Problem Session: Basic Topology

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Exercises no. 31, 33, 36–38

31. *Properties of the closure (cf. Lecture 2.40).*
Prove Proposition 2.40 from the lecture, that is, prove that the closure of an arbitrary subset A of a topological space $(X, \mathcal{O})$ has the following properties
   - $\bar{\emptyset} = \emptyset$, $\bar{X} = X$
   - $A \subset B \;\Rightarrow\; \bar{A} \subset \bar{B}$
   - $\overline{A \cup B} = \bar{A} \cup \bar{B}$
   - A is closed if and only if $\bar{A} = A$.
   - $\overline{\bar{A}} = \bar{A}$

33. *Boundary and closure of ε-balls in discrete metric spaces.*
Let X be a set containing at least two distinct points and let d be the discrete metric on X.
   (i) Let $\varepsilon > 0$ and $x \in X$. Compute the sets $B_\varepsilon(x)$, $S_\varepsilon(x)$ and $K_\varepsilon(x)$ (for the respective definitions see Exercise 32.)
   (ii) Prove that d induces (cf. Lecture Ex. 2.4(i)) the discrete topology on X.
   (iii) Compare the set $S_1(x)$ with $\partial B_1(x)$ and $K_1(x)$ with $\overline{B_1(x)}$. Discuss the differences between the present situation and the well-known picture in $(\mathbb{R}^n, d_2)$!

36. *Convergence in simple topological spaces.*
Which nets converge with respect to
   (i) the trivial topology ($\mathcal{O} := \{\emptyset, X\}$, also called the indiscrete topology) and
   (ii) the discrete topology?

37. *Closure via nets (cf. Lecture, 3.10).*
Prove 3.10(ii) from the lecture course, that is prove that

$$\bar{A} = \{x \in X |\ \text{ there exists a net } (x_\lambda)_\lambda \text{ in } A : x_\lambda \to x\}.$$

38. *The topology of pointwise convergence (cf. Lecture, Remark 1.18(ii)).*
We consider the (real) vector space of real-valued functions on $\mathbb{R}$, i.e.,

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$$

with the toplogy $\mathcal{O}$ induced by the sub-basis (cf. Lecture, 2.13.) $S_{t,a,b}$, where $t, a, b \in \mathbb{R}$ with $a \le b$

$$S_{t,a,b} := \{f \in \mathcal{F} |\ a < f(t) < b\}.$$

   (i) Prior to seriously starting to work on items (ii)–(v) answer the following question: What is the purpose of this exercise?
   (ii) Justify the name "topology of pointwise convergence" for $\mathcal{O}$ by showing that a sequence of functions $(f_n)_n$ converges pointwise if and only if it converges in $(\mathcal{F}, \mathcal{O})$.
   (iii) Prove that the constant function $f(x) = 1\ \forall x \in \mathbb{R}$ lies in the closure of the set

$$A := \{f \in \mathcal{F} |\ f(x) \neq 0 \text{ for only finitely many } x\}.$$

   (iv) Show that there is no sequence $(f_n)_n$ in A that converges to f. (*Hint:* Let C_n be the (finite!) set of all $x \in \mathbb{R}$ such that $f_n(x) \neq 0$ and consider the neighborhood $S_{t,1/2,3/2}$ of f for some $t \notin C_n\ \forall n$.)
   (v) Prove that $(\mathcal{F}, \mathcal{O})$ is not first countable hence not metrizable. (*Hint:* Use the characterization of the closure in metric spaces analogous to Lecture, 3.20(ii).)
   (vi) For good measure, construct a net in A that converges to f. (Such a net exists by Lecture, 3.10(ii)!) (*Hint:* Consider $\Lambda = \{M \subseteq \mathbb{R} | M \ \text{ finite}\}$ and f_Λ the characteristic function of the set Λ.)