0.1. Introducing the Theme

(i) Nomenclature: *Functional analysis* may be described as that part of analysis where the power of topology is used to study function spaces and operators between them.

In *normed* functional analysis the topology of the space is given by a norm, i.e., one studies normed vector spaces $(E, \|\ \ \|)$. The norm in turn is used to define a metric via the (usual) assignment $d(x, y) := \|x - y\|$. Then one has at hand all the notions of metric spaces (e.g. Cauchy sequence, uniform convergence) and topology (e.g. continuity, convergence, compactness).

In *normed, linear* functional analysis on then studies linear operators[1] on normed vector spaces and, in particular, derives the following main results:

- *Hahn-Banach Thm.:* Every bounded, linear functional on a subspace of a normed vector space can be extended to the entire space preserving its norm.

  This is the starting point for a rich theory of linear functionals, since the theorem guarantees the existence of "many" of these.

- *Uniform boundedness principle (Banach-Steinhaus Thm.):* Every family of pointwise bounded operators from a Banach space to a normed vector space is already uniformly bounded.

  This, in particular, implies the boundedness of pointwise limits of sequences of bounded operators (provided the domain is a Banach space).

- *Open mapping and Closed Graph Thms. with Banach's isomorphism Thm.:* The two former statements "Surjective bounded Operators of Banach spaces are open." and "Any closed, linear map of Banach spaces is continuous." are actually equivalent and imply the third one, which guarantees that any bijective, bounded operator of Banach spaces is an isomorphism.

More advanced topics are spectral theory of (compact) operators on Banach spaces. In particular, in this area, stronger results are available if the norm is induced by a scalar product via $\|x\| := \langle x|x\rangle^{1/2}$, i.e., in Hilbert spaces. There is a very rich spectral theory of (even unbounded) operators on Hilbert spaces.

(ii) Norms are not enough: On the other hand it turns out that many important spaces of analysis cannot be turned into normed vector spaces in a reasonable way. Here the phrase "in a reasonable way" refers to the following line of thought: Many of these spaces are naturally equipped with a notion of convergence (take e.g. (1) $\mathcal{C}(\mathbb{R})$ with uniform convergence on compact sets, or (2) $\mathcal{C}^\infty(K)$, with $K \subseteq \mathbb{R}$ compact and uniform convergence of all derivatives) and one can prove either that this notion of convergence cannot be induced by a norm at all (e.g.˙ case (1) above), or if the space is viewed as a subspace of a normed vector space it is not complete (e.g. case (2) above with $\mathcal{C}^\infty(K) \subseteq \mathcal{C}^k(K)$ for finite k). (Recall that if a space is not complete then there exist Cauchy sequences that do not converge within that space—a feature which is highly unfavorable from the analytic point of view, where one often relies on approximation procedures.)

Some of these spaces (in particular (1) and (2) above) can be given the structure of a complete metric space and it turns out that their topology can be generated by a *countable*

[1] Since we will exclusively deal with linear operators we will just call them operators.

family of semi-norms. Such spaces are called *Fréchet space.* Some further examples are

$$\mathcal{C}(\Omega),\ \mathcal{C}^k(\Omega),\ \mathcal{C}^\infty(K),\ \mathcal{C}^\infty(\Omega),\ \mathcal{S}(\mathbb{R}^n) \quad (\Omega \subseteq \mathbb{R}^n \text{ open, } K \subseteq \mathbb{R}^n \text{ compact, } k \in \mathbb{N}).$$

However, some important spaces are even "wilder" in the sense that they cannot even be turned into Fréchet spaces in a reasonable (see above) way. In particular, their topology is not induced by a *countable* family of semi-norms. This applies e.g. to many of the spaces used in distribution theory, most prominently the space $\mathcal{D}(\mathbb{R}^n)$ of test functions. More precisely,

$$\mathcal{D}(\mathbb{R}^n) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) :\ \operatorname{supp}(\varphi) \text{ compact}\}$$

consits of all smooth functions which vanish outside some bounded subset of Ω and is naturally equipped with the following notion of convergence, called uniform convergence in all derivatives on a fixed compact set

$$\varphi_n \to \varphi \ :\Leftrightarrow \ \begin{array}{l} (1)\ \exists K \text{ compact with } \operatorname{supp}(\varphi_n) \subseteq K\ \forall n \text{ and,} \\ (2)\ \|\partial^\alpha \varphi_n - \partial^\alpha \varphi\|_\infty \to 0 \quad \forall \alpha \in \mathbb{N}_0^n. \end{array}$$

So let's have a look at the "other end" of the scale of spaces we are dealing with.

(iii) Toplogical vector spaces: First recall that in a normed vector space the vector space operations (of addition and multiplication with scalars) are continuous. More generally, a vector space which at the same time carries a topology in such a way the two structures are compatible in the sense that the vector space operations are continuous, is called a *topological vector space.* There is in fact a theory of these spaces (as you will see during the course) but it turns out to be too general to prove several important results one would like to have from the analytical point of view. E.g. the (topological) dual space of a topological vector space can be trivial although the space is not, depriving one from using functionals in the fruitful way one is used to in the realm of normed vector spaces.

(iv) Locally convex vector spaces: It turns out that a sufficiently wide class of topological vector spaces is singled out by the condition

every point has a fundamental system of neighborhoods consisting of convex sets.

This class is called *locally convex vector spaces* and is large enough to contain Fréchet spaces as well as spaces such as $\mathcal{D}$. On the other hand the existence of a convex base of neighborhoods of the origin—which in fact is equivalent to the above condition— is strong enough for the Hahn-Banach theorem to hold, yielding a sufficiently rich theory of continuous linear functionals.

Alternatively locally convex vector spaces can be characterized as those topological vector spaces that allow for their topology to be generated by a (not necessarily countable) *family of semi-norms.* This remark clarifies the relation between locally convex and Fréchet spaces and moreover tells us that the main objective of this course is the study of

topological vector spaces, whose topology is induced by a family of semi-norms.