The exponential map of a $C^{1,1}$-metric

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Motivation

Semi-Riemannian geometry with metrics of low regularity

**GR:** field eqs. are hyperbolic PDEs → regularity issues are vital

**Recent interest** in low regularity
- [Le Floch and coworkers, 2007-...]
- [Chrusciel, Grant 2012] causality theory

**Folklore:** ”Everything” should work for $g \in C^{1,1}$, since geodesic equation still uniquely solvable

The exponential map of a smooth Semi-Riemannian metric is a diffeomorphism locally around each point.

- proof rests on the inverse function theorem for $\exp$, needs $g \in C^2$
- however, $\exp \in Lip$ only, inverse function theorem? Literatur?
The result

We prove

The exponential map of a $C^{1,1}$ semi-Riemannian metric retains maximal regularity.
Every point possesses a totally normal neighbourhood.

More precisely

**Theorem (KSS 2013)**

Let $(M, g)$ be a smooth manifold with a $C^{1,1}$-semi-Riemannian metric. Then locally around each point the exponential map is a bi-Lipschitz homeomorphism.

Strategy of proof

regularization techniques & comparison geometry
Sketch of Proof

step 1 (regularization): Componentwise convolution with a standard mollifier

\[ g_\epsilon := g \ast \rho_\epsilon \]

\((g_\epsilon)_\epsilon\) is a family of smooth Semi-Riemannian metrics with locally uniformly bounded Riemann curvature tensor.
We have:

- \(g_\epsilon \to g\) locally uniformly up to 1st derivative
- \(\| Riem [g_\epsilon]\|_E \leq K_1\), \(\| \Gamma g_\epsilon \|_E \leq K_2\) locally

Now consider the family of exponential maps \((\exp_{p}^{g_\epsilon})_\epsilon\)

step 2 (common domain): By standard ODE-theory \(\exp_{p}^{g_\epsilon}, \exp_{p}^{g}\) are defined on a common domain \(B_E(0, \mu) \subseteq T_p M\)
The Riemannian case

step 3 (Jacobi field estimates): Using the Rauch comparison Theorem we turn bounds on the sectional curvature into

\( v \in B_E(0, \mu), \ w \in T_p M \)

\[
e^{-c} \| w \|_E \leq \| T_v \exp_p^g (w) \|_E \leq e^c \| w \|_E
\]

This gives

- \( \exp_p^g \) is a local diffeo on \( B_E(0, \mu) \)
- bi-Lipchitz estimates (via mean value argument)

step 4 (Injectivity): A Theorem by Cheeger, Gromov, Taylor turns the estimates

\[
\| Riem[g_\epsilon] \|_{L^\infty(B(p,r))} \leq C_1, \ \ Vol_{g_\epsilon} (B(p, r)) \geq C_2
\]

into an injectivity radius estimate from below, i.e.,

\( \text{Inj}_{g_\epsilon}(M, p) \geq i(C_1, C_2). \)
The Semi-Riemannian case

step 3’ (Jacobi field estimates): Done by hand (sectional curvature unbounded) following ideas by [Chen, Le Floch 08]: Using the estimates on $\Gamma_{g_\epsilon}, Riem[g_\epsilon]$ one may bootstrap $\|J_\epsilon(s)\|_E$ to again obtain

$$e^{-c}\|w\|_E \leq \|Tv \exp^g_p(w)\|_E \leq e^c\|w\|_E$$

which gives

- $\exp^g_p$ is a local diffeo
- bi-Lipchitz estimates (via mean value argument)

step 4’: (Injectivity): Again done by hand
- using a homotopy lifting argument on some ball $B_E(0, r_5)$
- needs a tricky nesting of domains

$$\exp^g_p(B_E(0, r_5)) \subseteq B_E(p, r_4) \subseteq \exp^g_p(B_E(0, \tilde{r})) \subseteq \exp^g_p(B_E(0, r_3))$$
**Totally normal neighbourhoods**

$U \subseteq M$ is a *normal neighbourhood* around $p \in U$ if there exists $\tilde{U} \subseteq T_p M$ open and starshaped such that

$\exp_p : \tilde{U} \rightarrow U$ is a bi-Lipschitz homeomorphism

$U$ is called totally normal, if it is normal for all $p \in U$.

**Theorem (KSS 2013)**

Let $(M, g)$ be a smooth manifold with a $C^{1,1}$-semi-Riemannian metric. Then each point possesses a basis of totally normal neighbourhoods.

Adaptation of a classical proof by [Whitehead 1932].
From [Whitehead 1932] (geometry of paths) it already follows:

The exponential map of a $C^{1,1}$-semi-Riemannian metric is a homeomorphism locally around each point.

different methods, no Lipschitz property


Prospect of our work: A $C^{1,1}$-causality theory with easy to access methods.
References


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