Perceptions of continuity of pre-service teachers in Austria

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This paper discusses normative basic ideas (Grundvorstellungen) concerning the analytical concept of continuity and examines perceptions of continuity pre-service mathematics teachers develop before and after attending a didactical lecture called "School mathematics Analysis". In this course, the basic ideas of fundamental analytic notions are discussed in order to handle them in a mathematically correct and didactically appropriate way. To evaluate this intervention, the second author analyzed written statements on continuity of the pre-service teachers produced at the beginning and at the end of this course. These statements were classified in respect of their coherence with the (normative) basic ideas. Moreover, the connections with additional data of the pre-service teachers were analyzed and the answers of the two questionnaires were compared. An increase of the number of given basic ideas from the first to the second questionnaire can be determined.

Keywords: Calculus, continuity, fundamental concepts, basic ideas, correlations.

Introduction and problem outline

The concept of continuity is the "*bête noire* of analysis" (Tall & Vinner, 1981, p. 164) and it often plays a minor role in school curricula and in teacher training (Greefrath et al., 2016). In fact, meaningful and explicit calculations in this area are rare, which sharply contrasts other main concepts of analysis, as the limit, the derivative, and the integral. In addition, the differential quotient is the (function)-limit of the difference quotient, and so there is a bridge between the limit of a function and the derivate without even touching the notion of continuity (Bezuidenhout, 2001). However, for the perspective of the continuous extension of the difference quotient, it is necessary, to know the concept of continuity. Therefore, the Austrian mathematics curriculum in high schools (grades 9 to 12, i.e., students aged 15 to 18) lists "knowing and being able to explain the concept of continuity" and "knowing […] the relationship between differentiability and continuity" (RIS, 2023). Bezuidenhout (2001) covers the second aspect in a study. Here, we focus on the first issue. For an effective teaching of the continuity concept, it is essential that pre-service teachers hold proper and well-structured ideas of the notion. Propaedeutic ideas or ideas acquired in an everyday context can support the learning and understanding of the formal ε-δ-definition, but they can also be hampering if they are not suitable:

> Meanwhile, the quantifiers "for all", "there exists", which occur in epsilon-delta-definitions, have their own meanings in everyday language subtly different from those encountered in the definition of the limit concept. From such beginning arise conceptual obstacles which may cause serious difficulties. (Cornu, 1991, p. 153)

To avoid any misconceptions as mentioned by Vela (2011) and Schäfer (2011), reliable ideas and appropriate mental representations of fundamental mathematical concepts are an indispensable prerequisite for mathematics teachers to conduct high quality lessons that are, in particular, meaningful for their students (Bezuidenhout, 2001; Sichel, 2015; Vom Hofe & Blum, 2016).

The concept of *Grundvorstellungen* in the German tradition of subject-matter didactics "describes the relationships between mathematical content and the phenomenon of individual concept formation." (Vom Hofe & Blum, 2016, p. S230). Hence, it provides a conceptualization of mental representations of mathematical contents. Vom Hofe and Blum (2016) distinguish two aspects in the use of basic ideas: normative and descriptive ones. The first aspect provides an educational guideline towards a specific goal and sees basic ideas as adequate interpretations of attributes of a mathematical notion. The second aspect describes individual images and explanatory models of learners, which are only more or less adequate characterizations of the mathematical notion.

In their reference work, Greefrath et al. (2016) list the following three basic ideas attributed to the notion of a continuous real function: (1) Jump-freeness (JF): The graph of a continuous function has no jumps. (2) Representability (RE): The function graph can be drawn in one go without a break. (3) Predictability (PR): Small changes in the argument only lead to small changes of the values.

Clearly neither of these basic ideas immediately corresponds to a mathematically correct definition, so that they can only partly be viewed as normative. E.g. possessing no jump is only a necessary but not a sufficient condition for continuity. Other types of discontinuities are rather exotic from a student's point of view. For example, the function given by the term $\sin\left(\frac{1}{x}\right)$ cannot be continuously extended to $x = 0$, no matter which value is prescribed at $x = 0$. Hence, for our purpose, we consider such a wild oscillation as a "jump", too. Turning to the third basic idea, one needs sufficient experience to turn the PR basic idea into a satisfactory mathematical reformulation in terms of the ε-δ-definition. Finally, RE, again, is only aimed at ruling out jumps but fails to provide a meaningful understanding of the continuity of functions possessing a graph of infinite length over a finite interval, e. g. the function defined by $x \cdot \sin\frac{1}{x}$. Due to their mixed qualities, we can assume that parts of these three basic ideas are elements of the concept image (Tall & Vinner, 1981) of students who are faced with continuity (Hanke & Schäfer, 2017). To gain insights into the mental representations held on continuity by pre-service mathematics teachers, we investigate the following research questions:

1. Which basic ideas on continuity are present in pre-service teachers of secondary mathematics?
2. How do these basic ideas change through the attendance of a didactically oriented lecture course (that follows a first lecture course in analysis)?

An empirical study on pre-service teachers' basic ideas of continuity

Setting of the study and methodology

The study in question collects and classifies pre-service teachers' written statements on their ideas on the continuity of a real function, before as well as after a didactical intervention by the third author. It is comprised of the lecture course "School mathematics Analysis" (two weakly hours) that was accompanied by a tutorial (one weakly hour). This course followed a classical first course on one-dimensional analysis as it is common in German-speaking countries, and the majority of pre-service teachers took this course in the semester preceding the intervention. The later discussed the conception of basic ideas and made them explicit for the notions of a real sequence and its limit, the derivative of a function and the integral. Continuity on the other hand played only a minor role and was discussed solely as a warm up for differentiability. In that sense, the statements collected during our study have not been influenced by explicitly teaching the corresponding basic ideas on continuity and so they are clearly not the results of a teaching-to-the-test-scenario. However, the pre-service teachers' views on continuity may have changed through explicit discussions of the concept as a condition in central theorems of analysis. The pre-service teachers (mainly in the 5th semester) were invited to fill in a questionnaire in a pre-test-post-test setting, in which the first survey took place in the first, and the second in the final unit of the lecture "School mathematics Analysis" held in the winter term 2018/19. These points in time were chosen so that survey (1) was taken after the pre-service teachers' subject matter training in analysis but prior to their school mathematics training and survey (2) after the school mathematics training. In the first part of the questionnaire, we collected ideas of pre-service teachers on mathematical core notions of (one-dimensional) analysis in the form of written statements. In the second part, which was only used in survey (1), we also collected pre-service teachers' written answers on subject matter problems. In addition, in survey (2) we recorded the grade achieved in the subject matter course, the experience in working as a teacher (in school or attending a traineeship, which is part of their curriculum), the experience in giving private lessons, and the school type the pre-service teachers plan to teach in after graduation. We were able to compile a connected random sample with $n = 59$ subjects who had completed both questionnaires.

One item of both surveys that addressed the pre-service teachers' ideas on continuity was to continue the sentence "Under a *continuous function* I imagine …" (emphasis in the original, in German: "Unter einer *stetigen Funktion* stelle ich mir vor …"). Therefore, we consider the pre-service teachers' statements similar to communicative simulacra as defined by Hanke and Schäfer (2017).

To classify the answers, we first assigned the statements to the three basic ideas. In a second step, we evaluated the quality of the statements by comparing their formulations to the respective basic idea(s). We distinguished five quality categories: 0 means that the (normative) basic idea is not present. 1 means that the (normative) basic idea is recognizable, but unspecific or naively uttered. 2 means the (normative) basic idea is weakly pronounced or imprecisely formulated. 3 means that the (normative) basic idea is present, but not correctly formulated. Finally, 4 means that the (normative) basic idea is clearly and adequately expressed. For example, the answer "a graph that you can draw without putting down the pencil" belongs to RE and has quality 4, the statement "a function without large jumps or oscillations" addresses JF with quality 1. In our research group, prime examples were assigned to these quality levels to get a satisfying inter-rater reliability. If the assignment of an answer was subtle, we would discuss this statement until an agreement was reached. An evaluation of half the answers by two raters provides a substantial inter-rater reliability (Cohen's $\kappa = 0.72$).

Moreover, we calculated correlations between these quality levels and other relevant data collected in the course of the study, see above. Due to the ordinal nature of the data, we use the Kendall rank correlation coefficient (because of the large number of ties between the ranks).

Another question is if pre-service teachers are able to use the concept of continuity being faced with concrete exercises. In the first questionnaire, the statement "The function f with $f(x) = \frac{1}{x}$ $(x \neq 0)$ is discontinuous at $x_0 = 0$." has to be evaluated about its correctness. Is it true or false?

Finally yet importantly we additionally provide four functions in the first questionnaire: (1) $f\colon \mathbb{R} \mapsto [0,\infty),\ f(x) = |x-1|$, (2) $g\colon \mathbb{R} \mapsto \{-1,0,1,\},\ g(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$, (3) $h\colon \mathbb{R} \mapsto \mathbb{R}, h(x) = x \cdot \sin(x)$ and (4) $i\colon \mathbb{R} \mapsto \mathbb{R},\ i(x) = \begin{cases} x \cdot \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. The function graphs were also shown. For each function the pre-service teachers had to decide whether the function is (a) discontinuous at least at one point of its domain or (b) continuous, but not differentiable at least at one point of its domain or (c) differentiable on $\mathbb{R}$ (see also Hanke & Schäfer, 2017). In a broader sense, this part of our research can be interpreted as a contribution to investigating the concept usage (Moore, 1994).

Results

The statistical evaluation of our data was done by SPSS and R. First, we counted the number of basic ideas in each pre-service teacher statement in both surveys (Table 1). The results are in accordance with the data in Hanke and Schäfer (2017). The number of no answers decreases, and the number of recognizable basic ideas increases (a little). More interesting is the average quality of the statements: Table 2.

Table 1: Absolute frequencies of mentions of the basic ideas

	none	one	two	three
Survey (1)	9	43	7	0
Survey (2)	7	39	13	0

Table 2: Absolute frequencies of the quality levels

	1	2	3	4
Survey (1)	3	4	22	28
Survey (2)	4	8	29	24

The median in both cases is 3, the quartile deviation is 1 both in survey (1) and survey (2). A difference occurs in the distribution of the observed basic ideas: Table 3.

Table 3: Absolute frequencies of the quality levels depending on basic ideas

	0	1	2	3	4	*Σ*
Jump-freeness (1)	33	1	2	10	13	**26**
Representability (1)	31	2	2	9	15	**28**
Predictability (1)	56	0	0	3	0	**3**
Jump-freeness (2)	24	1	6	19	9	**35**
Representability (2)	47	1	0	1	10	**12**
Predictability (2)	41	2	2	9	5	**18**

The sum in the last column of Table 3 refers only to the items except the first ones (“0”). A chi-square test of homogeneity concerning exclusively the cumulative distributions, which can be found in the last column, shows a highly significant difference between the distributions of the pre-service teachers’ statements in the first and in the second survey ($p < 0.001$).

The basic ideas JF and PR are emphasised after the intervention, whereas the frequency of RE is declining. However, in a relative sense the quality of the statements addressing the RE increases strongly. In survey (1), the basic idea JF occurs less than RE, but in survey (2) almost three times as often, since not only the occurrence of RE has decreased, but also that of JF has strongly increased. Although in survey (2) the frequency increased by nine for JF, the number of statements with expression 4 fell from 13 to nine, while those with expression 3 increased strongly from ten to 19. Excluding the first column, no differences of average quality are significant, neither for all basic ideas taken together nor for each of them individually.

Additionally taking into account the first column in Table 3 (“0”), we find two significant differences: one in the RE ($p < 0.01$, Wilcoxon signed rank test for two matched samples) and the other one in the PR ($p < 0.001$). However, the average quality of the statements belonging to the RE decreases (Cohen’s $d = -0.4988$, which means a small effect. Here, we consider the variables as quasi-continuous (Bamberg et al., 2012).), and the average quality of the answers according to the PR increases (Cohen’s $d = 0.658$, this means a medium effect). This suggests that discussing the statement “A small wiggling at x_0 leads to a small wiggling at $f(x_0)$” in the lecture has a positive influence on the acceptance of this basic idea.

We observed significant correlations between the quality of the statements and the grade in the subject matter lecture and at the related tutorial, respectively. In survey (1) this holds for the statements in general excluding “0” ($p = 0.03$, Kendall’s $\tau = 0.2795$, this means a weak – positive – association) and the grades reached at the tutorial. We conjecture that for passing the subject matter tutorial, it was mainly necessary to handle definitions and theorems, but building up an adequate idea of continuity was not in the focus. In survey (2) we detected a weak negative correlation between quality of statements according to JF (including “0”) and the grade in the tutorial ($p = 0.04$, $\tau = -0.2595$). We have expected such a result. By linear regression analysis (For this purpose, we interpret the dependent variables of quality of the perceptions as metrically scaled. The independent variables x stand for the collected data of the pre-service teachers. We interpret them also as metrically scaled.) we obtain $f(x) = -0.61 \cdot x + 2.7$ with a significant slope ($p = 0.04$) and a highly significant intercept ($p < 0.001$). A positive correlation appears in survey (2): between the quality of statements assigned to RE (including “0”) and the grade achieved at the subject matter lecture ($p = 0.03$, $\tau = 0.2960$, this means also a weak association). Therefore, after the intervention, the descriptive basic idea of RE persists stronger for those who received a mediocre or bad grade.

To finish the topic of correlations between qualities of statements and the grades, we present the distribution of the grades of the subjects whose answers were assigned to two distinct basic ideas: Table 4. (The values “3.5” arose from the answers “3-4” for the lecture grade and “2-3” for the tutorial grade on a questionnaire. Half of these answers were assigned to one grade and half to the other.)

Table 4: Absolute frequencies of pre-service teachers who give two basic ideas of continuity

Grades	1	2	3	4
Lecture	5	1	3.5	3.5
Tutorial	8	3.5	3.5	0

We can refute the hypothesis of uniform distribution for the tutorial results ($p < 0.05$, quasi-exact chi-square-test for goodness of fit with Monte Carlo Methods), but not for the distribution concerning the lecture grades. Moreover, there is no significant tendency recognizable. Note that the reported frequencies are very small, although they were collected from both surveys.

Next, we consider the previous experience of the probands in working as a math teacher. We created an ordinal scale from "0" (no experience) and "1" and "2" (experience comes from attending different mandatory traineeships in behalf of their curriculum) to "3" (working as a teacher at a regular school with a special contract). We observe a significant negative correlation between this factor and the quality of statements according to PR including "0" in survey (2) ($p = 0.03$, $\tau = -0.2603$, this means a weak association). This can possibly be explained by the fact that this basic idea of continuity plays no role in mathematics education at school (Tall & Vinner, 1981). A linear regression analysis provides the function equation $f(x) = -0.54 \cdot x + 1.61$ with a significant slope ($p = 0.02$) and a highly significant intercept ($p < 0.001$).

We have found no further significant correlations between the quality of the statements and the grades received on the subject matter lecture and the tutorial, as well as teaching experience.

The final context we discuss concerns the plan, in which school forms the pre-service teachers want to teach in the future. We compiled another ordinal scale from "1" (only secondary level 1) and "2" (both, secondary level 1 and 2) to "3" (only secondary level 2). In survey (1), we detected a significant correlation between this individual attribute and the quality of the statements concerning the RE with "0" ($p = 0.02$, $\tau = -0.2827$). This weak correlation is negative. Our interpretation is that the imagination of a function graph, which can be drawn in one go, is more attractive for pre-service teachers who want to teach at lower school levels because of its low(er) level of abstraction. We obtain by calculating a linear regression the significant slope -0.90 ($p = 0.04$) and the highly significant intercept 3.33 ($p < 0.001$). We found another significant correlation in survey (2) between the quality of statements according to PR (excluding "0") and the school wish ($p = 0.02$, $\tau = -0.52115$). It is a strong negative association; therefore, we can conclude that pre-service teachers who only want to teach at the secondary level 2 are less aware of this basic idea. The phenomena of PR can also be observed also in everyday life experiences (a tiny cause brings only little effects with it), maybe, therefore these pre-service teachers do not value it.

Finally, we turn to the pre-service teachers' answers on the concrete exercises: 208 of 216 probands evaluate the statement "The function *f* with $f(x) = \frac{1}{x}$ $(x \neq 0)$ is discontinuous at $x_0 = 0$.", given in the first questionnaire. Only 70 answers "false". That are 34% of the given responses. Maybe the others thought due to $f(0)$ is not defined, the pole at $x_0 = 0$ is also a point of discontinuity of *f*. Takači et al. ask, "Are the functions [...] $g(x) = \frac{1}{x}$, $x \neq 0$, with its graph; [...] continuous over their domain and why? […]" (2006, p. 785). 23.8% of talented (in mathematics) grammar school students said "continuous", the rest "not continuous" (Takači et al., 2006). Tall and Vinner (1981) report a similar result: only six out of 41 first year mathematics students recognized the continuity of g.

Further, 208 of 216 pre-service teachers characterize the function *f* with $f(x) = |x-1|$ (see above), 168 (81%) correctly. 209 probands evaluated the function g (see above), and 147 did it in the right manner (70%). Almost the same percentage was observed concerning the function *h* (see above). The last function *i* was the most difficult one (see above): only 36% (71 of 195) picked the correct choice.

Discussion

First, as is seen from the results collected in Table 1, the number of basic ideas of continuity mentioned by pre-service teachers increases after the intervention. Assuming a cardinal nature of the quality level variable, we also detect a small increase of the average quality of the statements after the intervention (including the code "0"). Excluding the no-answers, there is almost no difference (there is a small decrease) in the arithmetic means of the quality before and after the intervention (Table 2). If we focus on each basic idea separately, the frequency of statements assigned to RE decreases, but their quality increases (not significantly). Counterbalancing this result, the frequencies of the statements according to the other two basic ideas both increase. However, their average qualities decrease slightly. The idea of PR was seemingly awaked by the School Mathematics Analysis-lecture (Hanke & Schäfer, 2017): the number of assignments multiplies by six (Table 3).

One limitation of our study is that the formulation "Under a *continuous function* I imagine …" does not invite the pre-service teachers to mention more than one basic idea. This may explain the small numbers observed in Table 4. A qualitative analysis per student could lead to more insights in detail. This would be a follow-up research possibility.

It is not easy to detect systematics in the significant correlations we found. We think this is due to the significant changes in the subjects' basic ideas, which become visible in the last column of Table 3.

Resume and outlook

The sharp increase in the frequency of the PR idea after the intervention suggests that the probands have developed a good first intuition of continuity (Table 3). Since we have not found any significant correlation with the subject-matter performance, it has to remain open, whether the test persons are also able to cast this first idea into a proper mathematical formulation. However, we consider this an important objective of the teacher training, which, however, likely needs a specific training.

One issue that stands out clearly is the underrepresentation of sequential continuity. This, of course, is a consequence of its absence in the basic ideas found in Greefrath et al. (2016). From our point of view this constitutes a major drawback in teacher education and in the classroom: The fact that a continuous function respects the limit of sequences is relevant in explicit calculations of limits (e.g. when calculating the limit of an exponential as $\lim_{n\to\infty} a^{x_n} = a^{\lim_{n\to\infty} x_n}$ with $a > 0$). Such calculations are often perceived by students as purely algebraic manipulations, and the analytic heart of the matter largely goes unseen. Sequential continuity also allows to discuss the structural property of the notion most vividly expressed via the formula $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$. This point of view, in particular, allows to connect to the concept usage of Moore (1994). Moreover, it could serve to establish relevant and meaningful calculations using continuity in the classroom, which are currently missing, see the introduction. Finally, sequential continuity is a viable tool to detect discontinuities, in many cases by simpler arguments than using the ε-δ-definition, with its heavy dependence on quantifiers.

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