Distributional description of impulsive gravitational waves

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In this contribution we present a recent application [13, 7] of the nonlinear theory of generalized functions to the theory of general relativity, i.e. we treat the geodesic and the geodesic deviation equation for impulsive pp-waves in the algebra of generalized functions. Formally these equations are ODEs nonlinear in the components of the metric tensor which is proportional to the Dirac δ-distribution; hence we have to deal with *nonlinear singular ODEs*. We derive existence and uniqueness results for these equations in Colombeau's algebra of generalized functions and calculate distributional limits of the unique solutions explicitly using the concept of association. These "distributional solutions" display the physically expected behavior, which shows that the geometry of impulsive pp-waves can be described consistently using distributions.

1 Impulsive pp-waves

In this section we briefly recall the basic features of impulsive plane fronted gravitational waves with parallel rays (impulsive pp-waves). PP-wave space times possess a covariantly constant null vector field, which in turn allows to write the metric in the form

$$ds^2 \;=\; H(u,x,y)du^2 - du\,dv + dx^2 + dy^2, \tag{1}$$

where u, v is a pair of null coordinates ($u = t - z,\ v = t + z$) and x, y are transverse (Cartesian) coordinates. We are especially interested in *impulsive* pp-waves where the profile function takes the form [10] $H(u,x,y){=}f(x,y)\,\delta(u)$, and δ is the Dirac measure. Hence the spacetime is flat everywhere except for the (null) hypersurface $u = 0$, where it has a δ-like shape, modelling a gravitational shock wave.
Physically these geometries arise as ultrarelativistic limits of Kerr-Newman space times [1]. More precisely, a boost is applied to e.g. a Schwarzschild black hole with

respect to an asymptotic observer where the boost velocity approaches the speed of light. Hence impulsive pp-waves model the gravitational field of highly energetic particles and play an important role in scattering processes at the Planck scale [8]. Penrose [10] in his "scissor and paste" approach constructed impulsive pp-waves by glueing together two halfes of Minkowski space time along the null hupersurface $u = 0$ with an (x, y)-dependent shift in the v-coordinate. Penrose also introduced a different coordinate system where the metric tensor is actually continuous. But, clearly, these new coordinates are not (even) continuously related to the ones used in (1) and therefore the differential structure of the manifold is changed.
In this work we shall use the distributional form of the metric, mainly for physical reasons (thereby following the approaches of [2, 4]), since in the ultrarelativistic limit (which is a mathematically delicate process, cf. [12]) the metric arises that way. We describe the geometry of impulsive pp-waves entirely in the distributional picture by solving the geodesic and the geodesic deviation equation in Colombeau's algebra of generalized functions. Using the concept of association we derive the distributional particle motion which, however, turns out to be equivalent to the particle motion derived by Podolsky and Vesely using the continuous form of the metric [11]. Hence from a physical point of view the different manifold structures are "identical".

2 The geodesic equation

We start with a pp-metric of the form

$$ds^2 = f(x^i)\,\delta(u)\,du^2 - du\,dv + (dx^i)^2 \,, \tag{2}$$

where x^i $(i = 1, 2)$ denote the transverse coordinates and f is the *smooth* wave profile. It is straightforward to derive the geodesic equation which (using u as affine parameter, thereby excluding only trivial geodesics parallel to the shock) takes the form

$$\begin{aligned} \ddot{v}(u) &= f(x^j(u))\,\dot{\delta}(u) + 2\,\partial_i f(x^j(u))\,\dot{x}^i(u)\,\delta(u) \,, \\ \ddot{x}^i(u) &= \frac{1}{2}\,\partial_i f(x^j(u))\,\delta(u) \,, \end{aligned} \tag{3}$$

where $\dot{}$ denotes the derivative with respect to u.
Note that the first line involves a product of the δ-function with $\dot{x}^i$ which we cannot even expect to be continuous (and in fact turns out to be proportional to the

Heaviside function, cf. [13]). Therefore we cannot treat system (3) within Schwartz' linear theory of distributions and we proceed by transferring the equations into the framework of Colombeau's (full) algebra [3, 9].

First of all this amounts to a regularization of the δ-distribution to make it an element of $\mathcal{G}(\mathbb{R})$. Since we are interested in the most general result we will not be satisfied with the canonical embedding $\iota(\delta)$, which corresponds to a "model δ-net" regularization, but use the largest class of "reasonable" (smooth) regularizations of the Dirac measure. Hence we define (cf. the definition of strict δ-nets in [9] ch. 2.7) a *generalized delta function* to be an element D of $\mathcal{G}(\mathbb{R}^n)$ possessing a representative $(D(\varphi,\,.\,))_{\varphi\in\mathcal{A}_0}$ such that $\exists p\in\mathbb{N}_0\ \forall\varphi\in\mathcal{A}_p(\mathbb{R}^n)\,\exists\eta=\eta(\varphi)>0:$

(i) $\quad \mathrm{supp}(D(\varphi_\varepsilon,\,.\,))\subseteq[-\varepsilon,\varepsilon]\ \ \forall\varepsilon\in(0,\eta)$

(ii) $\quad \int D(\varphi_\varepsilon,x)\,dx\to 1\ (\varepsilon\to 0)$

(iii) $\quad \exists C=C(\varphi)>0$ such that $\int |D(\varphi_\varepsilon,x)|\,dx\le C\ \ \forall\varepsilon\in(0,\eta)$

Note that condition (i) has been chosen to avoid technicalities in the calculation of association relations which –of course– remain valid if (i) is replaced by the condition $\mathrm{supp}(R(\varphi_\varepsilon,\,.\,))\to\{0\}\ (\varepsilon\to 0)$.

We proceed by transferring system (3) into the $\mathcal{G}$-setting. A main problem one has to face in this approach is that, since generalized solutions are constructed componentwise, i.e. from componentwise classical (smooth) solutions to (here) *nonlinear* equations, we have to ensure that the interval of existence does not shrink to a point in the limit $\epsilon\to 0$. Usually one puts strong restrictions on the right hand side of the equation to guarantee this behavior (cf. [5, 6]). However, this would seriously affect physical applicability of our considerations and so we are forced to make use of the special form of the equations. Indeed we have the following technical result which guarantees global existence of solutions for small ϵ (cf. Appendix of [13])

Lemma *Let $g:\mathbb{R}^n\to\mathbb{R}^n$ smooth and $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ be a net of smooth functions bounded uniformly in the L^1-norm and supp$(\rho_\epsilon)\in[-\epsilon,\epsilon]$. For any $x_0,\dot{x}_0\in\mathbb{R}^n$ and any $\varepsilon\in(0,1)$ consider the system*

$$\begin{aligned}\ddot{x}_\varepsilon(t) &= g(x_\varepsilon(t))\rho_\varepsilon(t)\\ x_\varepsilon(-1) &= x_0\\ \dot{x}_\varepsilon(-1) &= \dot{x}_0\end{aligned}\tag{4}$$

Let $b > 0$, $I = \{x \in \mathbb{R}^n : |x - x_0| \leq b + |\dot{x}_0|\}$ *and* $\alpha = \min\left\{\frac{b}{C\|g\|_{L^\infty(I)} + |\dot{x}_0|}, \frac{1}{2LC}, 1\right\}$ *with* L *a Lipschitz constant for* g *on* I*. Then (4) has a unique solution on* $J_\varepsilon = [-1, \alpha - \varepsilon]$*. Consequently, for* ε *sufficiently small* x_ε *is globally defined and both* x_ε *and* $\dot{x}_\varepsilon$ *are bounded, uniformly in* ε*, on compact sets.*

The proof uses the fact that the integral operator solving the equation is a contraction on the Banach space $\{f \in C(J_\varepsilon, \mathbb{R}^n) : |f(t) - x_0| \leq b + |\dot{x}_0|\}$. We have chosen initial conditions in $u = -1$, i.e. "before the shock" since geodesics starting at $u = 0$ do not allow any regularization independent treatment. Note that the bounds on x^i will also allow composition of the solutions constructed in Theorem 1 with arbitrary smooth functions f.

Now we are prepared to consider the geodesic equations in the Colombeau algebra. Denoting the $\mathcal{G}$-functions corresponding to x^i and v by capital letters we state the following

Theorem 1 *Let* $D \in \mathcal{G}(\mathbb{R})$ *be a generalized delta function,* $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ *and let* v_0*,* $\dot{v}_0$*,* x_0^i*,* $\dot{x}_0^i \in \mathbb{R}$ $(i = 1, 2)$*. The initial value problem*

$$\begin{aligned}
\ddot{V}(u) &= f(X^j(u))\,\dot{D}(u) + 2\,\partial_i f(X^j(u))\,\dot{X}^i(u)\,D(u) \\
\ddot{X}^i(u) &= \frac{1}{2}\,\partial_i f(X^j(u))\,D(u) \\
V(-1) &= v_0 \qquad X^i(-1) = x_0^i \\
\dot{V}(-1) &= \dot{v}_0 \qquad \dot{X}^i(-1) = \dot{x}_0^i
\end{aligned} \tag{5}$$

has a unique locally bounded solution $(V, X^1, X^2) \in \mathcal{G}(\mathbb{R})^3$*.*

To prove *existence* one has to verify that componentwise constructed classical solutions are moderate. This, however, is an easy consequence of the lemma above. To prove *uniqueness* observe that the lemma (with obvious modifications) remains valid if one adds a smooth function independent of ϵ to the right hand side of the first line in (4). This allows to establish the $\mathcal{N}$-estimates on the difference of two locally bounded solutions of (5).

Note that we only had to use conditions (i) and (iii) of the generalized delta function whereas (ii) is the essential ingredient to prove the association relations stated in

Theorem 2 *The unique solution (V, X^i) of the geodesic equation (5) satisfies the following association relations:*

$$\begin{aligned}
X^i &\approx x_0^i + \dot{x}_0^i(1+u) + \frac{1}{2}\partial_i f(x_0^j + \dot{x}_0^j)u_+ \\
V &\approx v_0 + \dot{v}_0(1+u) + f(x_0^j + \dot{x}_0^j)H(u) + \\
&\quad +\partial_i f(x_0^j + \dot{x}_0^i)\left(\dot{x}_0^i + \frac{1}{4}\partial_i f(x_0^j + \dot{x}_0^j)\right)u_+ \ ,
\end{aligned} \tag{6}$$

where H and u_+ denote the Heaviside and kink function, respectively.

Hence we see that even within the largest "reasonable" class of regularizations of the Dirac δ the geodesics of impulsive pp-waves (on the distributional level) are given by refracted straight lines. More precisely, the geodesics suffer a jump and a kink in the v- as well as a kink in the x^i-direction when crossing the shock hypersurface. The scale of the effect is entirely determined by the values of the profile function and it's first derivatives at the shock hypersurface.

3 Jacobi equation

In this section we treat the Jacobi (or geodesic deviation) equation which describes the behavior of infinitesimally nearby geodesics, thereby using essentially the same method as above. Despite the fact that these equations involve terms proportional even to "$H^2\delta$" and "δ^2" (see [13]) we still manage to end up with a regularization independent and physically reasonable distributional interpretation of again unique generalized solutions.

To keep formulas transparent we shall restrict ourselves to the special case of axi-symmetry, i.e. we demand that $f = f(r)$, where $r^2 = (x^1)^2 + (x^2)^2$. Furthermore, using initial conditions $x_0^2 = 0 = \dot{x}_0^2$ we will be able to work entirely in the $x^2 = 0$ hypersurface. Finally, we put $v_0 = 0 = \dot{x}_0^1$. Under these assumptions the Jacobi equation for the deviation vector field $N^a(u) = (N^u(u), N^v(u), N^x(u), N^y(u))$ over a geodesic takes the following explicit form in $\mathcal{G}$.

$$\begin{aligned}
\ddot{N}^v &= 2[N^x f'(X)D]^{\cdot} - N^x f'(X)\dot{D} + [N^u f(X)D]^{\cdot\cdot} - \\
&\quad -N^u f''(X)\dot{X}^2 D - N^u f'(X)\ddot{X}D \\
\ddot{N}^x &= [\dot{N}^u f'(X) + \frac{1}{2}N^x f''(X)]D + \frac{1}{2}f'(X)N^u\dot{D} \\
\ddot{N}^y &= \ddot{N}^u \ = \ 0 \ ,
\end{aligned} \tag{7}$$

where D again denotes a generalized delta function and X is the (unique) solution to system (5) with simplification as discussed above. Since these equations are linear in the components of the Jacobi field it is straightforward to derive the following result

Theorem 3 *Let n^a, $\dot{n}^a \in \mathbb{R}^4$. Then system (7) with initial conditions $N^a(-1) = n^a$ and $\dot{N}^a(-1) = n^a$ has a unique solution in $\mathcal{G}(\mathbb{R})^4$.*

The explicit calculation of association relations for the unique solutions to (7) is a long and tedious computation and we are only going to state a result for two special choices of initial conditions. We have the following

Theorem 4 *The unique solution of the geodesic deviation equation (7) with initial conditions $N^a(-1) = 0$ and $\dot{N}^a(-1) = (a, b, 0, 0)$ satisfies the following association relations:*

$$\begin{aligned} N^x &\approx \frac{1}{2} a f'(x_0)(u_+ + H(u)) \\ N^v &\approx b(1+u) + a[f(x_0)\delta(u) + \frac{1}{4} f'(x_0)^2 (H(u) + u_+)] \end{aligned} \tag{8}$$

For the initial conditions $N^a(-1) = 0$ and $\dot{N}^a(-1) = (0, a, b, 0)$ we have

$$\begin{aligned} N^x &\approx b(1+u) + \frac{1}{2}\, b\, f''(x_0)\, u_+ \ , \\ N^v &\approx a(1+u) + \frac{1}{4}\, b\, f'(x_0)\, f''(x_0)\, u_+ + b\, f'(x_0)\, H(u) \end{aligned} \tag{9}$$

Hence in the first case the Jacobi field suffers a kink, a jump and a δ-pulse across the shock overlapping the flat space behavior. The scale of the effect is proportional to the constant factor a which amounts to a "time advance" of nearby geodesics. In fact, the initial velocity a of the Jacobi field in u-direction causes nearby geodesics to cross the shock at different values of the parameter u. But as can be seen from (9) this "time advance" is not the only effect producing kinks and jumps. Equations (9) can be understood in terms of a (formal) Taylor expansion of the profile function near the point at which the refence geodesic crosses the shock, hence display the physically expected behavior. For further details see the discussion in [13].

We have described the geometry of impulsive pp-wave space times by an analysis of the geodesic and geodesic deviation equation in Colombaeu's algebra of generalized functions. Via the concept of association we derived a distributional picture, which perfectly coincides with physical expectations, thereby showing that a consistent distributional description of these geometries is indeed possible, as long as careful mathematical techniques are invoked to handle the nonlinear operations on the singular (i.e. distributional) wave profile instead of ad-hoc "multiplication rules" extending Schwartz linear theory.

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