Notions of curvature for non-smooth spacetimes

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FWF Österreichischer Wissenschaftsfonds

excellent = austria

1. Intro & motivation

2. Curvature for rough metrics
   - Linear distributional curvature
   - Nonlinear distributional curvature

3. Applications
   - Impulsive gravitational waves
   - Interlude: Causality
   - Singularity theorems
   - Aside: Synthetic curvature bounds & singularity thms.

Outline

Curvature for non-smooth spacetimes

- Curvature is the essential quantity in GR
  Einstein equations relate matter/energy to curvature of spacetime

$$\mathrm{Ric} - \frac{1}{2}\mathrm{R}\, g + \Lambda g \;=\; 8\,\pi\,\mathrm{G}$$

- non-smooth means spacetime metric below $g \in \mathcal{C}^2$
  or no spacetime at all (Lorentzian length/metric spaces, causal sets)

Why is this interesting?

- physically relevant *models* (matched spacetimes, impulsive wave, etc.)
- *PDE* point-of-view
- *singularities* vs *curvature blow-up* — *CCH* of Penrose
- approaches to *Quantum Gravity* (no metric, e.g. causal sets)

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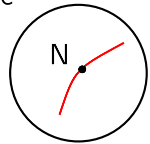
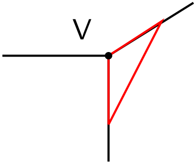
Basic geometric properties change if regularity drops

Example 1: Walking on a sphere vs. walking on a cube

Sphere



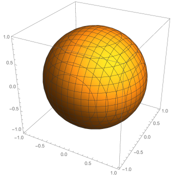
Cube

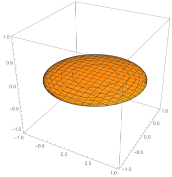


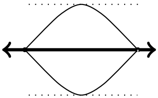
It is always shorter to deviate to the right face than to go along the edges.

Example 2: Squeezing the sphere

Convexity fails for metrics of Hölder regularity $g \in C^{1,\alpha}$ $(\alpha < 1)$.







Equator still geodesic but shorter to deviate into hemispheres.
[Hartman-Wintner 52]

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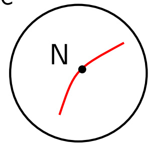

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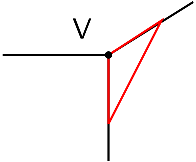

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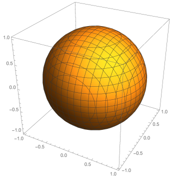

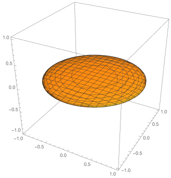

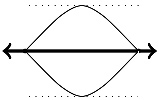

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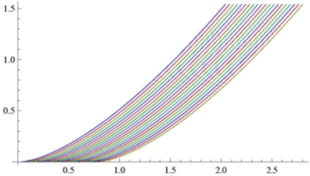
Lorentzian causality theory changes if regularity drops

Example 3: Lightcones bubble up [Chrusciel-Grant 12]



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Non-uniqueness of null geodesics
$\rightsquigarrow$ null cone has full measure.

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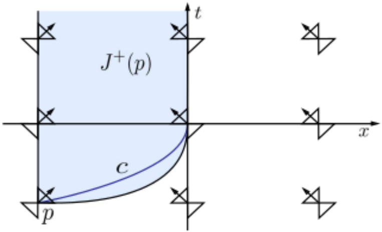

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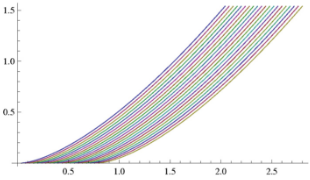

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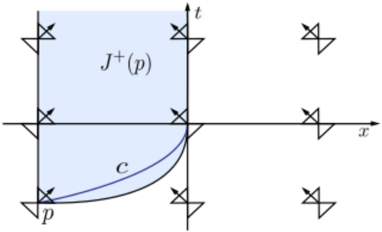

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Basic distributional geometry

- recall, distributions on manifolds & distributional tensor fields

[Schwartz, de Rham, ...]

$$\mathcal{D}'(M) := \Big(\Omega^n_c(M)\Big)'$$

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- distributional metrics [Marsden 68, Parker 79]

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- g gives no musical isomorphism $\mathcal{D}'{}^1_0 \ni X \mapsto X^\flat := g(X, .) \in \mathcal{D}'{}^0_1$
- index, geodesics, etc. of a distributional metric?
- only way to define, inverse, curvature, etc. is via smoothing

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Distributional connections

- [Marsden 68, Parker 79]

$$\nabla:\ \mathfrak{X}(M)\times\mathcal{D}'(M)^1_0\to{\mathcal{D}'}^1_0(M)\quad \text{w. usual properties}$$

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- [LeFloch-Mardare 07]

$$\nabla:\ \mathfrak{X}(M)\times\mathfrak{X}(M)\to{\mathcal{D}'}^1_0(M)\quad \text{w. usual properties}$$

+ extend to entire smooth tensor algebra
+ every $\mathcal{D}'$-metric has a 'Levi Civita connection'

- used from now on

+ becomes workable if $\nabla:\ \mathfrak{X}(M)\times\mathfrak{X}(M)\to(L^2)^1_0(M)$

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Curvature from a distributional connection?

- $\mathcal{C}^\infty$: $\mathrm{Riem}: \ \mathfrak{X}(M)^3 \to \mathfrak{X}(M), \ R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \ (*)$
- $\mathcal{D}'$-connection: $\nabla: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'^1_0(M)$

problem: $\nabla_Y Z \in \mathcal{D}'^1_0 \ \rightsquigarrow \ \nabla_X \nabla_Y Z$ not defined

- workaround: look for special distributional connections with

$$\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{A}(M) \subseteq \mathcal{D}'^1_0(M)$$

such that ∇ can be extended to $\mathcal{A}(M)$ in second slot

$$\begin{aligned} \nabla: \ \mathfrak{X}(M) \times \mathcal{A}(M) \quad &\to \quad \mathcal{D}'^1_0(M) \qquad (X \in \mathfrak{X}, Y \in \mathcal{A}, \theta \in \Omega^1) \\ \underbrace{\nabla_X Y}_{\in \mathcal{D}'^1_0}(\theta) \quad &:= \quad X\big(\underbrace{Y(\theta)}_{\in \mathcal{D}'}\big) - \underbrace{\nabla_X \theta}_{\in \mathcal{A}}(\underbrace{Y}_{\in \mathcal{A}}) \qquad \in \mathcal{D}'(M) \end{aligned}$$

- obvious choice $\mathcal{A} = (L^2_{\mathrm{loc}})^1_0$

For such an L^2-*connection*, the curvature tensor is defined via $(*)$.

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Curvature from a distributional metric

- want Levi-Civita connection to be L^2: need $\boxed{g \text{ in } H^1_{\text{loc}} \cap L^\infty_{\text{loc}}}$
- note: $H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ is an algebra
- notion of nondegeneracy: $|\det g| \geq C > 0$ on compact sets

Definition (Geroch-Traschen 87, LeFloch-Mardare 07, S-Vickers 09)

A distributional metric g is called *gt-regular* if it is of $H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$-regularity and it is uniformly nondegenerate: $\forall K$ cp. $\exists C > 0$: $|\det g| \geq C$ on K.

Geroch-Traschen class is the maximal "reasonable" distributional setting

+ allows to define curvature $\mathrm{Riem}[g]$, $\mathrm{Ric}[g]$, $\mathrm{R}[g]$ in distributions
+ is stable w.r.t. perturbations

Limitations

− Bianchi identities fail $\rightsquigarrow$ energy conservation ?

− $\dim\Big(\mathrm{supp}\,(\mathrm{Riem}[g])\Big) \geq n-1 \rightsquigarrow$ thin shells: yes, strings: no!

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Stability for the Geroch-Traschen class

g be gt-regular & g_ε be a (smooth) approximation. When do we have

$$\mathrm{Riem}[g_\varepsilon] \to \mathrm{Riem}[g], \text{ in } \mathcal{D}'^1_3(M) \text{ ?}$$

Yes if

- $g_\varepsilon \to g$ in H^1_{loc}, $g_\varepsilon^{-1} \to g^{-1}$ in L^∞_{loc}

  [LeFloch-Mardare 07]

  But for smoothings via convolution $g_n^{-1} \not\to g^{-1}$ in L^∞_{loc}

- $g_\varepsilon \to g$ in H^1_{loc}, $g_\varepsilon^{-1} \to g^{-1}$ in L^2_{loc} & $g_\varepsilon,\ g_\varepsilon^{-1}$ bounded in L^∞_{loc} $(*)$
- Existence of approximation with $(*)$

- if g continuous [Geroch-Traschen 87]
- if g not too far from continuous (*'stable'*): [S-Vickers 09]

  $\forall K$ cp. there is A^K continuous, such that

$$\max_{i,j} \operatorname{essup}_{x\in K} |g_{ij}(x) - A^K_{ij}(x)| \le C < \frac{\mu_K}{2n},$$

where $\mu_K := \min_{1\le i\le n} \operatorname{essinf}_{x\in K} |\lambda^i(x)|$, $\lambda_i, \dots, \lambda_n$ eigenvalues of g.

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- if g not too far from continuous (*'stable'*): [S-Vickers 09]

  $\forall K$ cp. there is A^K continuous, such that

$$\max_{i,j} \operatorname{essup}_{x\in K} |g_{ij}(x) - A^K_{ij}(x)| \le C < \frac{\mu_K}{2n},$$

  where $\mu_K := \min_{1\le i\le n} \operatorname{essinf}_{x\in K} |\lambda^i(x)|$, $\lambda_i, \dots, \lambda_n$ eigenvalues of g.

Stability for the Geroch-Traschen class

g be gt-regular & g_ε be a (smooth) approximation. When do we have

$$\mathrm{Riem}[g_\varepsilon] \to \mathrm{Riem}[g], \text{ in } \mathcal{D}'^1_3(M) \text{ ?}$$

Yes if

- $g_\varepsilon \to g$ in H^1_{loc}, $g_\varepsilon^{-1} \to g^{-1}$ in L^∞_{loc}

  [LeFloch-Mardare 07]

  But for smoothings via convolution $g_n^{-1} \not\to g^{-1}$ in L^∞_{loc}

- $g_\varepsilon \to g$ in H^1_{loc}, $g_\varepsilon^{-1} \to g^{-1}$ in L^2_{loc} & $g_\varepsilon,\ g_\varepsilon^{-1}$ bounded in L^∞_{loc} $(*)$
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Outline

Nonlinear distributions

Based on *algebras of generalised functions* [Colombeau 84, 85]

- differential algebras containing the vector space $\mathcal{D}'(M)$
- display maximal consistency w.r.t. classical analysis; preserve:
  - the product of $\mathcal{C}^\infty$ functions
  - Lie derivatives of distributions.
- regularization of distributions by nets of $\mathcal{C}^\infty$-functions
- asymptotic estimates in terms of ε (quotient construction)

Construction on manifolds $\boxed{\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)}$ [Kunzinger-S 02]

$$\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty : \ \forall K\ \forall P\ \exists l : \ \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-l})\}$$

$$\mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \ \forall K\ \forall m : \ \sup_{x\in K} |u_\varepsilon(x)| = O(\varepsilon^{m})\}$$

fine sheaf of differential algebras w.r.t. $L_X u := [(L_X u_\varepsilon)_\varepsilon]$

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Nonlinear distributional geometry

- *tensor fields*: fine sheaf of finitely generated, projective $\mathcal{G}(M)$-modules

$$\begin{aligned}\mathcal{G}^r_s(M) &:= \mathcal{E}^r_s(M)/\mathcal{N}^r_s(M)\\ &\cong L_{\mathcal{C}^\infty(M)}(\mathcal{G}^s_r(M), \mathcal{G}(M)) \;\cong\; \mathcal{G}(M)\otimes_{\mathcal{C}^\infty}\mathcal{T}^r_s(M)\\ &\cong L_{\mathcal{G}(M)}(\mathcal{G}^s_r(M), \mathcal{G}(M))\end{aligned}$$

- *embeddings*: injective sheaf morphism

$$\iota : \mathcal{T}^r_s(M) \hookrightarrow \mathcal{D}'^r_s(M) \hookrightarrow \mathcal{G}^r_s(M)$$

basically given by chart-wise, component-wise convolution

- *generalised metric*: $\boxed{g = [(g_\varepsilon)_\varepsilon] \in \mathcal{G}^0_2(M) \text{ symm.}\,\&\ \det(g) \text{ inv. in } \mathcal{G}}$

locally represented by sequence of smooth metrics g_ε with $|\det(g_\varepsilon)| \geq \varepsilon^m$ for some m on any compact set.

- *musical isomorphism*: $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := g(X, .\,) \in \mathcal{G}^0_1(M)$

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ζ_i cut-off functions, ψ_i charts, χ_i partition of unity, ρ_ε mollifier on cp. sets, ε small: only finite sum

in $\mathcal{G}$

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Generalised curvature

- every $\mathcal{G}$-metric has a *Levi Civita connection*

$$\nabla: \ \mathcal{G}^1_0(M) \times \mathcal{G}^1_0(M) \to \mathcal{G}^1_0(M)$$

- *usual formulas* hold (also for representatives/sequences), e.g.

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \quad \in \mathcal{G}^3_1(M)$$

- *compatibility* with gt-setting [S-Vickers 09]

$$\begin{array}{ccc} H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \ni g & \xrightarrow{\star\rho_\varepsilon} & [g_\varepsilon] \in \mathcal{G} \\ \mathrm{gt} \downarrow & & \downarrow \mathcal{G} \\ \mathrm{Riem}[g] & \xleftarrow{\mathcal{D}'-\lim} & \mathrm{Riem}[g_\varepsilon] \end{array}$$

technicalities: g stable, ρ admissible to have $|\det(g_\varepsilon)| \geq \varepsilon^m$

$\forall j: \int x^\alpha \rho_\varepsilon(x)\, dx = 0$ for all $1 \leq |\alpha| \leq j$ and ε small, and $\forall \eta > 0: \int |\rho_\varepsilon(x)|\, dx \leq 1 + \eta$ for ε small

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Application: Geodesics for impulsive gravitational waves

- exact models of short but violent burst of gravitational radiation
- non-expanding with Λ

$$\mathrm{d}s^2 = \frac{2\,\mathrm{d}\eta\,\mathrm{d}\bar{\eta} - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2H(\eta,\bar{\eta})\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2}{[\,1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\,]^2}$$

- geodesic equations ill-defined as distributions
- consistent solution concept in $\mathcal{G}[I, M]$

Theorem [Podolský-Schinnerl-Sämann-Švarc-S 18–24]

There are unique global generalised solutions of the i.v.p. for geodesics. They have limiting 'distributional geodesics' w. clear geometric meaning.

- analytically singular solutions but geodesically complete
- for $\Lambda = 0$ counterexamples to Ehlers-Kundt conjecture
- allows to make sense of notorious 'discontinuous transformation'

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- consis

Theorem … 18–24]

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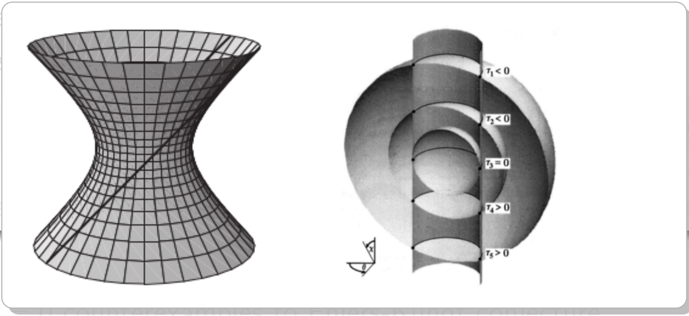

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$$\begin{aligned}
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\ddot Z_{p\varepsilon} - \frac{1}{2}H_{,p}\,\delta_\varepsilon \dot U_\varepsilon^2 &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\dot{\big(H\,\delta_\varepsilon\,U_\varepsilon\big)}\Big)\;\frac{Z_{p\varepsilon}}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}\\
\ddot V_\varepsilon - \frac{1}{2}\,H\,\delta_\varepsilon^{'}\,\dot U_\varepsilon^2 - \delta^{pq} H_{,p}\,\delta_\varepsilon\,\dot Z_q^\varepsilon\,\dot U_\varepsilon &= -\Big(e + \frac{1}{2}\,\dot U_\varepsilon^2\,\tilde G_\varepsilon - \dot U_\varepsilon\,\dot{\big(H\,\delta_\varepsilon\,U_\varepsilon\big)}\Big)\;\frac{V_\varepsilon + H\,\delta_\varepsilon U_\varepsilon}{3/\Lambda - U_\varepsilon^2 H\delta_\varepsilon}
\end{aligned}$$

where $\delta_\varepsilon = \delta_\varepsilon(U_\varepsilon(t))\,, \quad \delta'_\varepsilon = \delta'_\varepsilon(U_\varepsilon(t))\,, \quad e = 0, \pm 1\,,$

$\tilde G_\varepsilon = \tilde G_\varepsilon(U_\varepsilon(t), Z_{p\varepsilon}(t))\,, \quad H = H(Z_{p\varepsilon}(t))\,,$ and $H_{,p} = H_{,p}(Z_{q\varepsilon}(t))$

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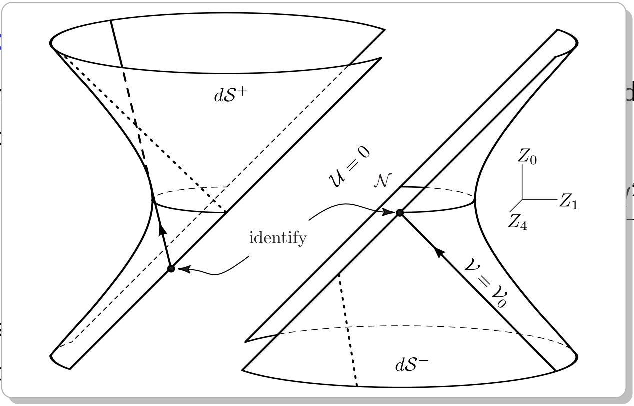

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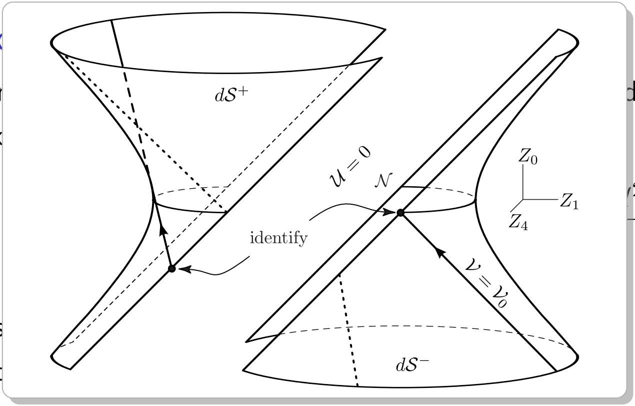

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Interlude: Advanced regularisation techniques & causality

- regularisation of continuous metrics plus control on the light cones
  [Chrusciel-Grant 12]

- allows to save bulk of causality for g Lipschitz
  [Chrusciel, Graf, Grant, Kunzinger, Minguzzi, Sämann, Stojkovic, S, Vickers, ... 14–22]

- recall: below Lipschitz some aspects auf causality break down
  - bubbling metrics [Chrusciel-Grant 12]
  - $I^+(p)$ need not be open [Grant-Kunzinger-Sämann-S 20]

- recall: convexity fails for $g \in C^{1,\alpha}$ $(\alpha < 1)$ [Hartman-Wintner 52]
  - causal solutions of geodesic equations do not maximise τ
  - $g \in C^0$: maximisers exists (compactness)
  - $g \in$ Lip: maximisers are $C^{1,1}$-curves, have causal character, solve the ODE in the Filippov-sense
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ω local timelike one-form

Lemma [Chrusciel-Grant 12, Graf 20, . . .]

For $g \in C(M)$ there are C^∞-Lorentzian metrics $\hat g_\varepsilon$ and $\check g_\varepsilon$ on M with

(i) $\check g_\varepsilon \prec g \prec \hat g_\varepsilon$.

(ii) $\check g_\varepsilon$, $\hat g_\varepsilon \to g$, and $(\check g_\varepsilon)^{-1}$, $(\hat g_\varepsilon)^{-1} \to g^{-1}$ as good as convolution

solve the ODE in the Filippov-sense

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Outline

Energy conditions and the singularity theorems

- energy conditions/curvature bounds lie at analytical core
$$\boxed{\mathrm{Ric}(X,X) \geq 0} \text{ for } X \text{ timelike/null}$$
- $g \in C^{1,1}$ / C^1 / Lip. $\implies \mathrm{Ric}[g] \in L^\infty_{\text{loc}}$ / $\mathcal{D}'^{(1)}$ / $\mathcal{D}'$
- focusing of geodesics via distributional energy conditions
$$\langle \mathrm{Ric}[g](X,X), \varphi \rangle \geq 0 \text{ for all non-neg. test } n\text{-forms } \varphi$$
- line of arguments:
  - Raychaudhuri/Ricati arguments for CG-regularising sequence $\check{g}_\varepsilon$
  - $\mathrm{Ric}[\check{g}_\varepsilon] \to \mathrm{Ric}[g]$ only distributional: too weak for positivity
  - but: $\Big(\mathrm{Ric}[g](X,X)\Big) \star \rho_\varepsilon \geq 0$ for non-neg. ρ_ε
  - $g \in C^1$: $\Big|\Big(\mathrm{Ric}[g](X,X)\Big) \star \rho_\varepsilon - \mathrm{Ric}[\check{g}_\varepsilon](X,X)\Big| \to 0$ loc. uniformly

    Friedrichs Lemma

    plus convergence of geodesics (ODE-theory)
  - $g \in$ Lip: $\Big|\Big(\mathrm{Ric}[g](X,X)\Big) \star \rho_\varepsilon - \mathrm{Ric}[\check{g}_\varepsilon](X,X)\Big| \to 0$ only L^p_{loc} $(p < \infty)$

    and $\mathrm{Ric}[\check{g}_\varepsilon] \geq -|\kappa|$

    plus worldvolume estimates: see talk by Melanie Graf

Optimal regularity & singularity theorems

work in progress [Kunzinger-Reintjes-S-Vega]

RT-equations [Reintjes-Temple 20–24]

$g \in W^{1,p}_{\mathrm{loc}}$ and $\mathrm{Riem}[g] \in L^p_{\mathrm{loc}}$ $(n < p < \infty) \Longrightarrow g \in W^{2,p}_{\mathrm{loc}}$
(in a $W^{2,p}$-compatible atlas)

- recall
  - $W^{1,p} \supsetneq W^{1,\infty} =$ Lip $(p < \infty))$
  - $W^{1,p} \subseteq C^{0,\alpha}$ with $\alpha = 1 - n/p$ (Morrey's inequality)
- $W^{2,p}$-change of diff. structure leaves causality invariant $(I^\pm,\ J^\pm)$
- Lemma: $g \in W^{1,p}$, $\mathrm{Riem}[g] \in L^p \Longrightarrow$ no causal bubbles
- Hawking & Penrose theorems for $g \in W^{1,p}$, $\mathrm{Riem}[g] \in L^p$
by using the C^1-results

Singularity theorems: current state of affairs

spacetime results

- Penrose, Gannon-Lee, Hawking-Penrose: C^1
  [Graf 20], [Schinnerl-S 22], [Kunzinger-Ohanyan-Schinnerl-S 23]
- Hawking: Lip. [Calisti-Graf-Hafemann-Kunzinger-S 25]
- Hawking & Penrose: $g \in W^{1,p}$, $\mathrm{Riem}[g] \in L^p$ [work in progress]

causal cone structures [Minguzzi 19]

synthetic results

- sectional curvature bounds [Alexander-Graf-Kunzinger-Sämann 22]
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causal cone structures [Minguzzi 19]

- upper semi-cont. distribution of cones on M
- causal core of singularity theorems may be established

Theorem (Causal Penrose) [Minguzzi 19]

Let (M, C) be a globally hyperbolic closed cone structure w. a non-compact stable Cauchy hypersurface Σ. Then there are no compact future trapped sets and if Σ is non-empty and compact there is an inextendible future null geodesic entirely in $E^+(S)$.

Singularity theorems: current state of affairs

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Outline

Synthetic curvature bounds & singularity thms.

- Synthetic approaches: Lorentzian length spaces
  - causal space $(X, d, \ll, \leq, \tau)$ with τ intrinsic
  - (timelike) sectional curvature bounds via triangle comparison
  - Ricci bds. via optimal transport (RCD-spaces, Lott-Villani, Sturm)
  - smooth metric measure spacetimes [McCann 20], [Mondino-Suhr 22]
  - [Cavaletti-Mondino 22–24] TCD(K,N) and TMCP(K,N) properties

Theorem(TMCP-Hawking) [Cavaletti-Mondino 22]

Let X be a timelike non-branching, globally hyperbolic LLS with TMCP. Let V be a Borel achronal future timelike complete subset with mean curvature bded above. Then every future timelike geodesic starting in V has a bounded maximal domain of existence.

  - synthetic (NEC) [McCann 23]
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  - first-order Sobolev calculus on metric measure spacetimes
    (maximal weak subslope of time functions akin L-modulus of diff.)
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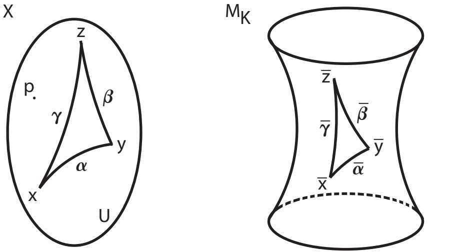

Faithful extension of sectional curvature bounds to “metric” Lorentzian setting

- first-order Sobolev calculus on metric measure spacetimes (maximal weak subslope of time functions akin L-modulus of diff.)

[Beran-Braun-Calisti-Gigli-McCann-Ohanyan-Rott-Sämann 24]

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Theorem (Synthetic Hawking) [Alexander-Graf-Kunzinger-Sämann 22]

Let $Y = (a, b) \times_f X$ be a warped product (X metric length space, $f \in C^\infty$, non-const.) with positive timelike sectional curvature. Then $a > -\infty$ or $b < \infty$ and hence Y is past/future timelike geodesically incomplete.

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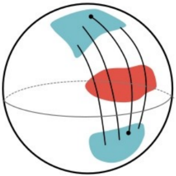

Idea: Ricci bounds encoded in convexity property of entropy functional along Wasserstein geodesics

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