Lorentzian Geometry and Low Regularity

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Talk dedicated to James Vickers
on the occasion of his 60th birthday

Overview

Semi-Riemannian geometry and general relativity with metrics of low regularity

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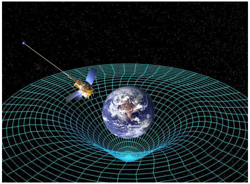
The basic physical setup of General Relativity

- Albert Einstein's theory of space, time and gravitation created exactly 99 years ago
- current description in physics of gravitation and the universe at large
- **geometric theory** due to Galileo's principle of equivalence: all bodies fall the same in a gravitational field
  $\rightsquigarrow$ gravitational field as property of the surrounding space
- Gravitational field influences how we measure lengths and angles hence the **curvature of space and time**



The basic mathematical setup of GR

Lorentzian geometry (basic geometric setup)

- smooth 4-dimensional space-time manifold M
- **smooth** space-time metric $\mathbf{g} \in \Gamma^0_2(M)$: at any T_pM symmetric, non-degenerate scalar product with signature $(-,+,+,+)$

Field equations (basic physical/analytical setup)

- Einstein Equations $\boxed{\mathbf{R}_{ij}[\mathbf{g}] - \frac{1}{2}\mathbf{R}[\mathbf{g}]\,\mathbf{g}_{ij} + \Lambda\,\mathbf{g}_{ij} = 8\pi\mathbf{T}_{ij}}$
- Ricci-tensor $\mathbf{R}_{ij}$, curvature scalar $\mathbf{R}$ built from **Riemann tensor** $R^m{}_{ikp} = \partial_k\Gamma^m{}_{ip} - \partial_p\Gamma^m{}_{ik} + \Gamma^a{}_{ip}\Gamma^m{}_{ak} - \Gamma^a{}_{ik}\Gamma^m{}_{ap}$ and Christoffel symbols $\Gamma^i_{jk} = \mathbf{g}^{il}\Gamma_{ljk} = \frac{1}{2}\mathbf{g}^{il}(\partial_k\mathbf{g}_{lj} + \partial_j\mathbf{g}_{kl} - \partial_l\mathbf{g}jk)$

$$\Rightarrow \mathbf{R}_{ij}, \mathbf{R} \;\sim\; \partial^2\mathbf{g} + (\partial\mathbf{g})^2$$

- coupled system of 10 quasi-linear PDEs of 2^{nd} order for $\mathbf{g}$

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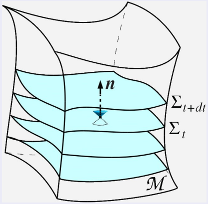
Why Low Regularity?

(1) Realistic matter—Physics

- want discontinuous matter configurations $\rightsquigarrow \mathbf{T} \notin \mathcal{C}^0 \implies \mathbf{g} \notin \mathcal{C}^2$
- finite jumps in $\mathbf{T} \rightsquigarrow \mathbf{g} \in \mathcal{C}^{1,1}$
- standard approach: $\mathbf{g}$ piecewise $\mathcal{C}^3$, globally only $\mathcal{C}^1$
- more extreme situations (impulsive waves): $\mathbf{g}$ piecew. $\mathcal{C}^3$, globally $\mathcal{C}^0$

(2) Initial value problem—Analysis

- $3+1$-split: $M = \Sigma \times \{t\}$; C. data $(\Sigma_0, \mathbf{g}_0, \mathbf{k})$ with $\Sigma_0 = \{t=0\}$, $\mathbf{g}(.,0) = \mathbf{g}_0$, $\partial_t \mathbf{g}(.,0) = \mathbf{k}$
- Local existence and uniqueness Thms. $(\mathbf{g}_0, \mathbf{k}) \in H^s \times H^{s-1}(\Sigma_0) \implies \mathbf{g} \in H^s(\Sigma)$
  - classical [CB,HKM]: $s > 5/2 \implies \mathbf{g} \in \mathcal{C}^1(\Sigma)$
  - recent big improvements [K,R,M,S]: $\mathbf{g} \in \mathcal{C}^0(\Sigma)$



GR and low regularity

Usually in the physics literature **g** is defined to be C^1.
BUT
"Unfortunately, this poses enormous problems [...] because the basic [...] properties of the spacetime [...] might not hold for general C^1-metrics. In order to avoid this annoying problem though—despite it being completely fundamental!—we will implicitly assume for most of this review that **g** is at least of class C^2."
[Garcia-Parrado,Senovilla,05]

geometry
regularity
standard results

g $\in \mathcal{C}^2$

- exponential map works
- existence of totally normal nbhds. $\Rightarrow$ geodesically convex nbhds.
- causality theory works [Chrusciel,11]
- needed for singularity thms. [Senovilla,98]

- things go wrong below $\mathcal{C}^2$
  - convexity goes wrong for $\mathbf{g} \in \mathcal{C}^{1,\alpha}$ $(\alpha < 1)$ [HW,51]
  - causality goes wrong, light cones "bubble up" for $\mathbf{g} \in \mathcal{C}^0$ [CG,12]
- treshold $\mathbf{g} \in \mathcal{C}^{1,1}$: Unique solvability of geodesics eq. suffices ???

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Distributional Approaches to GR

"Maximal" distributional setting [Geroch,Traschen,87]

using linear distributional geometry [Schwartz,DeRham,Marsden]

- $\mathbf{g} \in L^\infty_{\text{loc}} \cap H^1_{\text{loc}}$, $|\det(\mathbf{g})| \geq C > 0$ on cp. sets [LF,M,07],[S,Vickers,09]

$\rightsquigarrow$ $\nabla \in L^2_{\text{loc}}$ $\rightsquigarrow$ $\mathbf{Riem}[\mathbf{g}] \in \mathcal{D}'{}^0_2(M)$ plus H^1_{loc}-stability

BUT: $\dim(\text{supp}(\mathbf{Riem}[\mathbf{g}])) \geq 3 \rightsquigarrow$ shells: ok, strings: no!

Colombeau setting [Vickers,Kunzinger,S,...96–]

using nonlinear distr. geometry (special version) [GKOS,01]

- $\mathbf{g} \in \mathcal{G}^0_2$, $\det(\mathbf{g})$ invertible in $\mathcal{G}(M)$

$\rightsquigarrow$ $\mathbf{g}$ induces iso. $\mathcal{G}^1_0(M) \ni X \mapsto X^\flat := \mathbf{g}(X, .\,) \in \mathcal{G}^0_1(M)$

all curvature quantities defined by usual coordinate formulae

compatibility: C^2, Geroch–Traschen-setting [S.,Vickers,09]

Applications: An overview

- Curvature of cosmic strings
  [Clarke,Vickers,Wilson,96], [Vickers & Coworkers, 99–01]
- Geometry of impulsive pp-waves, geodesics, Penrose transform
  [Balasin,96], [Kunzinger,S.,98–04], [Grosser,Erlacher,11-13]
- (Ultrarelativistic) Kerr-Newman geometries
  [Balasin & Coworkers,96–03], [S.,98], [Heinzle,S.,02]
- Singular Yang-Mills theory [Kunzinger,S.,Vickers,05]
- Linear distributional geometry renewed [LeF,M,07]
  applications [LeFloch & Coworkers,07–]
- Wave equations in singular space times
  [Vickers,Wilson,00], [Grant,Mayerhofer,S,09], [Hanel,11]
  [Hörmann,Kunzinger,S.,12], [Hanel,Hörmann,Spreitzer,S.,13]
- Gen. global hyperbolicity, see talks of G. Hörmann & C. Sämann
- Geodesics in impulsive NP-waves, geodesics [S.,Sämann,12–]
- Reviews [S.,Vickers,06], [Nigsch,Sämann,13]

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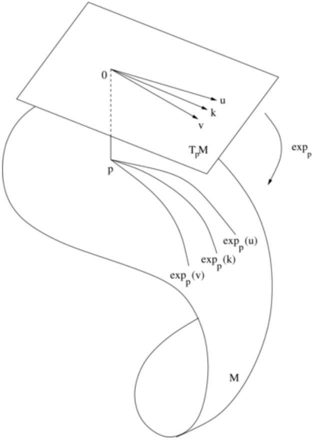
The exponential map in low regularity

The exponential map

- $exp_p: \ T_pM \ni v \mapsto \gamma_v(1) \in M$, where γ_v is the (unique) geodesic starting at p in direction of v
- maps rays through $0 \in T_pM$ to geodesics through $p \in M$

Regularity

- $\mathbf{g} \in \mathcal{C}^2 \Rightarrow exp_p$ local diffeo
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ loc. homeo [W32]
- $\mathbf{g} \in \mathcal{C}^{1,1} \Rightarrow exp_p$ bi-Lipschitz homeo [KSS14], [M13]



The exponential map for $\mathcal{C}^{1,1}$-metrics

Theorem (Max. reg. for *exp* [Kunzinger,S.,Stojković,14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then $\forall p \in M$ there exist open ngbhds. $\tilde{U}$ of $0 \in T_pM$ and U of p in M such that $\exp_p : \tilde{U} \to U$ is a bi-Lipschitz homeo.

Method of proof (details see M. Stojković's talk)

- **regularisation techniques:** approximate $\mathbf{g} \in \mathcal{C}^{1,1}$ by smooth $\mathbf{g}_\varepsilon$ $\Rightarrow \mathbf{g}_\varepsilon \to \mathbf{g} \in \mathcal{C}^1$ and $\mathbf{Riem}[\mathbf{g}_\varepsilon]$ bded, but $\mathbf{Riem}[\mathbf{g}_\varepsilon] \not\to \mathbf{Riem}[\mathbf{g}]$
- **comparison geometry:** new Lorentzian methods [Chen,LeFloch,08]

Alternative approach by [Minguzzi,13] uses

- Picard-Lindelöf approximations, inverse funct. thm. for Lip. maps

Merrits: [Minguzzi,13] gives somewhat stronger results but techniques do not extend below $\mathcal{C}^{1,1}$.

Consequences: Tools for $\mathcal{C}^{1,1}$-metrics

Recall: convexity fails for $\mathbf{g} \in \mathcal{C}^{1,\alpha}$ $(\alpha < 1)$

Theorem (Convexity [Kunzinger,S.,Stojković,14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then all points $p \in M$ possess a basis of convex (totally normal) neighborhoods.

For any pair p, q of points in a convex nbhd. $\mathcal{U}$ there is a unique geodesic entirely contained in $\mathcal{U}$ connecting p with q.

Theorem (Gauss Lemma [Kunzinger,S.,Stojković,Vickers,14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ then the exponential map is a radial isometry.

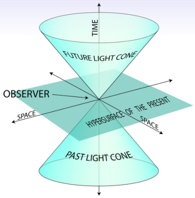
More precisely all $p \in M$ possess a basis of normal nbhds. U with $\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeo. and for almost all $x \in \tilde{U}$, if v_x, $w_x \in T_x(T_pM)$ and v_x is radial, then

$$\langle T_x \exp_p(v_x), T_x \exp_p(w_x) \rangle = \langle v_x, w_x \rangle.$$

Causality theory for $\mathcal{C}^{1,1}$-metrics

What is causality theory?

- essentially the theory of future & past
- tells how signals/fields propagate
  $\rightsquigarrow$ PDE, see talks of G.H. and C.S.



Theorem (Loc. causality [Kunzinger,S.,Stojković,Vickers,14])

If $\mathbf{g} \in \mathcal{C}^{1,1}$ *then the causality of* M *is locally Minkowskian.*

More precisely, all $p \in M$ *possess a basis of normal nbhds.* $exp_p : \tilde{U} \to U$ *a bi-Lipschitz homeomorphism and*

$$I^+(p,U) = exp_p(I^+(0) \cap \tilde{U}), \qquad J^+(p,U) = exp_p(J^+(0) \cap \tilde{U})$$

$$\partial I^+(p,U) = \partial J^+(p,U) = exp_p(\partial I^+(0) \cap \tilde{U}).$$

Main technique

Regularisations of the metric adapted to the causal structure [Chrusciel,Grant,12], [Kunz.,S.,Stojković,Vickers,14]

If $g \in \mathcal{C}^0$ then for any $\varepsilon > 0$ there exist smooth metrics $\check{\mathbf{g}}_\varepsilon$ and $\hat{\mathbf{g}}_\varepsilon$ with

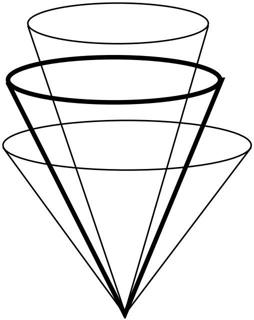
$$\check{\mathbf{g}}_\varepsilon \prec \mathbf{g} \prec \hat{\mathbf{g}}_\varepsilon,$$

$$d_h(\check{\mathbf{g}}_\varepsilon, \mathbf{g}) + d_h(\hat{\mathbf{g}}_\varepsilon, \mathbf{g}) < \varepsilon$$

where $d_h(\mathbf{g}_1, \mathbf{g}_2) :=$

$$\sup_{0 \neq X, Y \in TM} \frac{|\mathbf{g}_1(X,Y) - \mathbf{g}_2(X,Y)|}{\|X\|_h \|Y\|_h}$$

and h is some Riem. backgrd metr.



$$\mathbf{g} \prec \mathbf{h} \;:\Leftrightarrow$$
$$\mathbf{g}(X,X) \leq 0 \Rightarrow \mathbf{h}(X,X) < 0$$

$\mathcal{C}^{1,1}$: Further results and outlook

$\mathcal{C}^{1,1}$-causality theory works!

- Fundamental constructions (local causality, push up principles) of causality theory remain valid for $\mathbf{g} \in \mathcal{C}^{1,1}$.
- Accumulation curves of causal curves are causal.

  [Chrusciel,Grant,12]
- This allows to obtain all of standard causality theory for $\mathbf{g} \in \mathcal{C}^{1,1}$ following the classical proofs. [Kunzinger,S.,Stojković,Vickers,14]

Outlook

This (finally) puts us into a position (to try) to prove (Hawking's) singularity theorems for $g \in \mathcal{C}^{1,1}$.

see M. Kunzinger's talk

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Geodescis in impulsive gravitational waves

Nonexpanding impulsive gravitational waves

- model short but strong pulses of gravitational radiation propagating in Minkowski or (anti-)de Sitter universes
- relevant models of ultrarelativistic particle

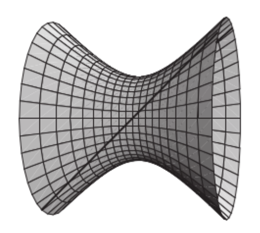
$= 0\}$

$\mathbf{g} \in \mathcal{C}^{0,1}$

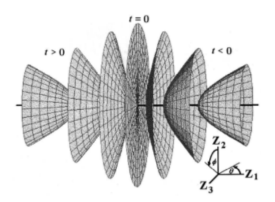
$V, Z, \bar{Z}))$

UdV

$_{,\bar{z}}))]^2$ (1)



anti-de Sitter universe



propagating wave

de Sitter universe

propagating wave

Geodesics: regularity, matching, completeness

$\mathcal{C}^1$-matching of the geodesics in impulsive grav. waves

- Physicists like to derive the geodesics by matching the geodesics of the background across the wave-surface.
- Only possible if geodesics — cross the wave-surface at all, and — are $\mathcal{C}^1$ across the wave-surface

Quest (Jiří Podolský)

Prove that the geodesics in these space-times are $\mathcal{C}^1$-curves.

Problem: Geodesic eqs. are ODEs with discontinuous r.h.s.

$$\ddot{\gamma}^j(t) + \Gamma^j_{kl}(\gamma(t))\, \dot{\gamma}^k(t)\, \dot{\gamma}^l(t) = 0$$

$\mathbf{g}_{ij} \in \mathcal{C}^{0,1} \Rightarrow \Gamma^j_{kl} \in L^\infty_{\mathrm{loc}}$

The case $\Lambda = 0$

- Metric (1) takes the simpler form

$$ds^2 = 2\,|dZ + U_+(H_{,Z\bar{Z}}dZ + H_{,\bar{Z}\bar{Z}}d\bar{Z})|^2 - 2\,dUdV \qquad (2)$$

⇝ geo. equations are non-autonomous with U as "time"-parameter

⇝ use Carathéodory's solution concept

Theorem (Geodescis for imp. pp-waves [Lecke,S.,Švarc,14])

The geodesic equations for the impulsive pp-wave metric (2) *has unique global solutions in the sense of Carathéodory with absolutely continuous velocities.*

Explicitly matched geodesics agree with
$\mathcal{D}'$-shadows of $\mathcal{G}$-solutions of [Kunzinger,S.,99a].
Complete picture emerges in combination with [Kunzinger,S.,99b].

The case $\Lambda \neq 0$

- U **not** a parameter $\rightsquigarrow$ no Carathéodory-sols. but **Filippov**-sols

Observation (Geodesics for general $\mathbf{g} \in \mathcal{C}^{0,1}$ [S.,14])

The geodesic equation for any locally Lipschitz metric has solutions in the sense of Filippov with absolutely continuous velocities.

- $\mathcal{C}^1$-matching needs uniqueness which does not hold in general
- BUT (1) is pw. $\mathcal{C}^\infty$, discont. across totally geodesic null-hypersrf.

Theorem (Nonexp. imp. w. [Podolský,Sämann,S.,Švarc,14])

The geodesic eq. for the nonexpanding imp. wave metric (1) *has unique global solutions in the sense of Filippov w. a.c. velocities.*

Next steps: $\mathcal{D}'$-picture for $\Lambda \neq 0$ (Lecke,Sämann,S.,Stojković)
expanding impulsive waves (Podolský,Sämann,S.,Švarc)

Some related Literature

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Thank you for your attention!

Expendic G-02
Sculpture, Aluminium
$283 \times 283 \times 24$cm
(c) Tomas Eller, 2009