Fourier Transform & Sobolev Spaces

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Abstract

We introduce the concept of weak derivative that allows us to define new interesting Hilbert spaces—the Sobolev spaces. Furthermore we discuss the Fourier transform and its relevance for Sobolev spaces.

We follow largely the presentation of [Werner, *Funktionalanalysis* (Springer 2005); V.1.9–V.2.14].

1 Test Functions & Weak Derivatives

Motivation 1.1 (test functions and weak derivatives). In this paragraph we want to extend the concept of derivative to introduce new Hilbert spaces of "weakly differentiable" functions.

Remark 1.2 (Notation). We are going to use the following notational conventions:

(*i*) Let $x \in \mathbb{R}^n$, we write $|x|$ for the euclidean norm of x, $|x| := \sum_{k=1}^n x_k^2$.
(*ii*) If $x, y \in \mathbb{R}^n$, we denote the standard scalar product simply by $xy := \sum_{k=1}^n x_k y_k$.
(*iii*) We employ the following notation for polynomials: we simply write xf for the function $x \mapsto f(x)x$ if $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.3 (test functions). Let $\Omega \subseteq \mathbb{R}^n$ be open. Then we define the set of *test functions* (or $\mathcal{C}^\infty$-*functions with compact support*) on Ω as

$$\mathcal{D}(\Omega) := \left\{\varphi \in \mathcal{C}^\infty(\Omega, \mathbb{C}) : \operatorname{supp}(\varphi) := \overline{\left\{x : \varphi(x) \neq 0\right\}} \subseteq \Omega \text{ is compact}\right\}$$

We call $\operatorname{supp}(\varphi)$ the *support* of φ.

Example 1.4 (of a test function). The function

$$\varphi(x) = \begin{cases} c \cdot \exp((|x|^2 - 1)^{-1}) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

is a test function on $\mathbb{R}^n$ (c.f. [A1, 10.12]). So there actually do exist such functions.

Lemma 1.5 ($\overline{\mathcal{D}(\Omega)} = L^p(\Omega)$). The space of test functions $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

Proof (sketch). Given $f \in L^p$ we have to find $f_\varepsilon \in \mathcal{D}$ such that $f_\varepsilon \to f \in L^p$. This can be done using convolution with so called *mollifiers*:

$$f_\varepsilon(x) := \int f(x - y)\varphi_\varepsilon(y)dy$$

where the mollifier is defined by $\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$ using φ from **example 1.4**.

Then we have $f_\varepsilon \in \mathcal{C}^\infty$ and $f_\varepsilon \to f$ in L^p by [Evans, *Partial Differential Equations*; Appendix C, Thm. 6]. The compact support can be achieved by an appropriate cut–off. $\boxtimes$

Motivation 1.6 (weak derivatives). Many functions that arise in applications are not differentiable in the classical sense but they are still integrable. This allows us to use the following trick called the concept of *weak derivatives*. Let $I \subseteq \mathbb{R}$ be an open interval, and to begin with let $f \in \mathcal{C}^1(\bar{I})$ and $\varphi \in \mathcal{D}(I)$, it then follows that

$$\int_I f'(x)\overline{\varphi(x)}dx = -\int_I f(x)\overline{\varphi'(x)}dx$$

using integration by parts (the boundary values vanish, since the support of φ is a compact subset of I). This formula can be rewritten using the scalar product of $L^2(I)$ as

$$\left\langle f' \,|\, \varphi \right\rangle = -\left\langle f \,|\, \varphi' \right\rangle.$$

We call a function g that satisfies $\left\langle g \,|\, \varphi \right\rangle = -\left\langle f \,|\, \varphi' \right\rangle$ a "weak" or "generalized" derivative of f.

The n-dimensional analogon can be derived similarly. Let $\Omega \subseteq \mathbb{R}^n$ open, $f \in \mathcal{C}^1(\overline{\Omega})$ and $\varphi \in \mathcal{D}(\Omega)$, then

$$\left\langle \frac{\partial}{\partial x_i} f \,\middle|\, \varphi \right\rangle = -\left\langle f \,\middle|\, \frac{\partial}{\partial x_i}\varphi \right\rangle$$

now by the theorem of Gauss and the compact support of φ. Similarly we obtain the following for m-times continuously differentiable functions f:

$$\left\langle D^\alpha f \,|\, \varphi \right\rangle = (-1)^{|\alpha|}\left\langle f \,|\, D^\alpha \varphi \right\rangle \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{1}$$

where α is a multi–index (i.e. $\alpha \in \mathbb{N}_0^n$), and $|\alpha| \leq m$ (for an introduction to the multi–index notation see [A2, 19.2]).

We now drop the assumption $f \in \mathcal{C}^m$ and turn equation (1) into a definition.

Definition 1.7 (weak derivative). Let $\Omega \subseteq \mathbb{R}^n$ be open, α a multi–index and $f \in L^2(\Omega)$. Then $g \in L^2(\Omega)$ is called *weak* or *generalized* α-th derivative of f, if

$$\left\langle g \,|\, \varphi \right\rangle = (-1)^{|\alpha|}\left\langle f \,|\, D^\alpha \varphi \right\rangle \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 1.8 (on weak derivatives). By the above discussion if $f \in \mathcal{C}^m$ then the "weak derivatives" $D^\alpha f$ ($|\alpha| \leq m$) coincide with the classical (usual) derivatives. Moreover, such a g is unique: suppose h is another weak derivative of f, then $\left\langle g - h \,|\, \varphi \right\rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ and by lemma 1.5 and $\left[\text{FA1, Prop. 2.34: } \left\langle f \,|\, \varphi \right\rangle = 0\ \forall \varphi \text{ in a dense set} \Rightarrow f = 0\right]$ it follows that $g = h$.

We denote *the* weak derivative of f by $D^{(\alpha)}f$, so we can write

$$\left\langle D^{(\alpha)} f \,|\, \varphi \right\rangle = (-1)^{|\alpha|}\left\langle f \,|\, D^\alpha \varphi \right\rangle \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Now we may define spaces of weakly differentiable functions.

Definition 1.9 (Sobolev spaces). Let $\Omega \subseteq \mathbb{R}^n$ be open. Then we define

(*i*) $W^m(\Omega) := \left\{ f \in L^2(\Omega) : D^{(\alpha)} f \in L^2(\Omega) \text{ exists } \forall |\alpha| \leq m \right\}$

(*ii*) $\left\langle f \,|\, g \right\rangle_{W^m} := \sum_{|\alpha| \leq m} \left\langle D^{(\alpha)} f \,|\, D^{(\alpha)} g \right\rangle_{L^2}$

(*iii*) $H^m(\overline{\Omega}) := \overline{\mathcal{C}^m(\overline{\Omega}) \cap W^m(\Omega)}$, where the closure is taken with respect to the norm induced by $\left\langle\,.\,|\,.\,\right\rangle_{W^m}$

(*iv*) $H_0^m(\Omega) := \overline{\mathcal{D}(\Omega)}$, again with respect to the induced norm on W^m.

These spaces are all called *Sobolev spaces*.

Remark 1.10 (on Sobolev spaces).

(*i*) $W^0 = L^2$ by definition and $H_0^0 = L^2$ by **lemma 1.5**.

(*ii*) Obviously W^m, H^m and H_0^m are vector spaces, and $\left\langle\,.\,|\,.\,\right\rangle_{W^m}$ is a scalar product.

(*iii*) (no proofs) If Ω is a bounded region with sufficiently smooth boundary, then $W^m(\Omega) = H^m(\overline{\Omega})$ $\left(\text{i.e. } W^m(\Omega) = \overline{\mathcal{C}^m(\overline{\Omega}) \cap W^m(\Omega)}\right)$. On the other hand, it always holds that $W^m(\Omega) = \overline{\mathcal{C}^m(\Omega) \cap W^m(\Omega)}$.

Proposition 1.11 (Sobolev spaces are (H)-spaces). W^m, H^m and H_0^m are Hilbert spaces for all $m \geq 0$.

Proof. Since H^m and H_0^m are closed subspaces of W^m is suffices to show the completeness of $W^m(\Omega)$. Let $(f_n)_n$ be a Cauchy sequence with respect to $\|\,.\,\|_{W^m}$. Since

$$\|f\|_{W^m}^2 = \sum_{|\alpha|\leq m} \|D^{(\alpha)} f\|_2^2 \qquad \forall f \in W^m,$$

the sequences $(D^{(\alpha)} f_n)_n$ are all $\|\,.\,\|_2$-Cauchy sequences. Therefore there exist $f_\alpha \in L^2(\Omega)$ such that

$$\|D^{(\alpha)} f_n - f_\alpha\|_2 \to 0 \qquad (n \to \infty, |\alpha| \leq m).$$

Set $f_0 := f_{(0,\dots,0)}$, we'll show that $D^{(\alpha)} f_0 = f_\alpha$, since this implies $f_0 \in W^m$ and $f_n \to f_0$ with respect to $\|\,.\,\|_{W^m}$ (by definition of that norm). Indeed, for $\varphi \in \mathcal{D}(\Omega)$, we have (using the continuity of the L^2-scalar product in both slots)

$$\begin{aligned}
\left\langle f_\alpha \,|\, \varphi \right\rangle &= \left\langle \lim_{n\to\infty} D^{(\alpha)} f_n \,\middle|\, \varphi \right\rangle = \lim_{n\to\infty} \left\langle D^{(\alpha)} f_n \,\middle|\, \varphi \right\rangle \\
&\overset{1.7}{=} \lim_{n\to\infty} (-1)^{|\alpha|} \left\langle f_n \,\middle|\, D^\alpha \varphi \right\rangle = (-1)^{|\alpha|} \left\langle \lim_{n\to\infty} f_n \,\middle|\, D^\alpha \varphi \right\rangle \\
&= (-1)^{|\alpha|} \left\langle f_0 \,\middle|\, D^\alpha \varphi \right\rangle \overset{1.7}{=} \left\langle D^{(\alpha)} f_0 \,\middle|\, \varphi \right\rangle
\end{aligned}$$

And so we obtain $f_\alpha = D^{(\alpha)} f_0$ by **lemma 1.5**. $\square$

Motivation 1.12 (Sobolev spaces and PDEs). Clearly the Sobolev spaces are nested, i.e., $W^m(\Omega) \subseteq W^{m-1}(\Omega)$, and the identity map $\mathrm{id} : W^m(\Omega) \to W^{m-1}(\Omega)$ is continuous [since the norm on W^{m-1} can be estimated by $\|\,.\,\|_{W^m}$].

In applications the following two results are of great importance:

Sobolev embedding theorem: *For $f \in W^m(\Omega)$ and $m > k + \frac{n}{2}$ there exists $g \in \mathcal{C}^k(\Omega)$ with $f = g$ almost everywhere.*

Rellich-Kondrachov theorem: *If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, then the embedding $H_0^m(\Omega) \to H_0^{m-1}(\Omega)$ is compact.*

These theorems are important tools for proving the existence of solutions of many (especially elliptical) partial differential equations.

We are going to prove these theorems in **section 3**. As a technical tool we introduce the Fourier transform, which is also an interesting topic in itself.

2 Fourier Transform

Motivation 2.1 (decay vs. smoothness). If $f \in L^2(\mathbb{R}^n)$ this means that f has a certain fall–off property at ∞. In the Sobolev space W^m we even ask for such a fall–off property for the (weak) derivatives of f. The Fourier transform allows us to translate derivatives into multiplication with polynomials (see **lemma 2.8** below). So we may relate the L^2-property of derivatives of f into stronger fall–off conditions on f itself. In this way the Sobolev spaces allow us to measures smoothness of functions in terms of their fall–off on the Fourier transform side. Finally the Sobolev embedding theorem links this fall–off back to classical $\mathcal{C}^k$-properties.

Definition 2.2 (Fourier transform). Let $f \in L^1(\mathbb{R}^n)$, then we define the *Fourier transform* $\mathcal{F} f$ of f by

$$(\mathcal{F} f)(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx \qquad \forall \xi \in \mathbb{R}^n. \tag{2}$$

[recall the notation $x\xi$ for the scalar product of x with ξ] The operator $\mathcal{F}$ is called the *Fourier transform*.

Proposition 2.3 ($\mathcal{F} \in L(L^1, \mathcal{C}_0)$). If $f \in L^1(\mathbb{R}^n)$, then $\mathcal{F} f \in \mathcal{C}_0(\mathbb{R}^n)$. The Fourier transform $\mathcal{F} : L^1(\mathbb{R}^n) \to \mathcal{C}_0(\mathbb{R}^n)$ is a continuous and linear operator with $\|\mathcal{F}\| \leq (2\pi)^{-\frac{n}{2}}$.

Proof. First we have

$$|(\mathcal{F} f)(\xi)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)| \underbrace{|e^{-ix\xi}|}_{=1} dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_1,$$

i.e. $\|\mathcal{F} f\|_\infty \leq (2\pi)^{-\frac{n}{2}} \|f\|_1$, which implies $\|\mathcal{F}\| \leq (2\pi)^{-\frac{n}{2}}$. Therefore $\mathcal{F}$ is a continuous operator from $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

To show that $\mathcal{F} f \in \mathcal{C}_0(\mathbb{R}^n)$ (i.e. $\mathcal{F} f$ is continuous and vanishes at infinity), let $(\xi^{(k)})$ be a sequence in $\mathbb{R}^n$ with $\xi^{(k)} \to \xi$. Using the continuity of $\exp(x)$ and the scalar product we obtain

$$|e^{-ix\xi^{(k)}} - e^{-ix\xi}| \to 0 \qquad \forall x \in \mathbb{R}^n.$$

Plugging this into **(2)** and using dominated convergence we see that

$$|(\mathcal{F} f)(\xi^{(k)}) - (\mathcal{F} f)(\xi)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)| \underbrace{|e^{-ix\xi^{(k)}} - e^{-ix\xi}|}_{\leq 2} dx \to 0,$$

since the integrand is dominated by the L^1-function $2|f|$. This shows that $\mathcal{F} f$ is continuous.

To obtain $\operatorname{im} \mathcal{F} \subseteq \mathcal{C}_0(\mathbb{R}^n)$ it suffices to show

$$\lim_{|\xi| \to \infty} |(\mathcal{F} \varphi)(\xi)| = 0 \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

since $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ by **lemma 1.5**. So let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $R > 0$ and $|\xi| \geq R$. Then there exists a coordinate ξ_j with $|\xi_j| \geq \frac{R}{\sqrt{n}}$ $\left[R \leq |\xi| = \sqrt{\sum_{k=1}^n |\xi_k|^2} \leq \sqrt{n} \max_{1 \leq k \leq n} |\xi_k|\right]$. Using integration by parts we finally obtain

$$|(\mathcal{F}\varphi)(\xi)| = \left|\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\varphi(x)e^{-ix\xi}dx\right| = \left|\frac{-1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_j}\varphi(x)\frac{1}{-i\xi_j}e^{-ix\xi}dx\right|$$
$$\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left|\frac{\partial}{\partial x_j}\varphi(x)\right|\underbrace{\left|\frac{1}{-i\xi_j}\right|}_{\leq\frac{\sqrt{n}}{R}}\underbrace{\left|e^{-ix\xi}\right|}_{=1}dx \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\left\|\frac{\partial}{\partial x_j}\varphi(x)\right\|_1\frac{\sqrt{n}}{R} \to 0 \qquad (R\to\infty),$$

which completes the proof. $\square$

Remark 2.4 (on proposition 2.3). This result means, that the Fourier transform of an L^1-function has some decay behaviour, but not enough to secure the L^1-property. We now hope to find an explicit inverse operator of the Fourier transform starting with a certain subset of $L^1(\mathbb{R}^n)$. It turns out that the rapidly decreasing *Schwartz functions* do the trick:

Definition 2.5 (Schwartz space). Let $f:\mathbb{R}^n\to\mathbb{C}$; f is called *rapidly decreasing*, if

$$\lim_{|x|\to\infty} x^\alpha f(x) = 0 \qquad \forall\alpha\in\mathbb{N}_0^n \tag{3}$$

[recall $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$]. The space

$$\mathcal{S}(\mathbb{R}^n) := \left\{f\in\mathcal{C}^\infty(\mathbb{R}^n) : D^\beta f \text{ is rapidly decreasing } \forall\beta\in\mathbb{N}_0^n\right\}$$

is called *Schwartz space*, its elements *Schwartz functions*.

Remark 2.6 (on Schwartz functions).

(*i*) Examples for Schwartz functions are $\gamma(x) = e^{-x^2}$ and every testfunction.

(*ii*) The following conditions are equivalent to (3):

$$\lim_{|x|\to\infty} P(x)f(x) = 0 \qquad \forall \text{ polynomials } P:\mathbb{R}^n\to\mathbb{C} \tag{4}$$

$$\lim_{|x|\to\infty} |x|^m f(x) = 0 \qquad \forall m\in\mathbb{N}_0 \tag{5}$$

(Obviously (3) implies the first condition, which in turn implies (5), because one only has to consider the polynomials $P(x) = (x_1^2+\ldots+x_n^2)^{\frac{m}{2}}$ with even m. Finally this again implies (3), because $|x^\alpha|\leq|x|^{|\alpha|}$.)

(*iii*) So by definition, Schwartz functions and all of their derivatives decay faster than the inverse of any polynomial. Using this we obtain $\boxed{\mathcal{S}(\mathbb{R}^n)\subseteq L^p(\mathbb{R}^n) \text{ for all } p\geq 1}$, since for $mp-(n-1)>1$ and $f\in\mathcal{S}(\mathbb{R}^n)$ we have:

$$\int_{\mathbb{R}^n}|f(x)|^p dx \leq \int_{\mathbb{R}^n}\left(\frac{c}{1+|x|^m}\right)^p dx$$
$$= c^p\omega_{n-1}\int_0^\infty\left(\frac{1}{1+r^m}\right)^p r^{n-1}dr < \infty,$$

where ω_{n-1} is the area of the n-sphere $\left\{x\in\mathbb{R}^n : |x|=1\right\}$ and c is a properly chosen constant (see [Otto Forster, *Analysis 3*; §14, Satz 8, Beispiel (14.10)]).

(*iv*) Because $\mathcal{D}(\mathbb{R}^n)$ is obviously a subspace of $\mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions is dense in $L^p(\mathbb{R}^n)$ for $1\leq p<\infty$ as well (**lemma 1.5**).

(*v*) A smooth function $f\in\mathcal{C}^\infty$ is a Schwartz function if and only if

$$\sup_{x\in\mathbb{R}^n}(1+|x|^m)|D^\beta f(x)| < \infty \qquad \forall m\in\mathbb{N}_0, \beta\in\mathbb{N}_0^n. \tag{6}$$

Clearly every Schwartz function has this property. For the other direction consider

$$x^\alpha D^\beta f \le |x|^m |D^\beta f(x)| = \frac{(|x|^m+|x|^{m+1})|D^\beta f(x)|}{1+|x|} \le \frac{((1+|x|^m)+(1+|x|^{m+1}))|D^\alpha f|}{1+|x|} \overset{(6)}{\le} \frac{c}{1+|x|} \to 0.$$

The motivation for the use of the factor $(1+|x|^m)$ is that it is always possible to divide by it—which is not the case for $|x|^m$.

Motivation 2.7 (Schwartz functions and Fourier transform). The importance of the Schwartz space lies in the fact, that the Fourier transform $\mathcal{F}$ is a bijection of $\mathcal{S}(\mathbb{R}^n)$ (we'll prove this in **proposition 2.14**), which was not the case with $L^1(\mathbb{R}^n)$.

Recall the notation x^α for the map $x\mapsto x^\alpha$, then we have for any $f\in\mathcal{S}(\mathbb{R}^n)$:

$$x^\alpha f\in\mathcal{S}(\mathbb{R}^n), D^\alpha f\in\mathcal{S}(\mathbb{R}^n) \qquad \forall \alpha\in\mathbb{N}_0^n. \tag{7}$$

This allows us to state the interesting and important interplay between differentiation and multiplication by polynomials using the Fourier transform:

Lemma 2.8 (exchange formulas). Let $f\in\mathcal{S}(\mathbb{R}^n)$ and α be a multi–index. Then

(i) $\mathcal{F}f\in\mathcal{C}^\infty(\mathbb{R}^n)$ and $D^\alpha(\mathcal{F}f) = (-i)^{|\alpha|}\,\mathcal{F}(x^\alpha f)$,
(ii) $\mathcal{F}(D^\alpha f) = i^{|\alpha|}\xi^\alpha\,\mathcal{F}f$.

Proof. *(i)* Formally we have:

$$\begin{aligned} D^\alpha(\mathcal{F}f)(\xi) &= \frac{\partial^\alpha}{\partial\xi^\alpha}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx \overset{(*)}{=} \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)\frac{\partial^\alpha}{\partial\xi^\alpha}e^{-ix\xi}dx \\ &= (-i)^{|\alpha|}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)x^\alpha e^{-ix\xi}dx = (-i)^{|\alpha|}\,\mathcal{F}(x^\alpha f)(\xi). \end{aligned}$$

We are allowed to pull the differential into the integral in step $(*)$ by dominated convergence and because $x^\alpha f\in\mathcal{S}(\mathbb{R}^n)\subseteq L^1(\mathbb{R}^n)$ (c.f. [Werner, Cor. A.3.3]).

(ii) is also a simple calculation using integration by parts (note that since $f\in\mathcal{S}(\mathbb{R}^n)$ all boundary terms vanish):

$$\begin{aligned} \mathcal{F}(D^\alpha f)(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(D^\alpha f)(x)e^{-ix\xi}dx = \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)\frac{\partial^\alpha}{\partial x^\alpha}e^{-ix\xi}dx \\ &= (-1)^{|\alpha|}(-i)^{|\alpha|}\xi^\alpha(\mathcal{F}f)(\xi) = i^{|\alpha|}\xi^\alpha(\mathcal{F}f)(\xi). \end{aligned}$$

$\square$

Remark 2.9 (on lemma 2.8). This lemma now shows why the Fourier transform is so interesting for PDEs: differentials are turned into multiplications, i.e. analytic operations are turned into algebraic operations and the other way round.

Lemma 2.10 ($\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$). If $f\in\mathcal{S}(\mathbb{R}^n)$, then also $\mathcal{F}f\in\mathcal{S}(\mathbb{R}^n)$.

Proof. In **lemma 2.8(*i*)** we showed that $\mathcal{F}f\in\mathcal{C}^\infty(\mathbb{R}^n)$ holds. So we just have to prove $\xi^\alpha D^\beta(\mathcal{F}f)(\xi)\to 0$ for $|\xi|\to\infty$. This also follows by **lemma 2.8**:

$$\xi^\alpha D^\beta(\mathcal{F}f)(\xi) = (-i)^{|\beta|}\xi^\alpha\,\mathcal{F}(x^\beta f)(\xi) = (-i)^{|\beta|}(-i)^{|\alpha|}\,\mathcal{F}(D^\alpha x^\beta f)(\xi),$$

since $D^\alpha(x^\beta f) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$, **proposition 2.3** gives the result. $\square$

Motivation 2.11 **($e^{-\frac{x^2}{2}}$).** Next we want to find out what happens to a function $f \in \mathcal{S}(\mathbb{R}^n)$ if we apply the Fourier transform twice—which is now allowed by **lemma 2.10**. To do this we first need to calculate the Fourier transform of

$$\gamma(x) = e^{-\frac{x^2}{2}} \qquad (x \in \mathbb{R}^n),$$

We also use $\gamma_a(x) = \gamma(ax)$ for $a > 0$. Recall [A3-PS, 182]:

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \gamma(x) dx = 1.$$

Lemma 2.12 **($\mathcal{F}$ of $e^{-\frac{x^2}{2}}$).** We have

$$(\mathcal{F}\gamma)(\xi) = e^{-\frac{\xi^2}{2}} = \gamma(\xi), \qquad (\mathcal{F}\gamma_a)(\xi) = \frac{1}{a^n}\gamma\Big(\frac{\xi}{a}\Big).$$

Proof. The second result is clear once we have proved the first one (just substitute ξ for $a\xi$). First we are going to calculate $\mathcal{F}\gamma$ for $n = 1$. In this case γ solves the ordinary differential equation

$$y' + xy = 0, \qquad y(0) = 1.$$

Thus, by **lemma 2.8**, we have

$$0 = \mathcal{F}(\gamma' + x\gamma) = i\xi\,\mathcal{F}\gamma + \Big(\frac{1}{-i}\,\mathcal{F}\gamma\Big)',$$

which means, that $\mathcal{F}\gamma$ solves the same differential equation; it even has the same initial value:

$$(\mathcal{F}\gamma)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1.$$

Because such initial value problems have unique solutions, it follows that $\gamma = \mathcal{F}\gamma$.
We show the case $n > 1$ using the result for $n = 1$:

$$\begin{aligned}
(\mathcal{F}\gamma)(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{n} e^{-x_k^2/2} \prod_{k=1}^{n} e^{-ix_k\xi_k} dx_1 \cdots dx_n \\
&= \prod_{k=1}^{n} \Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x_k^2}{2}} e^{-ix_k\xi_k} dx_k \Big) \\
&= e^{\frac{\xi_1^2}{2}} \cdots e^{\frac{\xi_n^2}{2}} = e^{\frac{\xi^2}{2}}.
\end{aligned}$$

$\square$

Lemma 2.13 **(a step closer to $\mathcal{F}^{-1}$).** Let $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$(\mathcal{F}\mathcal{F}f)(x) = f(-x) \qquad \forall x \in \mathbb{R}^n,$$

i.e., $\mathcal{F}^2$ is the reflection.

Proof. Because we have $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$ by **lemma 2.10**, the lefthand side of the equation is defined. If we tried to use Fubini's theorem directly to calculate the double-integral $(\mathcal{F}\mathcal{F}f)(x)$, we would obtain a non-convergent integral. Therefore we have to be more careful and use the "convergence-improving" property of the functions γ_a.

A first useful observation is the following implication of Fubini's theorem:

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{F} f)(x)g(x)dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)g(x)e^{-i\xi x}d\xi dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(\xi)e^{-i\xi x}d\xi dx = \int_{\mathbb{R}^n} f(x)(\mathcal{F} g)(x)dx.\end{aligned} \qquad \forall f,g \in \mathcal{S}(\mathbb{R}^n) \tag{8}$$

Here the change in the order of integration is allowed, because $(x,\xi) \mapsto f(\xi)g(x)e^{-i\xi x}$ is integrable ($e^{-i\xi x}$ is damped in both arguments). Now set $g(x) := e^{-ix\xi_0}\gamma(ax)$ with $\xi_0 \in \mathbb{R}^n$ and $a > 0$. Then we have:

$$(\mathcal{F} g)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\xi_0}\gamma(ax)e^{-ix\xi}dx = (\mathcal{F}\gamma_a)(\xi+\xi_0). \tag{9}$$

Now comes the actual proof:

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} (\mathcal{F} f)(x)\underbrace{e^{-ix\xi_0}\gamma(ax)}_{=g(x)}dx &\overset{(8)}{=} \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)(\mathcal{F} g)(x)dx \\ &\overset{(9)}{\underset{2.12}{=}} \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)\frac{1}{a^n}(\mathcal{F}\gamma)\Big(\frac{x+\xi_0}{a}\Big)dx \\ &\overset{u=\frac{x+\xi_0}{a}}{\underset{2.12}{=}} \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(au-\xi_0)\gamma(u)du.\end{aligned}$$

If we now let $a \to 0$ then we see, using dominated convergence, that the first expression tends to $(\mathcal{F}\mathcal{F} f)(\xi_0)$ while the last expression tends to $f(-\xi_0)$. Since the first integral is dominated by $|\mathcal{F} f|$ and the last one by $\|f\|_\infty\gamma$, which are both integrable. ◻

Proposition 2.14 ($\mathcal{F}$ is bijective, Plancherel formula). The Fourier transform is a bijection of $\mathcal{S}(\mathbb{R}^n)$; the inverse operator is given by

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(\xi)e^{ix\xi}d\xi.$$

Furthermore we have the so–called *Plancherel equation*

$$\big\langle \mathcal{F} f \,|\, \mathcal{F} g\big\rangle_{L^2} = \big\langle f \,|\, g\big\rangle_{L^2} \qquad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Proof. From the previous **lemma 2.13** we immediately obtain $\mathcal{F}^4 = \mathrm{id}_{\mathcal{S}(\mathbb{R}^n)}$, which implies that the Fourier transform is bijective and $\mathcal{F}^{-1} = \mathcal{F}^3$. Therefore we have

$$(\mathcal{F}^{-1} f)(x) = (\mathcal{F}^2(\mathcal{F} f))(x) \overset{2.13}{=} (\mathcal{F} f)(-x),$$

which implies the first claim. We prove the Plancherel equation using **(8)**:

$$\int_{\mathbb{R}^n} (\mathcal{F} f)(\xi)\overline{(\mathcal{F} g)(\xi)}d\xi = \int_{\mathbb{R}^n} f(x)(\mathcal{F}\,\overline{(\mathcal{F} g)})(x)dx,$$

if we write $h := \mathcal{F} g$, we obtain

$$(\mathcal{F}\,\overline{(\mathcal{F}\,g)})(x) = (\mathcal{F}\,\overline{h})(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\overline{h(\xi)}e^{-ix\xi}d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}}\overline{\int_{\mathbb{R}^n}h(\xi)e^{ix\xi}d\xi}$$
$$= \overline{(\mathcal{F}^{-1}\,h)(x)} = \overline{g(x)},$$

which immediately implies

$$\left\langle \mathcal{F}\,f \,|\, \mathcal{F}\,g\right\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \left\langle f \,|\, g\right\rangle_{L^2}.$$

□

Remark 2.15 ($\mathcal{F}$ on L^2).

(*i*) The Plancherel equation implies

$$\|\mathcal{F}\,f\|_2 = \|f\|_2 \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Which means that the Fourier transform $\mathcal{F}$ is bijective and isometric with respect to $\|\,.\,\|_2$ on the subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$.

(*ii*) Since by **remark 2.6(*iv*)** $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can extend $\mathcal{F}$ to an isometric operator on $L^2(\mathbb{R}^n)$, which is called the *Fourier(-Plancherel) transform*. **Proposition 2.14** implies that $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is also an isometric isomorphism, and that we have:

$$\left\langle \mathcal{F}\,f \,|\, \mathcal{F}\,g\right\rangle_{L^2} = \left\langle f \,|\, g\right\rangle_{L^2} \qquad \forall f, g \in L^2(\mathbb{R}^n). \tag{10}$$

(*iii*) It is important to understand, that $\mathcal{F}\,f$ for $f \in L^2(\mathbb{R}^n)$ cannot generally be calculated using the formula for $f \in L^1(\mathbb{R}^n)$: the integral in **(2)** need not exist.

Lemma 2.16 (exchange formula for W^m). Let $f \in W^m(\mathbb{R}^n)$. Then:

$$\mathcal{F}(D^{(\alpha)}f) = i^{|\alpha|}\xi^\alpha\,\mathcal{F}\,f \qquad \forall |\alpha| \le m.$$

Proof. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have:

$$\begin{aligned}\left\langle \mathcal{F}(D^{(\alpha)}f) \,|\, \mathcal{F}\,\varphi\right\rangle &\overset{(10)}{=} \left\langle D^{(\alpha)}f \,|\, \varphi\right\rangle \overset{(*)}{=} (-1)^{|\alpha|}\left\langle f \,|\, D^\alpha\varphi\right\rangle \\ &\overset{(10)}{=} (-1)^{|\alpha|}\left\langle \mathcal{F}\,f \,|\, \mathcal{F}\,D^\alpha\varphi\right\rangle \overset{2.8}{=} (-1)^{|\alpha|}\left\langle \mathcal{F}\,f \,|\, i^{|\alpha|}\xi^\alpha\,\mathcal{F}\,\varphi\right\rangle \\ &= i^{|\alpha|}\int_{\mathbb{R}^n}\xi^\alpha(\mathcal{F}\,f)(\xi)\overline{(\mathcal{F}\,\varphi)(\xi)}d\xi.\end{aligned}$$

In step $(*)$ we used the fact that the "integration by parts-trick" also works for $\varphi \in \mathcal{S}(\mathbb{R}^n)$: More precisely let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $(\varphi_n)_n \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. Then a simple calculation shows

$$\begin{aligned}\left\langle D^{(\alpha)}f \,|\, \varphi\right\rangle &= \left\langle D^{(\alpha)}f \,|\, \lim_{n\to\infty}\varphi_n\right\rangle = \lim_{n\to\infty}\left\langle D^{(\alpha)}f \,|\, \varphi_n\right\rangle \\ &\overset{1.7}{=} \lim_{n\to\infty}(-1)^{|\alpha|}\left\langle f \,|\, D^\alpha\varphi_n\right\rangle = (-1)^{|\alpha|}\left\langle f \,|\, \lim_{n\to\infty}D^\alpha\varphi_n\right\rangle \\ &= (-1)^{|\alpha|}\left\langle f \,|\, D^\alpha\varphi\right\rangle.\end{aligned}$$

Since the Fourier transform is a bijection of the Schwartz space, we have for $h(\xi) := \mathcal{F}(D^{(\alpha)}f)(\xi) - i^{|\alpha|}\xi^\alpha(\mathcal{F}\,f)(\xi)$ in particular that

$$\int_{\mathbb{R}^n} h(\xi)\overline{\psi(\xi)}d\xi = 0 \qquad \forall \psi \in \mathcal{D}(\mathbb{R}^n).$$

We still don't know whether $h \in L^2(\mathbb{R}^n)$. What we know is, that every restriction of h to an open ball $B_R := \left\{x \in \mathbb{R}^n : |x| < R\right\}$ is in $L^2(B_R)$. Thus the previous equation tells us

$$\left\langle h \,|\, \psi \right\rangle_{L^2(B_R)} = 0 \qquad \forall \psi \in \mathcal{D}(B_R).$$

And since $\mathcal{D}(B_R)$ is dense in $L^2(B_R)$, we know that $h = 0$ almost everywhere in B_R for all $R > 0$, so that we have $h = 0$ almost everywhere in $\mathbb{R}^n$—the result we were after. $\square$

Remark 2.17 (the other exchange formula). In **lemma 2.16** we only showed the analogon for weak derivatives of one of the exchange formulas of **lemma 2.8**. Since we only know that $\mathcal{F} f \in L^2(\mathbb{R}^n)$ for $f \in W^m(\mathbb{R}^n)$, the weak derivative $D^{(\alpha)}(\mathcal{F} f)$ need not exist.

Now we have all the tools we need to prove the promised big theorems:

3 Sobolev embedding theorem & Rellich-Kondrachov theorem

Theorem 3.1 (Sobolev embedding theorem). Let $\Omega \subseteq \mathbb{R}^n$ be open, and $m, k \in \mathbb{N}_0$ with $m > k + \frac{n}{2}$. If $f \in W^m(\Omega)$, then there exists a k-times continuously differentiable function on Ω that is equal to f almost everywhere. In other words, the class of equivalent functions $f \in W^m(\Omega)$ has a representative in $\mathcal{C}^k(\Omega)$.

Remark 3.2 (on theorem 3.1). **Theorem 3.1** essentially means that a function that is sufficiently often weakly differentiable is also classically differentiable. Seen differently, decay conditions on a function and its weak derivatives imply smoothness of the function (c.f. also **motivation 2.1**).

Proof (of **theorem 3.1***).* We will only prove this for $\Omega = \mathbb{R}^n$, for arbitrary open subsets $\Omega \subseteq \mathbb{R}^n$ see [Werner; V.2.12]. The idea is to use the proof of **lemma 2.8** where we showed that (read step **(*i*)** backwards)

$$x^\alpha g \in L^1(\mathbb{R}^n)\ \forall |\alpha| \leq k \quad \Rightarrow \quad \mathcal{F} g \in \mathcal{C}^k(\mathbb{R}^n).$$

If we use this result for $g := \mathcal{F} f$, we only need to show that

$$f \in W^m(\mathbb{R}^n),\ m > k + \frac{n}{2} \text{ and } |\alpha| \leq k \quad \Rightarrow \quad \xi^\alpha\, \mathcal{F} f \in L^1(\mathbb{R}^n), \tag{11}$$

since then we have $\mathcal{F}\mathcal{F} f \in \mathcal{C}^k(\mathbb{R}^n)$, which is already nearly the claimed result.
To show **(11)**, notice that **lemma 2.16** already implies $\xi^\alpha\, \mathcal{F} f \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq m$. In particular we have

$$\int_{\mathbb{R}^n} \xi_j^{2m} |\mathcal{F} f(\xi)|^2 d\xi < \infty \quad \forall j = 1, \ldots, n \qquad \text{and} \qquad \int_{\mathbb{R}^n} |\mathcal{F} f(\xi)|^2 d\xi < \infty.$$

Using $1 + \xi^2 = \|(1, \xi_1^2, \ldots, \xi_n^2)\|_1 \leq \|(1, \ldots, 1)\|_p \|(1, \xi_1^2, \ldots, \xi_n^2)\|_m = (n+1)^{\frac{1}{p}} (1 + \xi_1^{2m} + \cdots + \xi_n^{2m})^{\frac{1}{m}}$, (Hölder inequality with $\frac{1}{p} + \frac{1}{m} = 1$) we also obtain

$$\int_{\mathbb{R}^n} (1+\xi^2)^m |\mathcal{F} f(\xi)|^2 d\xi < \infty.$$

Plugging this inequality and $|\xi^\alpha| \le |\xi|^{|\alpha|} \le (1+\xi^2)^{\frac{|\alpha|}{2}}$ together, it follows that for all $|\alpha| \le k$

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi^\alpha \mathcal{F} f(\xi)| d\xi &\le \int_{\mathbb{R}^n} (1+\xi^2)^{\frac{|\alpha|}{2}} |\mathcal{F} f(\xi)| d\xi \le \int_{\mathbb{R}^n} (1+\xi^2)^{\frac{m}{2}} |\mathcal{F} f(\xi)| (1+\xi^2)^{-\frac{m-k}{2}} d\xi \\ &\le \Big(\int_{\mathbb{R}^n} (1+\xi^2)^m |\mathcal{F} f(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \frac{1}{(1+\xi^2)^{m-k}} d\xi\Big)^{\frac{1}{2}} < \infty.\end{aligned}$$

To explain the last step in some more detail we denote by ω_{n-1} the area of the unit sphere in $\mathbb{R}^n$ (see [Otto Forster, *Analysis 3*; §14, Satz 8, Beispiel (14.10)]), then we have

$$\int_{\mathbb{R}^n} \frac{1}{(1+\xi^2)^{m-k}} d\xi = \omega_{n-1} \int_0^\infty \frac{1}{(1+r^2)^{m-k}} r^{n-1} dr < \infty,$$

as long as $2(m-k)-(n-1) > 1$, i.e., $m > k + \frac{n}{2}$. Therefore we really have $\xi^\alpha \mathcal{F} f \in L^1(\mathbb{R}^n)$ which implies, as already mentioned, $\mathcal{F}\mathcal{F} f \in \mathcal{C}^k(\mathbb{R}^n)$.

Using $(\sigma g)(x) := g(-x)$ we now have by **lemma 2.13** that the equation $\sigma g = \mathcal{F}\mathcal{F} g$ holds for all $g \in \mathcal{S}(\mathbb{R}^n)$. Since σ and $\mathcal{F}$ are continuous and $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the equation must also hold for all $g \in L^2(\mathbb{R}^n)$. This shows that f has to be equal to $\sigma(\mathcal{F}\mathcal{F} f) \in \mathcal{C}^k(\mathbb{R}^n)$ almost everywhere, which is the claimed result for $\Omega = \mathbb{R}^n$. ◻

Theorem 3.3 (Rellich-Kondrachov theorem). Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open, then the identity map from $H_0^m(\Omega)$ to $H_0^{m-1}(\Omega)$ is compact.

Proof. First we prove the statement for $m = 1$. To see that the identity map from $H_0^1(\Omega)$ to $H_0^0(\Omega) = L^2(\Omega)$ (see **remark 1.10**) is compact, we need to show that all bounded sequences $(f_j)_j$ in $H_0^1(\Omega)$ have a subsequence that converges with respect to $\|\,.\,\|_2$. Since the space of tesfunctions $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ (by definition), we can assume that $f_j \in \mathcal{D}(\Omega)$. By setting $f_j = 0$ outside its compact support we may extend f_j to a compactly supported function on all of $\mathbb{R}^n$, hence view f_j as an element of $\mathcal{D}(\mathbb{R}^n)$.

Since a bounded sequence in $\mathcal{D}(\Omega)$ is also bounded in $L^2(\mathbb{R}^n)$, we can use [FA2, Theorem 5.26 resp. Corollary 5.28] to obtain the existence of a weakly converging subsequence of $(f_j)_j$ in $L^2(\Omega)$. We again write $(f_j)_j$ for this sequence. The next step is to show its Cauchy property—then we're done since $L^2(\Omega)$ is of course complete.

Using the Plancherel equality **(10)** we obtain

$$\begin{aligned}\|f_k - f_j\|_2^2 = \|\mathcal{F} f_k - \mathcal{F} f_j\|_2^2 &= \int_{\mathbb{R}^n} |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi \\ &= \int_{|\xi|\le R} |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi + \int_{|\xi|>R} |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi. \end{aligned} \tag{12}$$

First we'll show that the second integral can be made small by choosing an appropriate $R > 0$. Using the exchange formula of **lemma 2.16** we have for $l = 1, \dots, n$ and $f \in W^1(\mathbb{R}^n)$ that

$$\|\xi_l \mathcal{F} f\|_2 \overset{2.16}{=\!=} \Big\|\mathcal{F}\Big(\frac{\partial}{\partial x_l} f\Big)\Big\|_2 \overset{(10)}{=\!=} \Big\|\frac{\partial}{\partial x_l} f\Big\|_2 \le \|f\|_{W^1(\mathbb{R}^n)} \tag{13}$$

(here the derivatives are weak derivatives). This means that the sequence $(|\xi|\,\mathcal{F} f_j)_j$ is bounded in $L^2(\mathbb{R}^n)$, so for all $\varepsilon > 0$ we may find $R > 0$ such that

$$\int_{|\xi|>R} |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi \le \frac{1}{R^2} \underbrace{\int_{|\xi|>R} |\xi|^2 |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi}_{<c, \text{ by (13)}} < \varepsilon \qquad \forall j, k \in \mathbb{N}.$$

Now we have to deal with the first integral in (12). Set $e_\xi(x) := e^{ix\xi}$ for $x \in \Omega$. Since Ω is bounded, we have $e_\xi \in L^2(\Omega)$. Thus

$$f \mapsto \left\langle f \,|\, e_\xi \right\rangle = \int_\Omega f(x) e^{-ix\xi} dx = \mathcal{F}(f\chi_\Omega)(\xi)$$

defines a continuous functional on $L^2(\Omega)$. Since $(f_j)_j$ converges weakly in $L^2(\Omega)$ we obtain that $((\mathcal{F} f_j)(\xi))_j$ is convergent for all $\xi \in \mathbb{R}^n$. Using this we want to conclude that

$$\int_{|\xi|\le R} |\mathcal{F} f_k(\xi) - \mathcal{F} f_j(\xi)|^2 d\xi \to 0 \qquad (j, k \to \infty).$$

This follows using dominated convergence: the integrand is dominated by $\sup_j \|\mathcal{F} f_j\|_\infty$ $\big($by [FA1, Satz 1.34(iii)] the sequence $(f_j)_j$ is also bounded in $L^1(\Omega)$, which allows us to use proposition 2.3 to obtain that $(\mathcal{F} f_j)_j$ is bounded with respect to $\|\,.\,\|_\infty\big)$. This proves the statement for $m = 1$.

Now let $m > 1$ and $(f_j)_j$ a bounded sequence in $H_0^m(\Omega)$, then all of the sequences $(D^{(\alpha)} f_j)_j$ for $0 \le |\alpha| \le m-1$ are bounded in $H_0^1(\Omega)$. Therefore, by the above, they contain L_2-Cauchy–subsequences. Applying this argument to all multi–indices we obtain a subsequence of $(f_j)_j$ that converges in $H_0^{m-1}(\Omega)$ since $H_0^{m-1}(\Omega)$ is complete (by definition). $\square$

Remark 3.4 (description of W^m using $\mathcal{F}$). Lemma 2.16 implies the following characterization:

$$W^m(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{\frac{m}{2}} \mathcal{F} f \in L^2(\mathbb{R}^n) \right\}.$$

$\big[f \in W^m \Leftrightarrow D^{(\alpha)} f \in L^2\ \forall |\alpha| \le m \Leftrightarrow \mathcal{F}(D^{(\alpha)} f) \in L^2\ \forall |\alpha| \le m \Leftrightarrow \xi^\alpha \mathcal{F} f \in L^2\ \forall |\alpha| \le m \Leftrightarrow (1 + |\xi|^2)^{\frac{|\alpha|}{2}} \mathcal{F} f \in L^2\ \forall |\alpha| \le m\big]$

This characterization would allow us to extend the definition of the Sobolev spaces W^m to positive real exponents $m \in \mathbb{R}^+$ [if we also want to allow negative m we have to use tempered distributions].