorem 1.4** (Interpolation theorem of Riesz - Thorin).

Let \(1 \leq p_0, p_1, q_0, q_1 \leq \infty\) and \(0 < \theta < 1\). Define \(p\) and \(q\) by

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]

If \(T\) is a linear map such that

\[
T : L^{p_0} \to L^{q_0} \text{ with } \|T\|_{L^{p_0} \to L^{q_0}} = N_0
\]

and

\[
T : L^{p_1} \to L^{q_1} \text{ with } \|T\|_{L^{p_1} \to L^{q_1}} = N_1.
\]

Then we have

\[
\|Tf\|_q \leq N_1^{1-\theta} N_0^\theta \|f\|_p \quad \forall f \in L^{p_0} \cap L^{p_1}
\]

if \(\mathbb{K} = \mathbb{C}\) and

\[
\|Tf\|_q \leq 2N_1^{1-\theta} N_0^\theta \|f\|_p \quad \forall f \in L^{p_0} \cap L^{p_1}
\]

if \(\mathbb{K} = \mathbb{R}\). In particular, the operator \(T\) can be extended to a continuous linear map \(T : L^p \to L^q\) with

\[
\|T\| \leq cN_1^{1-\theta} N_0^\theta
\]

where \(c = 1\) if \(\mathbb{K} = \mathbb{C}\) and \(c = 2\) if \(\mathbb{K} = \mathbb{R}\).

**Remark 1.5.** Before we prove the theorem we are going to have a closer look on its assertion. If we consider the spaces \(L^p\) as functions of \(\frac{1}{p}\) we may reinterpret the theorem by saying that

\[
C := \left\{ (p, q) \mid T : L^\frac{1}{p} \to L^\frac{1}{q} \right\}
\]

is a convex set. Indeed, if we take two points from this set the theorem of Riesz-Thorin shows us that their connection line is also contained in the set.

Furthermore inequality (2) tells us that the mapping

\[
(\alpha, \beta) \mapsto \log \|T\|_{\frac{1}{\alpha} \to \frac{1}{\beta}}
\]

is convex (which is to say the points lying on and above the graph form a convex set).

**Proof.** To begin with note that due to our assumptions \(Tf \in L^{q_0} \cap L^{q_1}\) for \(f \in L^{p_0} \cap L^{p_1}\) and by Lemma 1.2 we have that \(Tf \in L^q\) for such \(f\). We treat the case \(\mathbb{K} = \mathbb{C}\) first:
Case 1: $p < \infty$ and $q > 1$

- Since the integrable step functions are dense in all $L^p$-spaces they are dense in $L^{p_0} \cap L^{p_1}$. So it is sufficient to show ineq. (2) for all such functions.
- We will do so by showing that

\[
\left| \int (Tf)g \right| \leq N_0^{1-\theta} N_1^{\theta}
\]

for all integrable step functions $f, g$ with $\|f\|_p = \|g\|_{p'} = 1$, where as usual $1 = \frac{1}{p} + \frac{1}{q'}$.

Indeed ineq. (4) tells us that the functional

\[
l : L^{q'} \rightarrow \mathbb{C}
\]

\[
g \mapsto \int (Tf)g
\]

obeys $\|l\| \leq N_0^{1-\theta} N_1^{\theta}$. (Note that here we again used the fact that integrable step functions are dense in $L^{q'}$ ($q' < \infty$ since $q > 1$).)

By [FA1, 2.45] we know that $l \in (L^{q'})' \sim L^q$ is the isometrically isomorphic image of $Tf$ so $\|Tf\|_q \leq N_0^{1-\theta} N_1^{\theta}$.

To show ineq. (4) we define the step functions $f$ and $g$ by

\[
f = \sum_{j=1}^{J} a_j \chi_{A_j}, \quad g = \sum_{k=1}^{K} b_k \chi_{B_k}
\]

and

\[
\|f\|_p^p = \sum_{j=1}^{J} |a_j|^p \mu(A_j) = 1, \quad \|g\|_{q'}^{q'} = \sum_{k=1}^{K} |b_k|^{q'} \mu(B_k) = 1
\]

where $\mu$ is the Lebesgue measure on $\mathbb{R}^n$ and $A_j$ resp. $B_k$ are pairwise disjoint.

For $z \in \mathbb{C}$ let $p(z)$ and $q'(z)$ be defined as

\[
\frac{1}{p(z)} = \frac{1}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1}{q'_0} + \frac{z}{q'_1}
\]

so $p(0) = p_0$, $p(\theta) = p$ and $p(1) = p_1$ as well as $q'(0) = q'_0$, $q'(\theta) = q'$, $q'(1) = q'_1$.

Using the convention $\frac{0}{0} = 0$ we set

\[
f_z = |f|^{p/p(z)} \frac{f}{|f|} \text{ and } g_z = |g|^{q'/q'(z)} \frac{g}{|g|}.
\]

Now $f_z$ and $g_z$ are integrable step functions, especially $f_z \in L^{p_1}$ which implies that $Tf_z$ is defined since $f_z$ is again a step function.

Finally we define $F : \mathbb{C} \rightarrow \mathbb{C}$ as $F(z) = \int (Tf_z)g_z d\nu$

By eqs. (5) we have

\[
F(z) = \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{|a_j|}{|a_j|} \frac{a_j}{|a_j|} \frac{b_k}{|b_k|} \frac{b_k}{|b_k|} \int_{B_k} T \chi_{A_j} d\nu.
\]
This shows that $F$ is a linear combination of terms of the form $\gamma^z$ with $\gamma > 0$. So $F$ is analytic and satisfies the assumptions of Prop. 1.3, since every function $\gamma^z$ is bounded in $S$ (see Prop 1.3) by

\[ |\gamma^{x+iy}| = \gamma^x \leq \max \{1, \gamma\} \quad \forall x + iy \in S. \]

Next we estimate $|F(iy)|$ and $|F(1 + iy)|$. We have

\[ |F(iy)| \leq \|Tf_{iy}\|_{q_0} \|g_{iy}\|_{q_0} \leq N_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q_0} \]

and furthermore

\[ \|f_{iy}\|_{p_0} = \sum_{j=1}^J |a_j|^{\frac{p_0}{p_j}} \mu(A_j)^{\epsilon q_{0j} - 1} = \sum_{j=1}^J |a_j|^p \mu(A_j) = \|f\|_{p_j} = 1, \]

where we have used

\[ |a_j|^{\frac{p_0}{p_j}} = |a_j|^{\frac{\mu(A_j)^{\epsilon q_{0j}}}{\mu(A_j)^{\epsilon q_{0j}}}} = |a_j|^p \] \hspace{1cm} (6)

Analogously we obtain $\|g_{iy}\|_{q_0} = 1$.

Summing up we have

\[ \sup_{y \in \mathbb{R}} |F(iy)| \leq N_0 \]

and repeating the same calculation with $1 + iy$ replacing $iy$ we obtain

\[ \sup_{y \in \mathbb{R}} |F(1 + iy)| \leq N_1. \]

Now finally Prop. 1.3 yields

\[ \left| \int Tfg \, dv \right| = |F(\theta)| \leq \sup_{y \in \mathbb{R}} |F(\theta + iy)| \leq N_0^{1-\theta} N_1^\theta. \]

So we have estimated ineq. (4) and we are done.

**Case 2: $p = \infty$**

This assumption immediately implies that $p_0 = p_1 = \infty$. If $q = q_0 = q_1 = 1$ there’s nothing to show. So let $q > 1$. Now $f$ need not be integrable and we may choose $f = f_z \forall z$. Analogously we can handle the case $q = 1, p < \infty$ (now $g_z = g$).

It remains to show ineq. (3), i.e., the case $\mathbb{K} = \mathbb{R}$. But luckily this follows from ineq. (2) and the following argument. Let $U : L_\mathbb{C}^p \rightarrow L_\mathbb{R}^p$ be a continuous linear operator between real $L^p$ spaces. Furthermore define its canonical extension as $U_{\mathbb{C}}(f + ig) = Uf + iUg$. This map is $\mathbb{C}$-linear, $U_{\mathbb{C}} : L_\mathbb{C}^p \rightarrow L_\mathbb{C}^p$, and the following inequality holds

\[ \|U_{\mathbb{C}}\| = \sup_{\|f + ig\| = 1} \|U_{\mathbb{C}}(f + ig)\| \leq \sup_{\|f + ig\| = 1} (\|U(f)\| + \|U(g)\|) \]

\[ \leq \sup_{\|f\| = 1} \|U(f)\| + \sup_{\|g\| = 1} \|U(g)\| \leq 2\|U\|. \]
Applying this to the assumption of the Theorem we obtain for $T$ by using the extension $T_C$ and ineq. (2)

$$\|Tf\|_q = \|T_C f\|_q \leq \|T_C\|^{1-\theta}_{p_0-q_0} \|T_C\|^\theta_{p_1-q_1} \|f\|_p$$

$$\leq 2 \|T\|^{1-\theta}_{p_0-q_0} \|T\|^\theta_{p_1-q_1} \|f\|_p = 2N^{1-\theta} N^\theta_1 \|f\|_p$$

for $f$ real valued. 

\[ \square \]

**Example 1.6.** We want to give an example that eq. 2 doesn’t hold in $\mathbb{R}$. 
Let us have a look at $T(s,t) = (s + t, s - t)$, which has norms $\|T\|_{\infty \rightarrow 1} = 2$ and $\|T\|_{2 \rightarrow 2} = \sqrt{2}$. For $\theta = \frac{1}{2}$, $p = 4$, $q = \frac{3}{2}$ we would get

$$\|(s + t, s - t)\|_{\frac{3}{2}} \leq 2^{\frac{3}{4}} \|(s, t)\|_4$$

which isn’t true for $s = 2$ and $t = 1$.

2. Applications

As a first relevant application we discuss properties of the convolution.

**Definition 2.1.** Let $T = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : 0 \leq t \leq 2\pi\}$.

The convolution of $f, g \in L^1(T)$ is defined by

$$(f * g)(e^{is}) = \int_0^{2\pi} f(e^{it})g(e^{i(s-t)}) \frac{dt}{2\pi}.$$ 

**Remark 2.2.** For $f, g \in L^1(\mathbb{T})$ the convolution $f * g$ is measurable and the following inequalities hold

$$\int_0^{2\pi} |(f * g)(e^{is})| \frac{ds}{2\pi} \leq \int_0^{2\pi} \int_0^{2\pi} |f(e^{it})| |g(e^{i(s-t)})| \frac{dt}{2\pi} \frac{ds}{2\pi}$$

$$= \int_0^{2\pi} |f(e^{it})| \int_0^{2\pi} |g(e^{i(s-t)})| \frac{ds}{2\pi} \frac{dt}{2\pi}$$

$$= \int_0^{2\pi} |f(e^{it})| \frac{dt}{2\pi} \|g\|_1$$

$$= \|f\|_1 \|g\|_1.$$ 

(7)

In particular, $f * g \in L^1(\mathbb{T})$ and fixing $f \in L^1$ and writing $Tfg = f * g$ we have $T_f : L^1 \rightarrow L^1$ with $\|T_f\|_{L^1 \rightarrow L^1} \leq \|f\|_1$. Similarly we have

$$\sup_{s \in [0,2\pi]} |(f * g)(e^{is})| \leq \sup_{s \in [0,2\pi]} \int_0^{2\pi} |f(e^{it})| |g(e^{i(s-t)})| \frac{dt}{2\pi}$$

$$= \int_0^{2\pi} |f(e^{it})| \sup_{s \in [0,2\pi]} |g(e^{i(s-t)})| \frac{dt}{2\pi}$$

$$= \int_0^{2\pi} |f(e^{it})| \frac{dt}{2\pi} \|g\|_\infty$$

$$= \|f\|_1 \|g\|_\infty.$$ 

(8)

This yields $f * g \in L^\infty(\mathbb{T})$ for $g \in L^\infty(\mathbb{T})$ and with the notation as above $T_f : L^\infty \rightarrow L^\infty$ again with $\|T_f\|_{L^\infty \rightarrow L^\infty} \leq \|f\|_1$. 
**Proposition 2.3** (Young’s inequality). Let \(1 \leq p, q \leq \infty\) and \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0\). If \(f \in L^p\) and \(g \in L^q\) then \(f \ast g \in L^r\) and we have

\[
\|f \ast g\|_r \leq \|f\|_p \|g\|_q.
\]

**Proof.** \(T\) is finite, so \(f \ast g\) is defined since \(L^p[T], L^q[T] \subset L^1[T]\).

**Step 1:** Let \(f \in L^1\) be fixed. Due to ineq. (7) in Remark 2.2 we have

\[
\|f \ast g\|_1 = \|f\|_1 \|g\|_1 \text{ resp. } \|Tf\|_{1 \to 1} \leq \|f\|_1.
\]

Furthermore ineq. (8) in the Remark above yields

\[
\|Tf\|_{\infty \to \infty} \leq \|f\|_1.
\]

Using theorem 1.4 with \(\theta = \frac{1}{q}\) where \(q'\) is the conjugated exponent of \(q\) we get

\[
\|T\|_{q \to q} \leq \|f\|_1.
\]

That means

\[
(9) \quad \|f \ast g\|_q \leq \|Tf\|_{q \to q} \|g\|_q \leq \|f\|_1 \|g\|_q \quad \forall f \in L^1, g \in L^q.
\]

**Step 2:** Let now \(g \in L^q\) be fixed. The Hölder inequality for \(f \in L^{q'}\) and \(e^{ix} \in T\) leads us to

\[
|f \ast g(e^{ix})| \leq \int_0^{2\pi} |f(e^{it})g(e^{i(s-t)})| \frac{dt}{2\pi}
\]

Hölder

\[
\leq (\int_0^{2\pi} |f(e^{it})|^{q'} \frac{dt}{2\pi})^{\frac{1}{q'}} (\int_0^{2\pi} |g(e^{it})|^q \frac{dt}{2\pi})^{\frac{1}{q}}
\]

\[
= \|f\|_{q'} \|g\|_q,
\]

which implies \(\|f \ast g\|_\infty \leq \|f\|_{q'} \|g\|_q\).

Using ineq. (9) and ineq. (10) for the operator \(Tg f = f \ast g\) we have

\[
\|Tg\|_{1 \to q} \leq \|g\|_q \text{ and } \|Tg\|_{q' \to \infty} \leq \|g\|_q.
\]

We now choose \(\theta\) such that \(\frac{1}{p} = \frac{1}{r} + \frac{\theta}{q'}\), then \(\theta = \frac{q}{p}\). Moreover \(0 \leq \theta \leq 1\) since \(\frac{1}{p} + \frac{1}{q} \geq 1\). And with this choice of \(\theta\) we have

\[
\frac{1 - \theta}{q} + \frac{\theta}{\infty} = \frac{1}{p} - \frac{1}{q'} = \frac{1}{q} - 1 + \frac{1}{p} = \frac{1}{r}.
\]

So applying Thm. 1.4 we obtain

\[
\|Tg\|_{p \to r} \leq \|g\|_q,
\]

which means that

\[
\|f \ast g\|_r \leq \|f\|_p \|g\|_q \quad \forall f \in L^p[T], g \in L^q[T].
\]

\(\square\)
Another application is the following.

**Proposition 2.4.** Using the same assumptions as in Theorem 1.4 and supposing that \( T : L^{p_0} \to L^{\infty} \) is compact, we get that \( T : L^{p} \to L^{q} \) is compact.

**Proof.** See [D. Werner, Funktionalanalysis (Springer, 2005), p. 79, Satz II.4.5].  \( \square \)