On the impulsive limit of gravitational waves

Proceedings of the Spanish Relativity Meeting

Bilbao, 1999

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Impulsive pp-waves are commonly described either by a distributional space-time metric or, alternatively, by a continuous one. Although both pictures are physically equivalent the transformation relating them clearly has to be discontinuous which causes two basic problems: First, it changes the manifold structure and second the transformation of the metric is ill-defined within classical distribution theory. In this report we briefly review a mathematical framework [1] providing a sensible description of the situation at hand.

1 Impulsive pp-waves

Plane fronted gravitational waves with parallel rays (pp-waves) discovered by Brinkmann already in 1923 [2] are space-time geometries most conveniently characterized by the existence of a covariantly constant null vector field, which allows to write the metric in the form [3]

$$ds^2 \;=\; H du^2 - du dv + dx^2 + dy^2, \tag{1}$$

where $H(u,x,y)$ is a function of the retarded time coordinate u and the transverse Cartesian coordinates x and y. A pp-wave of the form $H = \rho(u) f(x,y)$ is called a *sandwich pp-wave* if ρ is non-vanishing only in some finite region $u_0 \leq u \leq u_1$ of space-time. R. Penrose [4] introduced *impulsive pp-waves* as an idealization of sandwich waves of infinitely short duration (say u_0, $u_1 \to 0$) but still producing a nontrivial effect in the sense that ρ equals the Dirac-δ, i.e., the metric taking the form

$$ds^2 \;=\; f(x,y)\,\delta(u)\,du^2 - du\,dv + dx^2 + dy^2\,. \tag{2}$$

This space-time is hence flat everywhere except for the null hyper plane $u = 0$ where a δ-like impulse is located.

Physically impulsive pp-waves arise—besides as the impulsive limit of sandwich waves—in the *ultrarelativistic limit of black hole space-times* of the Kerr-Newman family [5,6,7] and as multipole solutions of the Weyl family [8]. Moreover, these geometries play an important role in the description of scattering processes at the Planck scale [9].

Of course such a space-time, due to the appearance of a distribution in the metric, is not "really allowable" in the sense of classical general relativity. Also (written in this form) the metric is not even regular in the sense of Geroch and Traschen [10] hence lies outside the "largest reasonable" class of distributionally tractable space-times. However, this special situation is "tame" in the sense that one may calculate the Ricci tensor within classical distribution theory since it is linear in the metric coefficients. Hence the Einstein vacuum equations may be formulated giving $\Delta f = 0$ on the wave surface $u = 0$. Moreover, the geometry of impulsive pp-waves was described recently [11,12] entirely in the distributional picture. The key idea was to use a careful regularization procedure which allowed to handle the nonlinear singular geodesic and geodesic deviation equations in a mathematically satisfactory way.

On the other hand, impulsive pp-waves are frequently described by a different space-time metric which is actually continuous [4,13,14]. In the special case of an impulsive *plane* (linearly polarized) wave, i.e., $f = 1/2(x^2 - y^2)$, it takes the form

$$ds^2 = (1 + u_+)^2 dX^2 + (1 - u_+)^2 dY^2 - du dV \,. \tag{3}$$

where u_+ denotes the "kink" function $u_+ = \theta(u)\, u$. This form of the metric has the advantage that only the curvature tensor involves distributions while the metric can be treated "classically."

Clearly a transformation relating the metrics (2) and (3) cannot even be continuous, hence strictly speaking it changes (even the topological structure of) the manifold. In the case of a plane wave this discontinuous change of variables is given by [4]

$$\begin{aligned} \sqrt{2}\zeta &= X(1 + u_+) + iY(1 - u_+) \\ v &= V + \frac{1}{2}X^2(u_+ + \theta(u)) + \frac{1}{2}Y^2(u_+ - \theta(u)) \\ u &= u \,. \end{aligned} \tag{4}$$

However, the two (mathematically) distinct space-times are equivalent from a physical point of view, i.e., the geodesics and the particle motion agree [15]. More precisely, if one computes the geodesics for the continuous metric and then (formally) applies the discontinuous transformation one obtains exactly the geodesics as derived in the distributional picture [11]. An analogous statement holds true also for the deviation vector field.

Nevertheless, it has to be noted that the actual calculation of the continuous metric (3) from the distributional form (2) as well as the transformation of the geodesics involve undefined nonlinear operations with distributions which are dealt with by using ad hoc multiplication rules; in this case $\theta^2 = \theta$, $\theta u_+ = u_+$, $u_+\delta = 0 = u_+^2\delta$ and $\theta\delta = 1/2\,\delta$. Such a seemingly harmless operational approach to introducing nonlinearities into linear distribution theory may work out in certain cases but in just as many instances it will lead seriously astray. Using the above, i.e., exactly the "rules" appearing in the situation under consideration we get $(\theta\theta)\delta = \theta\delta = 1/2\,\delta \neq 1/4\,\delta = \theta(\theta\delta)$. Examples of the difficulties that may result from such manipulations are found both in mathematics [16] and in physics [17]. Especially in the context of nonlinear differential equations involving generalized functions (in this case: the geodesic equations) it turns out that the method of choice for determining which multiplications work out and which don't is to first regularize the singularities, then carry out the nonlinear operations and, finally, compute the distributional limits, if they exist [18].

2 Mathematical description

Following the ideas outlined above a precise meaning has been given to the beforementioned "physical equivalence," interpreting the discontinuous change of variables as the (distributional) limit of a family of smoothened transformations [1]. The key idea is not to use an arbitrary regularization of the discontinuous transformation but to take into account the following geometrical consideration: the coordinate lines in the new variables defined in (4) are exactly the geodesics of the metric (2) with vanishing initial speed in the $x-, y-$ and $v-$directions.

We shall briefly outline the route taken in the original article [1] and refer for all details to the latter. We start with the distributional form (2) of the metric, which we turn into a sandwich wave by regularizing the δ distribution in a completely general manner, i.e., by replacing it by a *generalized δ-function* ρ_ε which we define by the following set of conditions [18]

$$\begin{aligned} &\text{(a)} \qquad \mathrm{supp}(\rho_\varepsilon) \to \{0\} \quad (\varepsilon \to 0)\,, \\ &\text{(b)} \qquad \int \rho_\varepsilon(x)\, dx \to 1 \quad (\varepsilon \to 0) \ \text{ and} \\ &\text{(c)} \qquad \int |\rho_\varepsilon(x)|\, dx \leq C \quad \text{for small } \varepsilon\,. \end{aligned} \tag{5}$$

Hence we are left with the following (sequence of) sandwich waves $ds_\varepsilon^2 = f(x^i)\,\rho_\varepsilon(u)\,du^2 - du\,dv + \sum_{i=1}^2 (dx^i)^2$. The corresponding geodesic equations are given by

$$\begin{aligned} \ddot{x}^i_\varepsilon(u) &= \frac{1}{2}\,\partial_i f(x^j_\varepsilon(u))\,\rho_\varepsilon(u) \\ \ddot{v}_\varepsilon(u) &= f(x^j_\varepsilon(u))\,\dot{\rho}_\varepsilon(u) \;+\; 2\,\partial_i\, f(x^j_\varepsilon(u))\;\dot{x}^i_\varepsilon(u)\,\rho_\varepsilon(u), \end{aligned} \tag{6}$$

where we have used u as an affine parameter. One may show [12] that these equations are uniquely solvable (even) within the framework of Colombeau's algebras of generalized functions [19,18], which provide a suitable setting for singular differential equations.

Since we are only interested in geodesics with the special initial conditions $x^i_\varepsilon(-1) = x^i_0$, $v_\varepsilon(-1) = v_0$ and $\dot{x}^i_\varepsilon(-1) = 0 = \dot{v}_\varepsilon(-1)$, we adopt the notation $x^i_\varepsilon(x^i_0, u)$ and $v_\varepsilon(v_0, x^i_0, u)$ for these geodesics with vanishing initial speed. The unique solution was shown to possess the following distributional limit, *independent of the concrete shape of the regularization* [12]

$$\begin{aligned} x^i_\varepsilon(x^j_0, u) &\to x^i_0 + \frac{1}{2}\,\partial_i f(x^j_0)\,u_+ \\ v_\varepsilon(v_0, x^j_0, u) &\to v_0 + f(x^j_0)\,\theta(u) + \frac{1}{4}\partial_i f(x^j_0)\,\partial^i f(x^j_0)\,u_+\,. \end{aligned} \tag{7}$$

Next we carry out a transformation analogous to (4), i.e., $T_\varepsilon : (u, v, x^i) \to (u, V, X^i)$ (where $X^i = (X, Y)$) given implicitly by

$$\begin{aligned} x^i &= x^i_\varepsilon(X^j, u) \\ v &= v_\varepsilon(V, X^j, u) \end{aligned} \tag{8}$$

which in the limit $\varepsilon \to 0$ and in the special case of a plane wave actually reduces to (4). It may indeed be shown that T_ε is a diffeomorphism in a space-time region which contains the shock-hypersurface $u = 0$ and is independent of ε. In the new coordinates (u, V, X^i) the metric tensor takes the form

$$ds_\varepsilon^2 = -dudV + (2\sum_{i=1}^{2}(\dot{x}^i_\varepsilon \partial_j x^i_\varepsilon) - \partial_j v_\varepsilon)dudX^j + \sum_{i=1}^{2}(\partial_j x^i_\varepsilon dX^j)^2\,, \tag{9}$$

where $\dot{}$ and ∂_i denote derivatives with respect to u and X^i, respectively. Finally, we may calculate the weak limit of (9) as

$$ds_\varepsilon^2 \xrightarrow{\mathcal{D}'} -dudV + (\delta_{ij} + \frac{1}{2}u_+\partial_{ij} f)\,(\delta^j_k + \frac{1}{2}u_+\partial^j_k f)\,dX^i dX^k\,, \tag{10}$$

which is precisely the continuous or so-called Rosen form of a general impulsive pp-wave [13].

3 Conclusions and outlook

We may sum up our main result in the following diagram

$$\begin{tikzcd}
ds^2 = f(x^i)\delta(u)du^2 - dudv + \sum(dx^i)^2 \arrow[r, "\text{reg.}"] \arrow[d, "??"] & ds^2_\varepsilon = f(x^i)\rho_\varepsilon(u)du^2 - dudv + \sum(dx^i)^2 \arrow[d, "T_\varepsilon"] \\
(10) = \lim\limits_{\varepsilon\to 0} ds^2_\varepsilon & ds^2_\varepsilon = -dudV + (2\sum\limits_{i=1}^{2}(\dot{x}^i_\varepsilon \partial_j x^i_\varepsilon) - \partial_j v_\varepsilon) du dX^j + \sum\limits_{i=1}^{2}(\partial_j x^i_\varepsilon dX^j)^2 \arrow[l, "\mathcal{D}'\text{-limit}"']
\end{tikzcd}$$

The left hand arrow represents the "pullback" of the distributional metric (2) under the discontinuous coordinate change

$$\begin{aligned}
x^i &= X^i + \frac{1}{2}\partial_i f(X^j)u_+ \\
v &= V + f(X^j)\theta(u) + \frac{1}{4}\partial_i f(X^j)\partial^i f(X^j)u_+ \,.
\end{aligned} \tag{11}$$

Note that (11) is the generalization of the Penrose transformation (4) to the case of an arbitrary wave profile f [13]. Although undefined within Schwartz distribution theory, we have interpreted this transformation consistently as the composition of a regularization procedure, a smooth transformation of the regularized metric and a distributional limit. Finally, the family of smoothened transformations constitutes a generalized coordinate transformation in the sense of Colombeau, whose macroscopic (i.e., distributional) aspect is precisely given by the (generalized) Penrose transformation (11).

Physically speaking our regularization approach consists in viewing the impulsive wave as a limiting case of a sandwich wave with an arbitrarily regularized wave profile. From this point of view the two forms of the impulsive metric arise as the (distributional) limits of the sandwich wave in different coordinate systems. It is a well known fact [20] that given a one-parameter family of space-times one may obtain different limits by introducing a suitable (parameter dependent) change of coordinates. This effect is mainly due to the fact that the transformation in the limit sends finite points to infinity, hence is unbounded. However, in our case the limit of the transformation, although discontinuous, remains bounded and the mentioned ambiguities do not arise. This is in fact not surprising since our change of coordinates is adjusted to the geometry of the space-time which (in the sense of geodesic completeness) is nonsingular, allowing only for a *finite* jump of the geodesics.

Current research is concerned with a generalization of the above framework to the case of spherical impulsive waves [4]. Recently Podolský and Griffiths [21] have explicitly given the (discontinuous) transformation relating the continuous form found by Hogan [22] to the distributional form. The latter however, viewed as the (formal) impulsive limit of a Robinson-Trautman type-N solution involves the square of the Dirac-δ in the space-time metric.

Acknowledgement

It is a pleasure to thank the organizers of the conference for providing such a motivating and friendly environment. Also I'm very much indebted to M. Kunzinger for all his support. This work was supported in part by the Austrian Academy of Science, Ph.D. programme, grant #338 and by Research Grant P12023-MAT of the Austrian Science Foundation (FWF).

References

1. M. Kunzinger, R. Steinbauer, Class. Quant. Grav. **16**, 1255-1264 (1999).
2. M. W. Brinkmann, Proc. Natl. Acad. Sci. U.S.A. **9**, 1-3 (1923).
3. D. Kramer, H. Stephani, M. MacCallum, E. Herlt, *Exact Solutions of Einstein's Field equations,* (Cambridge University Press, Cambridge, 1980).
4. R. Penrose, in *General Relativity, Papers in Honour of J. L. Synge,* edited by L. O'Raifeartaigh (Clarendon Press, Oxford, 1972), pp. 101-115.
5. P. C. Aichelburg, R. U. Sexl, J. Gen. Rel. Grav. **2**, 303-312 (1971).
6. C. O. Loustó, N. Sánchez, Phys. Lett. B **232**, 462-466 (1989).
7. H. Balasin, H. Nachbagauer, Class. Quant. Grav. **12**, 707-713 (1995).
8. J. Podolský, J. B. Griffiths, gravitational waves," Phys. Rev. D. **58**, 124024 (1998).
9. E. Verlinde, H. Verlinde, Class. Quant. Grav. **10**, Suppl. 175-184 (1993).
10. R. Geroch, J. Traschen, Phys. Rev. D **36**, 1017-1031 (1987).
11. R. Steinbauer, J. Math. Phys. **39**, 2201-2212 (1998).
12. M. Kunzinger, R. Steinbauer, J. Math. Phys. **40**, 1479-1489 (1999).
13. P. C. Aichelburg, H. Balasin, Class. Quant. Grav. **14**, A31-A41, (1997).
14. J. Podolský, K. Vesely, Phys.Lett. **A241**, 145-147 (1998).
15. R. Steinbauer, in *Proceedings of the VIII Romanian Conference on General Relativity,* edited by I. Cotaescu, D. Vulcanov, (Mirton Publishing House, Timisoara, 1999) (also available as gr-qc/9809054).
16. O. Hájek, Bull. AMS **12**, 272-279 (1985).
17. R. Steinbauer, J. Math. Phys. **38**, 1614-1622 (1997).
18. M. Oberguggenberger, "Multiplication of Distributions and Applications to Partial Differential Equations," (Longman Scientific and Technical, New York, 1992).
19. J. F. Colombeau, "Elementary Introduction to New Generalized Functions," (North Holland, Amsterdam, 1985).
20. R. Geroch, Commun. Math. Phys. **13**, 180-193, (1969).
21. J. Podolský, J. B. Griffiths, Class. Quant. Grav. **16**, 2917-1946 (1999).
22. P. A. Hogan, Phys. Rev. Lett. **70** (2), 117-118 (1993).