Diffeomorphism invariant construction of nonlinear generalized functions

M. Grosser[1], M. Kunzinger[1], R. Steinbauer[1,2], H. Urbantke[2], and J. Vickers[3]

[1]Institut für Mathematik, Universität Wien, Strudlhofg. 4, A-1090 Wien, Austria
[2]Institut für Theoretische Physik, Universität Wien, Boltzmanng. 5, A-1090 Wien, Austria
[3]Department of Mathematics, University of Southampton, SO17 1BJ, England
roland.steinbauer@univie.ac.at

Received 15 November 1999, accepted dd.mm.yyyy by ue

Abstract. We briefly report on the diffeomorphism invariant construction of algebras of generalized functions (Colombeau algebras), in particular on the global approach presented in [1].

Keywords: nonlinear generalized functions, Colombeau Algebras, diffeomorphism invariance
PACS: 04.20.Cv, 02.30.Sa, 02.90

Colombeau algebras (cf. [2]) provide a mathematically rigorous framework for simultaneously treating singular (i.e., distributional) objects, nonlinear operations and differentiation, while at the same time displaying maximal consistency properties with respect to classical operations. More precisely, a Colombeau algebra is a commutative and associative differential algebra canonically containing the space of distributions (embedded basically via convolution with a general class of mollifiers) and preserving the pointwise product of smooth functions. Since their introduction in the mid 1980s they have served as a valuable tool in treating nonlinear PDEs involving either singular data or singular coefficients or both (see e.g. [3]) which lies beyond the scope of classical linear distribution theory. Also, Colombeau algebras have been applied in the investigation of space time geometries of low differentiability (for a survey see [4]).

Up to recently, however, no diffeomorphism invariant formulation of the theory of nonlinear generalized functions (apart from the so-called special variant of the construction in the sense of [3], p. 109, which, however, does not provide a canonical embedding of $\mathcal{D}'$) had been available.

Generally, Colombeau algebras are constructed according to the following scheme. We start with a differential algebra $\mathcal{E}$ (containing $\mathcal{D}'$) referred to as the basic space. Elements $R \in \mathcal{E}$ take a pairs (φ, x), where φ denotes a test function with unit integral and $x \in \mathbb{R}$, to complex numbers $R(\varphi, x)$. The Colombeau algebra $\mathcal{G}$ is then defined as the quotient of the subspaces of moderate resp. negligible elements of $\mathcal{E}$, i.e., $\mathcal{G} := \mathcal{E}_M/\mathcal{N}$. Whether or not $R \in \mathcal{E}$ is moderate (resp. negligible) is determined by *testing* the asymptotic behavior of R on scaled test functions $\varphi_\epsilon(\xi) := (1/\epsilon^n)\varphi(\xi/\epsilon)$ as the scaling parameter ϵ tends to 0; for moderateness (resp. negligibility) we require slow growth (resp. fast decay), i.e., $\mathrm{O}(\epsilon^{-N})$ for some fixed N (resp. $\mathrm{O}(\epsilon^n)$ for all n) in all derivatives on compact sets. If we wish the action of a diffeomorphism μ on R to extend the usual action on distributions we are forced to transform the test function φ to

$$\phi(\epsilon,x)(\xi) := \varphi\left(\frac{\mu^{-1}(\epsilon\xi+x)-\mu^{-1}x}{\epsilon}\right)\cdot|\det D\mu^{-1}(\epsilon\xi+x)|,$$

hence to use bounded paths $(\epsilon,x)\mapsto\phi(\epsilon,x)\in\mathcal{D}$ instead of single φ's as *test objects*.

The dependence of ϕ on x necessitates differentiability of R with respect to the first slot as well. Based on the approaches in [5] and [6], a diffeomorphism invariant local theory has first been obtained in [7] and [8]. Basically, diffeomorphism invariance of the algebra itself is achieved via diffeomorphism invariance of the testing procedure. The construction provides a fine sheaf $\Omega\to\mathcal{G}(\Omega)$ of differential algebras on $\mathbb{R}^n$ canonically containing $\mathcal{D}'$ and $\mathcal{C}^\infty$ and preserving partial derivatives. The required theory of differentiation in infinite dimensional spaces is based on the concept of calculus in convenient vector spaces in the sense of [9], which considerably facilitates solving nonlinear PDEs.

Moreover, applications especially in the theory of relativity have underscored the need for a theory of algebras of generalized functions on a manifold M that enjoys two additional features: first, it should be *geometric* in the sense that its basic objects should be defined intrinsically on the manifold itself. Second, $\mathcal{G}(M)$ should be a differential algebra with Lie derivatives commuting with the embedding of $\mathcal{D}'(M)$.

Such an algebra indeed was constructed by the authors of [1]. The key notion to allow for such a global approach is that of *smoothing kernels* which are the analogue of unbounded paths $\frac{1}{\epsilon}\phi(\epsilon,x)(\frac{\xi-x}{\epsilon})$ in the local theory allowing for the definition of an intrinsic testing procedure. Moreover, using various characterization results of [8] the localization of the global theory was shown to be precisely equivalent to the local theory of [7].

M. K. and R. S. were supported by research grants P10472-MAT and P12023-MAT of the Austrian Science Foundation (FWF), respectively.

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