Introduction to

Partial Differential Operators on Manifolds

and

Normally Hyperbolic Operators

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In this note we give a pedagogical account on linear partial differential operators (PDOs) on manifolds and on vector bundles. Thereby we focus on the algebraic approach, in which one defines a PDO as linear operator which interacts in a specific way with the $\mathcal{C}^\infty$-module structure of spaces of sections on vector bundles. To connect to the local theory we also prove Peetre's theorem which says that local operators on spaces of sections locally look just like usual PDOs (of finite order). Then we introduce normally hyperbolic operators, give some examples, and prove the Weitzenböck formula. Finally, we give a brief outlook on some aspects of the theory of normally hyperbolic operators.

$\boxed{1}$ DIFFERENTIAL OPERATORS ON MANIFOLDS

In this first section we introduce linear partial differential operators (PDOs) on manifolds, that is PDOs acting on smooth functions on a manifold M as well as PDOs taking smooth sections of some vector bundle E over M to smooth sections of some other vector bundle F over the same manifold M.
There are several equivalent approaches to PDOs on manifolds.

(1) The most basic one consists in pasting together locally defined PDOs, i.e., in defining a PDO as a map taking smooth functions (sections) to smooth functions (sections) that in any (vector bundle-)chart looks like a PDO (of finite order) on an open subset of Euclidean space (see [Hör90, Sec. 6.4]).

A more elegant way using purely intrinsic notions and avoiding charts is provided by using either of the following two characterisations of PDOs.

(2) Hörmander's characterisation: A linear, continuous operator between spaces of smooth functions is a PDO of order at most m if and only if

$$\mathbb{R} \ni \tau \mapsto e^{-i\tau\varphi}\, P(e^{i\tau\varphi})(x)$$

is a polynomial in τ of degree at most m for all smooth φ and all points x (see [Hör64], and also [Kah80, Thm. 6.3], [CP82, Sec. 1.7]; for operators acting on sections replace $P(e^{i\tau\varphi})$ by $P(e^{i\tau\varphi}u)$ for an arbitrary section u).

(3) Peetre's Theorem: A linear (not necessarily continuous!) operator between spaces of smooth functions (sections) is local (i.e., does not increase supports) if and only if it can be written as a usual PDO (of finite order) locally around each point (see [Pee60] and also [Kah80, Ch. 6]).

Finally there is an entirely algebraic approach to PDOs which actually works in the more general situation of operators acting between modules over some associative, commutative algebra with unit.

(4) Algebraic approach: PDOs (of finite order) are linear maps between spaces of smooth functions (or sections) that interact with the $\mathcal{C}^\infty$-module structure in a specific way ([Her73, Ch. 1] and also [Nic07, Sec. 10.1]).

Since the algebraic approach is probably the least well-known to analysts we will focus on it. However, to give a coherent introduction and to show that the algebraically defined PDOs "really are" PDOs we also include some aspects of (3).

We start by fixing the basic notions.

1.1. PDOs on open subsets of Euclidean space.

Let $\Omega \subset \mathbb{R}^n$ be open. A *partial differential operator* on Ω is a linear mapping P on a suitable space of functions (or distributions) defined on Ω of the form

$$u \mapsto Pu := \sum_{\alpha} a_\alpha D^\alpha u. \tag{1.1}$$

Here $\alpha \in \mathbb{N}_0^n$ is a multi-index and we have set $D_j := -i\partial_j$ $(1 \leqslant j \leqslant n)$ and used multi-index notation, that is $D^\alpha = D_1^{\alpha_1} \cdot \dots \cdot D_n^{\alpha_n}$.
Note carefully that we do not suppose the sum in (1.1) to be finite! Instead we assume that the coefficients $(a_\alpha)_\alpha$ are a locally finite family of functions on Ω, meaning that their supports $(\mathrm{supp}(a_\alpha))_\alpha$ form a locally finite family of sets, i.e., each point $x \in \Omega$ has a neighbourhood U that intersects only finitely many of the sets $\mathrm{supp}(a_\alpha)$. This implies that there exists an integer m such that for all functions u defined on U we have

$$Pu := \sum_{\alpha \leqslant m} a_\alpha D^\alpha u.$$

Equivalently, all but finitely many of the a_α vanish identically on any compact subset of Ω.
We will sometimes denote P by $P(x, D)$ to emphasise that the coefficients depend on $x \in \Omega$. If all the functions $a_\alpha \in \mathcal{C}^\infty(\Omega)$, we will say P has *smooth coefficients* and we will exclusively deal with such PDOs. In this case it is obvious that

$$P(x, D) : \ \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega).$$

is a linear and continuous operator.
For any PDO P on Ω we define its *order* as

$$m := \sup\{|\alpha| : \ \exists \alpha \text{ with } a_\alpha \not\equiv 0\}.$$

Observe that the order may be infinite. An example of such an operator on $\mathbb{R}$ is the following: choose $a_j \in \mathcal{D}(j, j+1)$ with $a_j(j + \frac{1}{2}) = 1$ and set $P(x, D) = \sum_{j \in \mathbb{N}} a_j(x) \frac{d}{d_j}$. However, as we have seen above *locally* every PDO has a finite order.

If P is (globally) of finite order m, the sum in (1.1) ranges only over $|\alpha| \leqslant m$, i.e.,

$$u \mapsto Pu := \sum_{|\alpha| \leqslant m} a_\alpha D^\alpha u \qquad \forall u \in \mathcal{C}^\infty(\Omega). \tag{1.2}$$

In this case we define the *principal symbol* P_m of P by

$$P_m(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \qquad (x \in \Omega,\ \xi \in \mathbb{R}^n), \tag{1.3}$$

which is a homogeneous polynomial of degree m in ξ with coefficients in $\mathcal{C}^\infty(\Omega)$.

1.2. Generalisation to systems. The above notions easily generalise to systems of PDEs: A linear differential (s, t)-system on Ω consists of an operator

$$P(x, D) = \sum_\alpha a_\alpha(x)\, D^\alpha, \tag{1.4}$$

where $\Omega \ni x \mapsto a_\alpha(x)$ now is a locally finite family of smooth maps on Ω taking values in the space of matrices $L(\mathbb{R}^s, \mathbb{R}^t)$. Clearly such an operator is a continuous mapping $P : \mathcal{C}^\infty(\Omega, \mathbb{R}^s) \cong \mathcal{C}^\infty(\Omega)^s \to \mathcal{C}^\infty(\Omega, \mathbb{R}^t) \cong \mathcal{C}^\infty(\Omega)^t$ and we also speak of an (s, t)-PDO.
If P is of finite order m, the principal symbol P_m of P at a point $(x, \xi) \in \Omega \times \mathbb{R}^n$ is now defined by

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha,$$

which is a matrix in $L(\mathbb{R}^s, \mathbb{R}^t)$.
Of course all of the above can be formulated in the case of complex coefficients in the same way.

§ 1.1. PEETRE'S THEOREM

We now follow the route outlined in (3) above to some extend and start with a discussion of the basic notion of locality, which we do in the context of linear operators acting on smooth sections of vector bundles.
To this end we introduce some notation: Let $E \to M$ and $F \to M$ be real (or—with the obvious changes in what follows—complex) vector bundles of fiber dimensions s and t over the same smooth manifold M, which we always suppose to be Hausdorff and paracompact. We will denote the respective projections by π_E and π_F or simply by π if

there is no danger of misunderstanding. The fibers of E at $x \in M$ will be denoted by $\pi_E^{-1}(x)$ or E_x and likewise for F. The spaces of smooth sections of E and F are denoted by $\Gamma(E)$ and $\Gamma(F)$, respectively.

1.3. Local operators.

Definition. A linear operator $P : \Gamma(E) \to \Gamma(F)$ is called *local* if it does not increase supports, that is

$$\operatorname{supp}(Pu) \subseteq \operatorname{supp}(u) \quad \forall u \in \Gamma(E).$$

For a local operator we may define its *restriction to open subsets* $U \subseteq M$ in a consistent way, i.e., such that

$$(Pu)|_U = P(u|_U) \qquad \forall u \in \Gamma(E). \tag{1.5}$$

Indeed, suppose $u, v \in \Gamma(E)$ agree on U, then by locality $\operatorname{supp}(Pu - Pv) \subseteq \operatorname{supp}(u - v) \subseteq M \setminus U$, hence

$$u|_U = v|_U \implies (Pu)|_U = (Pv)|_U. \tag{1.6}$$

Now let $u \in \Gamma(U, E)$ be a local section on U. To define the action of P on u choose an open set $V \subseteq \bar{V} \subseteq U$ and $\chi \in \mathcal{D}(U)$ with $\chi|_V = 1$. Then $\chi u \in \Gamma(U, E)$ is compactly supported hence by trivial extension $\chi u \in \Gamma(E)$ and we may set

$$Pu := P(\chi u).$$

By (1.6) we have $(Pu)|_V$ is independent of the choice of χ. Now to define the restriction of P to U use a covering by relatively compact subsets V_α and corresponding cut-off functions χ_α. By (1.6) we have $P(\chi_\alpha u)|_{V_\alpha \cap V_\beta} = P(\chi_\beta u)|_{V_\alpha \cap V_\beta}$. Hence the $P(\chi_\alpha u)|_{V_\alpha}$ form a coherent family and so define a smooth section $Pu \in \Gamma(U, F)$. This finally allows us to define the restriction $P|_{\Gamma(U)}$, or $P|_U$ for short, of P such that (1.5) holds.

1.4. Local expressions.

In order to study local operators $P : \Gamma(E) \to \Gamma(F)$ in some more detail we introduce some notation. Let (U, ϕ) be a chart of M such that both bundles E and F are trivial over U (i.e, $E|_U$ is diffeomorphic to $U \times \mathbb{R}^s$ and $F|_U$ to $U \times \mathbb{R}^t$). Then there exist diffeomorphisms

$$\begin{aligned} \pi_E^{-1}(U) \ni z &\mapsto (\phi(\pi_E(z)), v(z)) \in \phi(U) \times \mathbb{R}^s \\ \pi_F^{-1}(U) \ni z &\mapsto (\phi(\pi_F(z)), w(z)) \in \phi(U) \times \mathbb{R}^t \end{aligned}$$

and v resp. w are fiberwise linear (i.e., v is a linear isomorphism from each fiber $\pi_E^{-1}(x)$ to $\mathbb{R}^s$ and likewise for w.) Hence the mappings

$$u \mapsto v \circ u \circ \phi^{-1} \quad \text{and} \quad u \mapsto w \circ u \circ \phi^{-1}$$

are isomorphisms from $\Gamma(U,E)$ to $\mathcal{C}^\infty(\varphi(U))^s$ resp. from $\Gamma(U,F)$ to $\mathcal{C}^\infty(\varphi(U))^t$. Now by (1.5) there exists a linear mapping $Q: \mathcal{C}^\infty(\phi(U))^s \to \mathcal{C}^\infty(\varphi(U))^t$ such that

$$w \circ (Pu|_U) \circ \phi^{-1} = Q(v \circ (u|_U) \circ \phi^{-1}). \tag{1.7}$$

We will call Q the *local expression* of P on U (w.r.t. v and w). Our next aim is to show that the local expressions of a local operator are just differential operators (of finite order).

1.5. Peetre's Theorem.

Theorem. Let $P: \Gamma(E) \to \Gamma(F)$ be a linear and local operator. Then for each point x in M there exists a chart (U,ϕ), such that both bundles are trivial over U, an integer m, and a family of smooth functions $a_\alpha: \phi(U) \to L(\mathbb{R}^s,\mathbb{R}^t)$ $(|\alpha| \leqslant m)$ such that the local expression Q of P on U takes the form

$$Q(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha. \tag{1.8}$$

Observe that (1.8) says that the local expressions of a local operator precisely are (s,t)-PDOs of finite (but possibly different) orders, cf. 1.2.

For the proof we shall use the following result.

Lemma. Let $P: \mathcal{C}^\infty(\Omega;\mathbb{R}^s) \to \mathcal{C}^\infty(\Omega;\mathbb{R}^t)$ be linear and local and let $x_0 \in \Omega$. Then there exists an open neighbourhood U of x_0 in Ω, $m \in \mathbb{N}$ and $C > 0$ such that

$$\|P\varphi\|_\infty \leqslant C \max_{|\alpha|\leqslant m} \|\partial^\alpha \varphi\|_\infty =: C\|\varphi\|_{(m)} \qquad \forall \varphi \in \mathcal{D}(U\setminus\{x_0\};\mathbb{R}^s). \tag{1.9}$$

Proof. Let $x_0 \in U_0$ be open and relatively compact in Ω. We suppose that (1.9) does not hold. Then

$$\exists U_1 \subseteq \overline{U}_1 \subseteq U_0 \setminus \{x_0\},\ \exists \varphi_1 \in \mathcal{D}(U_1,\mathbb{R}^s): \ \|P\varphi_1\|_{(0)} > 2^2\|\varphi_1\|_{(1)}.$$

Replacing U_0 by $U_0 \setminus \overline{U}_1$ our indirect assumption implies

$$\exists U_2 \subseteq \overline{U}_2 \subseteq U_0 \setminus (U_1 \cup \{x_0\}),\ \exists \varphi_2 \in \mathcal{D}(U_2,\mathbb{R}^s): \ \|P\varphi_2\|_{(0)} > 2^{2\cdot 2}\|\varphi_2\|_{(2)}.$$

Inductively we obtain in this way

$$\exists U_k \subseteq \overline{U}_k \subseteq U_0 \setminus (\bigcup_{i=1}^{k-1} U_k \cup \{x_0\}),\ \exists \varphi_k \in \mathcal{D}(U_k,\mathbb{R}^s): \ \|P\varphi_k\|_{(0)} > 2^{2k}\|\varphi_k\|_{(k)}. \tag{1.10}$$

Now we set

$$\varphi := \sum_{k=1}^\infty \frac{1}{2^k\|\varphi_k\|_{(k)}}\, \varphi_k.$$

Since the supports of the φ_k's are disjoint, this expression is defined. Moreover, it converges for each $\| \ \|_{(l)}$ hence defines a smooth function φ with $\operatorname{supp}(\varphi) \subseteq \bar{U}_0$.
Now since restriction is well defined we obtain

$$(P\varphi)|_{U_k} = P(\varphi|_{U_k}) = \frac{1}{2^k \|\varphi_k\|_{(k)}} P\varphi_k.$$

Moreover, (1.10) implies that

$$\forall k \in \mathbb{N}\ \exists y_k \in U_k \subseteq U_0 :\ |P(\varphi_k)(y_k)| > 2^{2k} \|\varphi_k\|_{(k)}$$

and hence

$$|P(\varphi)(y_k)| = |\frac{1}{2^k \|\varphi_k\|_{(k)}} P\varphi_k(y_k)| > 2^k,$$

which contradicts the fact that $P\varphi$ as a smooth function on Ω is bounded on the relative compact subset U_0. □

Proof of the Theorem. Fix $x \in M$ and choose a relatively compact chart neighbourhood U of x that is trivialising for both bundles. Then as detailed in 1.4 the local expression Q of P on U is an operator from $\mathcal{C}^\infty(\phi(U))^s$ to $\mathcal{C}^\infty(\phi(U))^t$. Moreover, Q can be written as $Q(f_1, \dots, f_s) = (\sum_{i=1}^s Q_{ki}(f_i))_{k=1}^t$ with $Q_{ki} = \pi_k \circ Q \circ j_i$, where j_i is the injection $\mathcal{C}^\infty(\phi(U)) \ni f \mapsto (0, \dots, 0, f, 0, \dots, 0) \in \mathcal{C}^\infty(\phi(U))$ and and π_k is the k-th projection $\mathcal{C}^\infty(\phi(U))^t \to \mathcal{C}^\infty(\phi(U))$. Since each Q_{ki} is clearly linear and local we may assume w.l.o.g. that $r = 1 = s$.
To begin with let $V \subseteq \phi(U)$ be open and suppose that

$$\exists C > 0,\ m \in \mathbb{N}_0 :\ \|Q\varphi\|_{(0)} \leqslant C \|\varphi\|_{(m)} \qquad \forall \varphi \in \mathcal{D}(V).$$

Now fix $x \in V$ and set $Q_x :\ \varphi \mapsto (Q\varphi)(x)$. Then

$$|Q_x \varphi| = |Q\varphi(x)| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \varphi\|_\infty,$$

which shows that Q_x is a distribution of order at most m.
Moreover, let $\operatorname{supp}(\varphi) \subseteq V \setminus \{x\}$. Locality implies that $\operatorname{supp}(Q\varphi) \subseteq V \setminus \{x\}$ so that $Q_x\varphi = 0$. Hence we obtain $\operatorname{supp}(Q_x) \subseteq \{x\}$. Now the structure theorem of distributions tells us that Q_x is a linear combination of derivatives of δ_x of order at most m, i.e., there exists coefficients $\tilde{a}_\alpha(x) \in \mathbb{C}$ such that $Q_x = \sum\limits_{|\alpha| \leqslant m} \tilde{a}_\alpha(x) \partial^\alpha \delta_x$ that is

$$Q\varphi(x) = Q_x(\varphi) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \tilde{a}_\alpha(x) \partial^\alpha \varphi(x) =: \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial^\alpha \varphi(x) \qquad \forall \varphi \in \mathcal{D}(V). \tag{1.11}$$

By (1.6) we may employ a cut-off around x to see that (1.11) even holds for all $f \in \mathcal{C}^\infty(V)$. (Indeed let $\chi \in \mathcal{D}(V)$ with $\chi \equiv 1$ on a neighbourhood of x then for any $f \in \mathcal{C}^\infty(V)$ we may write $Qf(x) = Q(\chi f)(x) = \sum a_\alpha(x) \partial^\alpha (\chi f)(x) = \sum a_\alpha(x) \partial^\alpha f(x)$.)
We next show that all a_α depend smoothly on x: Insert $f = x^\nu$ for some multi-index ν into (1.11) we obtain

$$Q(x^\nu)(x) = \sum_{\alpha \leqslant \nu} \alpha! \binom{\nu}{\alpha} a_\alpha(x) x^{\nu - \alpha} \qquad \forall |\nu| \leqslant m.$$

Now for $\nu = 0$ we find $\mathcal{C}^\infty(V) \ni P(1)(x) = a_0(x)$. Going on by induction on $|\nu|$ we obtain that all $a_\alpha \in \mathcal{C}^\infty(V)$.
Finally we prove the theorem. Since $\phi(U)$ is relatively compact we may cover it by finitely many (V_i, x_i) $(1 \leqslant i \leqslant l)$, where each (V_i, x_i) is of the form of (U, x_0) in the above lemma. Using a subordinate partition of unity we obtain

$$\exists C\ \exists m:\ \|Q\varphi\|_{(0)} \leqslant C\|\varphi\|_{(m)} \qquad \forall \varphi \in \mathcal{D}(\phi(U) \setminus \{x_1, \ldots, x_l\})$$

(Here $m = \max\{m_i : 1 \leqslant i \leqslant l\}$ where m_i are the integers of the lemma, and C depends on the C_i and the norms of the partition functions.) So by the above (1.11) holds for all $f \in \mathcal{C}^\infty(\phi(U)\setminus\{x_1, \ldots, x_l\})$. Now let $f \in \mathcal{C}^\infty(\phi(U))$ be arbitrary then (1.11) holds for $f|_{(\phi(U)\setminus\{x_1,\ldots,x_l\})}$ but since both sides of the equation are continuous they also have to agree on the finitely many exception points $x_1, \ldots, x_l$. So we obtain (1.11) for all $f \in \mathcal{C}^\infty(\phi(U))$ and we are done. □

1.6. PDOs on vector bundles.

Recalling that Peetre's theorem states that a local operator $P:\ \Gamma(E) \to \Gamma(F)$ locally just is a PDO of finite order, the following definition is sensible and also particularly elegant.

Definition. A linear and local operator $P:\ \Gamma(E) \to \Gamma(F)$ is called a PDO. If both bundles are trivial, i.e., $E = F = M \times \mathbb{R}$ (or $\mathbb{C}$) we call P a PDO on M.

To introduce the notion of *order* in this setting we first define the order m at a point $x \in M$ as the largest number $|\alpha|$ such that there is some $a_\alpha(x) \neq 0$ in a local expression for P in a neighbourhood of x. From the chain rule and the Leibniz rule it is clear that m is independent of the choice of local expressions, hence well-defined.
The order of P is defined as the supremum of the orders of P at all points in M. Now Peetre's theorem tells us that every PDO locally has a finite order. However, if the manifold M is non-compact the order may be infinite and we are precisely in the same situation as on subsets of Euclidean space, cf. 1.1.
If P is of finite order m one may define its *principal symbol* via the local expressions—in which case one has to invoke a calculation in charts to see that this notion is well-defined, see [CP82, Sec. 1.7]. Alternatively one may use Hörmander's characterisation of PDOs of finite order mentioned in (2) above, or define the symbol at a point via inserting functions into P that vanish to m-th order at that point (see [Kah80, Ch. 6]).
We will, however, proceed to the algebraic approach to PDOs where the notions of order and symbol are defined in an intrinsic and geometric way. Only later we will see that the operators obtained in this way precisely are the PDOs of finite order of this section.

$\boxed{2}$ THE ALGEBRAIC APPROACH TO PDOS ON MANIFOLDS

We now start our account on the algebraic approach to PDOs which to the author's best knowledge first appeared in [Her70] but might have been folk knowledge already before. We will mainly follow the account in [Nic07, Sec. 10.1].

2.1. Basic definitions. Let again E and F be real (or complex) vector bundles over the same manifold M and denote the respective spaces of sections by $\Gamma(E)$ and $\Gamma(F)$. We denote the space of linear operators taking sections in E to sections in F by $\mathrm{Op}(E,F)$, i.e., we set

$$\mathrm{Op}(E,F) = \{T : \Gamma(E) \to \Gamma(F), \text{linear}\}.$$

However, the space of sections $\Gamma(E)$ possesses more structure than just that of a vector space: $\Gamma(E)$ is a module over the ring of smooth functions $\mathcal{C}^\infty(M)$, i.e., we have the operation of pointwise multiplication $\mathcal{C}^\infty(M) \times \Gamma(E) \to \Gamma(E)$. So we can ask for more algebraic compatibility of operators than just linearity: We will identify PDOs as elements of $\mathrm{Op}(E,F)$ that interact with the $\mathcal{C}^\infty(M)$-module structure of $\Gamma(E)$ and $\Gamma(F)$ in a specific way.
To begin with we set

$$\mathrm{PDO}^0(E,F) := \mathrm{Hom}(E,F) = \{P \in \mathrm{Op}(E,f) : \ P \text{ is } \mathcal{C}^\infty(M)\text{-linear}\},$$

i.e., $P \in \mathrm{Op}(E,F)$ belongs to $\mathrm{PDO}^0(E,F)$ if for all $f \in \mathcal{C}^\infty(M)$ and all $u \in \Gamma(E)$ we have $P(fu) = fP(u)$ or in terms of commutators

$$[P,f](u) := P(fu) - fP(u) = 0.$$

More structurally, each $f \in \mathcal{C}^\infty(M)$ defines a map

$$\begin{aligned} \mathrm{ad}(f) : \mathrm{Op}(E,F) &\to \mathrm{Op}(E,F) \\ P &\mapsto \mathrm{ad}(f)P := [P,f] = P \circ f - f \circ P. \end{aligned}$$

Observe that in the above formula f actually denotes the operator of ($\mathcal{C}^\infty(M)$ module-) multiplication with the function f. Now we may rephrase the definition of PDO^0 as

$$\mathrm{PDO}^0(E,F) = \{P \in \mathrm{Op}(E,F) : \ \mathrm{ad}(f)P = 0 \quad \forall f \in \mathcal{C}^\infty(M)\} =: \ker \mathrm{ad}$$

and go on iteratively.

Definition. (PDOs) For $m \in \mathbb{N}$ we define the spaces

$$\begin{aligned} \mathrm{PDO}^{(m)}(E,F) &:= \ker \mathrm{ad}^{m+1} \\ &= \{P \in \mathrm{Op}(E,F) : \; P \in \ker \mathrm{ad}(f_0)\mathrm{ad}(f_1)\cdots \mathrm{ad}(f_m) \quad \forall f_i \in \mathcal{C}^\infty(M)\} \end{aligned}$$

and we set

$$\mathrm{PDO}(E,F) := \bigcup_{m\geqslant 0} \mathrm{PDO}^{(m)}(E,F).$$

Observe that

$$\mathrm{PDO}^{(m)}(E,F) = \{P \in \mathrm{Op}(E,F) : \; \mathrm{ad}(f)P \in \mathrm{PDO}^{(m-1)}(E,F) \quad \forall f \in \mathcal{C}^\infty(M)\},$$

a point of view which is particularly useful in induction proofs as we soon shall see.
In case $E = F$ we write $\mathrm{PDO}^{(m)}(E)$ instead of $\mathrm{PDO}^{(m)}(E,E)$. In case E is the trivial bundle $M \times \mathbb{R}$ (or $M \times \mathbb{C}$), elements of $\mathrm{PDO}^{(m)}(E)$ are maps on $\mathcal{C}^\infty(M)$ and are called scalar operators and their space will be denoted by $\mathrm{PDO}^{(m)}(M)$. Also, we will sometimes just write $\mathrm{PDO}^{(m)}$ if the bundels are clear from the context.

2.2. Some examples.

To get acquainted with the above definitions and to explore their content we will now look at some explicit examples of geometric operators in PDO.

Example. (The gradient) Let (M,g) be a semi-Riemannian manifold. We define the operator

$$\begin{aligned} \mathrm{grad} : \mathcal{C}^\infty(M) &\to \mathcal{X}(M) \equiv \Gamma(TM) \\ u &\mapsto \mathrm{grad}(u) := (du)^\#. \end{aligned}$$

Here $(du)^\#$ denotes the metric equivalent vector field of the differential du of u, i.e., $du(X) = g((du)^\#, X)$ for all $X \in \mathcal{X}(M)$. In coordinates we hence have

$$\mathrm{grad}(u)^i \partial_i = g^{ij}(\partial_j u)\partial_i,$$

with g^{ij} denoting the components of the inverse metric and summation convention in effect.
We check that $\mathrm{grad} \in \mathrm{PDO}^{(1)}(M \times \mathbb{R}, TM)$: For all $f \in \mathcal{C}^\infty(M)$ we have

$$\mathrm{ad}(f)\mathrm{grad}(u) = [\mathrm{grad}(u), f] = \mathrm{grad}(fu) - f\mathrm{grad}(u) = \mathrm{grad}(f)u = (df)^\# u, \tag{2.1}$$

which shows that $\mathrm{ad}(f)\mathrm{grad}$ is just (the operator of) multiplication with $\mathrm{grad}(f)$. This clearly is a $\mathcal{C}^\infty(M)$-linear operation, hence $\mathrm{ad}(f)\mathrm{grad} \in \mathrm{PDO}^0$ and so $\mathrm{grad} \in \mathrm{PDO}^{(1)}$.

Example. (Covariant derivatives) A covariant derivative or linear connection on E is a linear map

$$\nabla :\ \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that for all $f \in \mathcal{C}^\infty(M)$ and all $u \in \Gamma(E)$ we have

$$\nabla(fu) = df \otimes u + f\nabla u.$$

Here $T^*M \otimes E$ denotes the tensor product of the bundles T^*M and E, i.e., $T^*M \otimes E$ is the bundle over M with fibers $T^*_x M \otimes E_x$. We recall a standard result on sections of product bundles $E \otimes F$ stating $\Gamma(E \otimes F) = \Gamma(E) \otimes \Gamma(F)$, where the tensor product on the right is over the module $\mathcal{C}^\infty(M)$.
A simple example of a connection is the Levi-Civita connection of a semi-Riemannian metric, which in the present terminology is a connection on TM and by standard methods can be extended to all tensor bundles $T^r_s M$.
To check that $\nabla \in \mathrm{PDO}^{(1)}$ we calculate

$$(ad(f)\nabla)u = [\nabla, f]u = \nabla(fu) - f\nabla u = df \otimes u. \tag{2.2}$$

Hence $ad(f)\nabla$ is just tensor multiplication with the one-form df, which is clearly $\mathcal{C}^\infty$-linear.

Example. (The divergence) Again on a semi-Riemannian manifold (M, g) we define the divergence by

$$\begin{aligned} \mathrm{div} :\ \mathfrak{X}(M) &\to \mathcal{C}^\infty(M) \\ X &\mapsto \mathrm{div}X := \mathrm{tr}(\nabla X). \end{aligned}$$

Here tr denotes the trace of $(1, 1)$-tensors and ∇ denotes the Levi-Civita connection of g. Hence locally we have

$$\mathrm{div}(X) = \nabla_i X^i = \partial_i X^i + \Gamma^i_{ki} X^k = |\det g|^{-\frac{1}{2}} \partial_i (|\det g|^{\frac{1}{2}} X^i)$$

where Γ^i_{jk} denotes the Christoffel symbols (of the second kind) and $\det g$ the determinant of g.
We now check that $\mathrm{div} \in \mathrm{PDO}^{(1)}(TM, M \times \mathbb{R})$. Indeed we have for $f \in \mathcal{C}^\infty(M)$

$$ad(f)\mathrm{div}X = [\mathrm{div}X, f] = \mathrm{div}(fX) - f\mathrm{div}X = df(X), \tag{2.3}$$

hence $ad(f)\mathrm{div}$ is just the dual action of the one-form df which clearly is $\mathcal{C}^\infty$-linear.

Example. (Scalar operators in $\mathrm{PDO}^{(1)}$) We consider $P \in \mathrm{PDO}^{(1)}(M)$. By definition we have for all $f \in \mathcal{C}^\infty(M)$ that $[P, f] \in \mathrm{PDO}^0(M)$ which allows us to define a linear map

$$\begin{aligned} \phi : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ f &\mapsto \phi(f) := [P, f](1). \end{aligned}$$

We next show that ϕ is a derivation on $\mathcal{C}^\infty(M)$. Indeed for $f,\ g \in \mathcal{C}^\infty(M)$ we have

$$\begin{aligned} \phi(fg) &= [P, fg](1) = P(fg) - fg\,P(1) = P(fg) - fP(g) + fP(g) - fg\,P(1) \\ &= [P, f](g\,1) + f\,P(g\,1) - fg\,P(1) = g[P, f](1) + f[P, g](1) = g\phi(f) + f\phi(g). \end{aligned}$$

Being a derivation, ϕ is given by a vector field on M, i.e., there exists $X \in \mathcal{X}(M)$ such that

$$\phi(f) = Xf \quad \forall f \in \mathcal{C}^\infty(M).$$

To determine P completely we set $\mu := P(1) \in \mathcal{C}^\infty(M)$. Then we obtain for all $u \in \mathcal{C}^\infty(M)$

$$Pu = P(u\,1) = P(u1) - uP(1) + uP(1) = [P, u](1) + uP(1) = Xu + \mu u,$$

hence P is sum of a derivation on $\mathcal{C}^\infty(M)$ and a zero order term.
It even holds that the space of PDOs of order at most one on M is the direct sum of the space $\mathrm{Der}(\mathcal{C}^\infty(M))$ of derivations on $\mathcal{C}^\infty(M)$ and the space of homomorphisms on $\mathcal{C}^\infty(M)$, i.e.,

$$\mathrm{PDO}^{(1)}(M) = \mathrm{Der}(\mathcal{C}^\infty(M)) \oplus \mathrm{Hom}(\mathcal{C}^\infty(M)).$$

Indeed, by the above we only have to show that $\mathrm{Der}(\mathcal{C}^\infty(M)) \cap \mathrm{Hom}(M)$ is trivial. To see this assume $P \in \mathrm{Der}(\mathcal{C}^\infty(M)) \cap \mathrm{Hom}(M)$. Then we find for $f,\ g \in \mathcal{C}^\infty(M)$

$$fgP(1) = P(fg) = P(f)g + fP(g) = 2fgP(1),$$

which forces $P(1)$ to vanish. But then we have $P(f) = fP(1) = 0$ for all $f \in \mathcal{C}^\infty(M)$ hence $P = 0$.

2.3. Connecting to the "classical approach".

We now are going to bridge the gap to the approach to PDOs of 1.6. The first step is to show locality of elements in $\mathrm{PDO}(E, F)$.

Lemma. (Locality) Any $P \in \mathrm{PDO}^{(m)}(E, F)$ is local.

Proof. We proceed by induction on m.
$m = 0$: Let $P \in \mathrm{PDO}^0$ and $u \in \Gamma(E)$. For any $U \subseteq M$ open that contains $\mathrm{supp}(u)$ we may find a function $f \in \mathcal{C}^\infty(M)$ such that $f = 1$ on $\mathrm{supp}(u)$ and $f = 0$ outside U. So $fu = u$ and we have

$$P(u) = P(fu) = fP(u)$$

and so $P(u) = 0$ outside U. Since U was arbitrary we obtain $\operatorname{supp}(P(u)) \subseteq \operatorname{supp}(u)$.
$m \mapsto m+1$: Let now $P \in \mathrm{PDO}^{(m+1)}$, $f \in \mathcal{C}^\infty(M)$, $u \in \Gamma(E)$. From $P(fu) = [P,f](u) + fP(u)$ we find that

$$\operatorname{supp}(P(fu)) \subseteq \underbrace{\operatorname{supp}([P,f](u))}_{\substack{\subseteq \operatorname{supp}(u) \\ \text{by induction}}} \quad \cup \quad \underbrace{\operatorname{supp}(f(P(u)))}_{\substack{\subseteq \operatorname{supp}(f) \cap \operatorname{supp}(P(u)) \\ \subseteq \operatorname{supp}(f)}} \quad .$$

Now choosing U and f as in the case $m = 0$ we again find that $\operatorname{supp}(P(u)) \subseteq \operatorname{supp}(u)$.□

The above lemma clearly tells us that any $P \in \mathrm{PDO}$ is a PDO in the sense of 1.6 above. But actually we can say more: PDO is the space of PDOs of finite order in the sense of the definitions in 1.6 above.

Proposition. (PDO(E,F) is the space of PDOs of finite order) The space $\mathrm{PDO}^{(m)}(E,F)$ precisely is the space of PDOs (in the sense of 1.6) of (finite) order at most m.

Proof. Let $P \in \mathrm{PDO}^{(m)}(E,F)$. Then by the above lemma P is local, hence by Peetre's theorem around each point x has a local expression that is a PDO of order $m(x)$. But we have $m(x) \leqslant m$ since otherwise the local expression Q would violate $\operatorname{ad}(f_0)\ldots\operatorname{ad}(f_m)Q = 0$ and so would P.
Conversely every PDO of (finite) order m (in the sense of 1.6) has all local expressions Q of order at most m. These clearly satisfy $\operatorname{ad}(f_0)\ldots\operatorname{ad}(f_m)Q = 0$ (just do a little calculation) hence by (1.7) P does and so $P \in \mathrm{PDO}^{(m)}$. □

2.4. Order and principal symbol.

Our next aim is to define the principal symbol for $P \in \mathrm{PDO}^{(m)}$. To this end we derive two essential properties of the operator

$$\operatorname{ad}(f_1)\cdots\operatorname{ad}(f_m)\,P \ \in \mathrm{PDO}^0$$

for $f_1, \ldots, f_m \in \mathcal{C}^\infty(M)$. The first one is that it does not change if we permute the functions $f_1, \ldots, f_m$, and the second one is that it localises to points.

Lemma. Let $P \in \mathrm{PDO}^{(m)}(E,F)$.

(i) (The ad(f)'s commute) For any pair f, $g \in \mathcal{C}^\infty(M)$ we have

$$\operatorname{ad}(f)(\operatorname{ad}(g)P) = \operatorname{ad}(g)(\operatorname{ad}(f)P).$$

(ii) (Pointwise character) Let f_i, g_i $(1 \leqslant i \leqslant m)$ such that at $x_0 \in M$ we have $df_i(x_0) = dg_i(x_0)$ for all $1 \leqslant i \leqslant m$. Then

$$(\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_m)P)|_{x_0} = (\mathrm{ad}(g_1) \cdots \mathrm{ad}(g_m)P)|_{x_0}.$$

Proof. Let P, f, g, as well as f_i and g_i as in the statement.

(i) $\mathrm{ad}(f)(\mathrm{ad}(g)P) = [[P, g], f] = [[P, f], g] + [P, [f, g]] = [[P, f], g] = \mathrm{ad}(g)(\mathrm{ad}(f)P)$.

(ii) Since $\mathrm{ad}(\text{constant}) = 0$ we may assume w.l.o.g. that $f_i(x_0) = g_i(x_0)$ for all i.
Actually it suffices to show that

$$(\mathrm{ad}(f_1)\mathrm{ad}(f_2) \cdots \mathrm{ad}(f_m)P)|_{x_0} = (\mathrm{ad}(g_1)\mathrm{ad}(f_2) \cdots \mathrm{ad}(f_m)P)|_{x_0},$$

since we then may go on by induction.
Set $\phi = f_1 - g_1$ and $Q = \mathrm{ad}(f_2) \cdots \mathrm{ad}(f_m)P \in \mathrm{PDO}^{(1)}$. We have to show that

$$(\mathrm{ad}(\phi)Q)|_{x_0} = 0.$$

Since ϕ vanishes at x_0 to second order we may write $\phi = \sum_j \alpha_j \beta_j$ with the α's and β's smooth functions vanishing at x_0. But then we find

$$(\mathrm{ad}(\phi)Q)|_{x_0} = (\sum_j \mathrm{ad}(\alpha_j\beta_j)Q)|_{x_0} = \sum_j ([Q, \alpha_j]\beta_j)|_{x_0} + \sum_j (\alpha_j[Q, \beta_j])|_{x_0} = 0.$$

$\square$

Summing up we have that the mapping

$$(f_1, \ldots, f_m) \mapsto \frac{1}{m!}(\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_m)P)|_{x_0} \in \mathrm{Hom}(E_{x_0}, F_{x_0})$$

is multilinear and symmetric in all slots and that it only depends on $df_i(x_0) =: \xi_i \in T^*_{x_0}M$. Hence it induces a symmetric multilinear map on $(T^*_{x_0}M)^m$

$$(\xi_1, \ldots \xi_m) \mapsto \sigma(P)_{x_0}(\xi_1, \ldots, \xi_m) := \frac{1}{m!}(\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_m)P)|_{x_0},$$

where the f_i are arbitrary smooth functions with $df_i(x_0) = \xi_i \in T^*_{x_0}M$. Now recall that $\sigma(P)_{x_0}$ as a symmetric and multilinear map by polarisation is completely determined by the homogeneous polynomial

$$\sigma_m(P)_{x_0}(\xi) := \sigma(P)_{x_0}(\underbrace{\xi, \ldots, \xi}_{m\text{-times}}).$$

Hence taking $f \in \mathcal{C}^\infty(M)$ with $df(x_0) = \xi_0$ and using the abbreviation $\mathrm{ad}(f)^m = \mathrm{ad}(f)\cdots\mathrm{ad}(f)$ (m-times) we have the mapping

$$\boxed{\begin{aligned} \sigma_m(P)_{x_0} : T^*_{x_0}M &\to \mathrm{Hom}(E_{x_0}, F_{x_0}) \\ \sigma_m(P)_{x_0}(\xi_0) &:= \tfrac{1}{m!}\,\mathrm{ad}(f)^m\, P|_{x_0}. \end{aligned}}$$

By varying the base point x_0 we obtain a mapping

$$\sigma_m(P) : T^*M \to \mathrm{Hom}(E, F)$$

which respects fibers, i.e., $\sigma_m(P)(\xi) : E_{\pi(\xi)} \to F_{\pi(\xi)}$. Also along the fibers of T^*M the map $\xi \to \sigma_m(P)(\xi)$ looks like a homogeneous polynomial of degree m with coefficients in $\mathrm{Hom}(E_{\pi(\xi)}, F_{\pi(\xi)})$.
Even more geometrically we may say that

$$\sigma_m(P) \in \mathrm{Hom}(\pi^*E, \pi^*F),$$

where π^*E and π^*F are the pullback bundles of E resp. F along $\pi : T^*M \to M$, the projection of the cotangent bundle. Roughly speaking this is a bundle over the manifold T^*M where the fiber over $\xi \in T^*M$ is precisely the fiber $E_{\pi(\xi)}$ of E over $\pi(\xi)$, the footpoint of ξ in T^*M. For more details on pullback bundles see [Die72, 16.19].

Finally we are in the position to make the following definition.

Definition. (Order and Principal symbol) We say that $P \in \mathrm{PDO}^{(m)}(E, F)$ is of *order* m if $\sigma_m(P) \not\equiv 0$. In this case we call $\sigma_m(P)$ the *principal symbol* of P. We denote the space of all PDOs of order m by $\mathrm{PDO}^m(E, F)$.

Let us now look at some examples.

Example. (The gradient) Recall the gradient $\mathrm{grad} f = (df)^\#$ from 2.2. We now look at its principal symbol to see $\mathrm{grad} \in \mathrm{PDO}^1(M \times \mathbb{R}, TM)$. Indeed by (2.1) we have for $\xi_0 \in T^*_{x_0}M$ and $f \in \mathcal{C}^\infty(M)$ with $df(x_0) = \xi_0$

$$\sigma_1(\mathrm{grad})_{x_0}(\xi_0) = \mathrm{ad}(f)\mathrm{grad}|_{x_0} = \xi_0^\#, \tag{2.4}$$

i.e., the principal symbol at (x_0, ξ_0) is just multiplication with the metric equivalent vector $\xi_0^\#$ of ξ_0 (Even more explicitly we have $\sigma_1(\mathrm{grad})_{x_0}(\xi_0) : \mathbb{R} \ni a \mapsto a\xi_0^\# \in T_{x_0}M$.)
Varying the base point we obtain

$$\sigma_1(\mathrm{grad}) : \; T^*M \ni \xi \mapsto \xi^\#. \; \in \mathrm{Hom}(M \times \mathbb{R}, TM).$$

Example. (Covariant derivatives) Any covariant derivative in E by (2.2) is of order one with principal symbol

$$\sigma_1(\nabla)(\xi) = \xi\otimes \tag{2.5}$$

i.e, tensor multiplication by ξ.

Example. (The divergence) The divergence $\operatorname{div}X = \operatorname{tr}(\nabla X)$ from 2.2 is also of order one with principal symbol (cf. (2.3))

$$\sigma_1(\operatorname{div})(\xi) = \xi(.), \tag{2.6}$$

i.e., the dual action of the one-form ξ.

2.5. Composition.

As the final issue of this section we provide an essential tool to construct new PDOs from old.

Proposition. (Composition of PDOs) Let E, F and G be vector bundles over M and let $P \in \mathrm{PDO}^{(m)}(F,G)$, $Q \in \mathrm{PDO}^{(l)}(E,F)$. Then we have

$$P \circ Q \in \mathrm{PDO}^{(m+l)}(E,G) \quad \text{and} \quad \sigma_{m+l}(P \circ Q) = \sigma_m(P) \circ \sigma_l(Q).$$

Proof. We prove the first statement by induction on $m+l$. For $m+l=0$ the assertion is obvious. For the inductive step $m+l \mapsto m+l+1$ we just take $f \in \mathcal{C}^\infty(M)$ and calculate

$$\begin{aligned} [P\circ Q, f] &= P\circ Q\circ f - f\circ P\circ Q \\ &= P\circ Q\circ f - P\circ f\circ Q + P\circ f\circ Q - f\circ P\circ Q = [P,f]\circ Q + P\circ[Q,f]. \end{aligned} \tag{2.7}$$

By induction hypothesis both operators on the r.h.s. are of order $m+n$, hence we are done.
To prove the second statement we calculate for $\xi \in T^*M$ with $\pi(\xi) = x$

$$\begin{aligned} \sigma_{m+l}(P\circ Q)(\xi) &= \frac{1}{(m+l)!}\,\mathrm{ad}(f)^{m+l}P\circ Q|_x \\ &= \frac{1}{(m+l)!}\sum_k \binom{m+l}{k}\mathrm{ad}(f)^k P\circ \mathrm{ad}(f)^{m+l-k}Q|_x \\ &= \frac{1}{(m+l)!}\,\frac{(m+l)!}{m!\,l!}\,\mathrm{ad}(f)^m P\circ \mathrm{ad}(f)^l Q|_x = \sigma_m(P)\circ\sigma_l(Q). \end{aligned}$$

□

3 NORMALLY HYPERBOLIC OPERATORS

Here we introduce operators which generalise the wave equation on $\mathbb{R}^{1+n}$ in the geometric setting. Our presentation is influenced by [BGP07, Sec. 1.5] and [BK96].

3.1. The basic definition.

To begin with observe that the principal symbol of the wave operator is intimately connected to the Minkowski metric $\eta = \mathrm{diag}(-1, 1, \dots, 1)$ since we have for $(\tau, \xi) \in \mathbb{R}^{1+n}$

$$\sigma_2(\Box)(\xi, \tau) = -\tau^2 + |\xi|^2 = \eta((\tau, \xi), (\tau, \xi)).$$

We now define a class of operators of second order with their principal symbol given by (minus) the (inverse) of some Lorentzian metric.

Definition. (Normally hyperbolic operators) Let E be a vector bundle over a manifold M. An operator $P \in \mathrm{PDO}^2(E)$ is called *normally hyperbolic* if there exists a Lorentzian metric g on M such that

$$\sigma_2(P)_x(\xi) = -g_x(\xi, \xi)\,\mathrm{Id}_{E_x} \qquad \forall x \in M,\ \xi \in T_x^*M,$$

where Id_{E_x} denotes the identity operator on the fiber E_x.

Using local coordinates on M and a local trivialisation of E a normally hyperbolic operator may be written as (using summation convention)

$$P = -g^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + B(x),$$

where again $g^{ij}(x)$ are the components of the inverse metric at $x \in M$ and A^i and B are matrix valued coefficients depending smoothly on x.

3.2. Examples.

Our first example of a normally hyperbolic operator will be the metric d'Alembertian of a Lorentzian metric, which sometimes is also called Laplace Beltrami operator or simply wave operator.

Example. (The metric d'Alembertian) Let (M, g) be a Lorentzian manifold. We define the metric d'Alembertian of g by

$$\begin{aligned} \Box_g : \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M) \\ \Box_g u &:= -\operatorname{div}\operatorname{grad} u. \end{aligned}$$

By the formulas from 2.2 we have in local coordinates

$$\Box_g u = -\operatorname{div}\operatorname{grad} u = \nabla_i g^{ij}\partial_j u = g^{ij}\nabla_i\nabla_j u,$$

where we have used that the Levi-Civita connection is metric and that it agrees with the chart derivative on functions. Also note the important formula

$$\Box_g u = |\det g|^{-\frac{1}{2}}\partial_i(|\det g|^{\frac{1}{2}} g^{ij}\partial_j u).$$

We now check that $\Box_g$ is normally hyperbolic. In view of Prop. 2.5, $\Box_g \in \mathrm{PDO}^{(2)}$ and we may calculate the principal symbol using (2.4) and (2.6) ($\xi \in T^*M$)

$$\sigma_2(\Box_g)(\xi) = -\sigma_1(\operatorname{div}) \circ \sigma_1(\operatorname{grad})(\xi) = -\xi(\xi^\#) = -g(\xi, \xi).$$

The operator to be discussed next is the connection d'Alembert operator which sometimes is also called Bochner-Laplace operator. In some sense this is *the* key example since we will see below that any normally hyperbolic operator can be brought into a normal form involving this operator.

Example. (The connection d'Alembertian) Let E be a vector bundle over M and let ∇ be a linear connection on E. Furthermore let g be a Lorentzian metric on M.

Now ∇ together with the Levi-Civita connection ∇_g of g induces a connection on $T^*M \otimes E$.

In explicit terms this connection is given by

$$\begin{aligned} \nabla^{T^*M\otimes E} =: \tilde{\nabla} :\ \Gamma(T^*M \otimes E) \cong \Omega^1(M) \otimes \Gamma(E) &\to \Gamma(T^*M \otimes T^*M \otimes E) \\ \tilde{\nabla}(\eta \otimes u) &= \nabla_g(\eta) \otimes u + \eta \otimes \nabla u. \end{aligned}$$

Observe also that we have $\tilde{\nabla}(f(\eta \otimes u)) = df \otimes \eta \otimes u + f\tilde{\nabla}(\eta \otimes u)$.

Now we define the connection d'Alembertian as the composition of the following three operators

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\tilde{\nabla}} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{\mathrm{tr}_g \otimes \mathrm{Id}_E} \Gamma(E),$$

i.e., we set

$$\Box^\nabla := -\mathrm{tr}_g \otimes \mathrm{Id}_E(\nabla^{T^*M\otimes E} \circ \nabla).$$

We now check that $\Box^\nabla$ is normally hyperbolic. First observe that as the composition of two operators in $\mathrm{PDO}^{(1)}$ and one in PDO^0 the connection d'Alembertian is in $\mathrm{PDO}^{(2)}$. We now calculate $\sigma_2(\Box^\nabla)$ using Prop. 2.5 and (2.5)

$$\sigma_2(\Box^\nabla)(\xi) = -(\mathrm{tr}_g \otimes \mathrm{Id}_E) \circ \sigma_1(\tilde{\nabla}) \circ \sigma_1(\nabla)(\xi) = -(\mathrm{tr}_g \otimes \mathrm{Id}_E)(\xi \otimes \xi \otimes .) = -g(\xi, \xi)\mathrm{Id}_E.$$

Example. (Further examples) Many well known operators of geometry fall into the class of normally hyperbolic operators. We list a few.

(i) The Yamabe operator on an n-dimensional Lorentzian manifold is

$$Y = \Box_g + \frac{n-2}{4(n-1)} R,$$

where R denotes the scalar curvature.

(ii) The Hodge Laplace operator on an oriented Lorentzian manifold is

$$\triangle_k = dd^* + d^*d.$$

(iii) The square of the Dirac operator on a Lorentzian spin manifold.

3.3. The Weitzenböck formula.

In this final part we will prove that each normally hyperbolic operator can be written as the connection d'Alembertian for a suitable connection plus zero order terms. We first give the precise statement.

Theorem. (Weitzenböck formula) Let P be a normally hyperbolic operator on $E \to M$. Then there exists a unique connection ∇ on E and a unique $B_P \in \Gamma(\mathrm{Hom}(E,E))$ such that

$$\boxed{P = \Box^\nabla + B_P.} \tag{3.1}$$

Equation (3.1) is usually refered to as the Weitzenböck formula and B_p is called the Weitzenböck remainder of P.

Proof. We start with a little calculation. Let ∇ any connection on E. Using the notation of 3.2 we then have for all $f \in \mathcal{C}^\infty(M)$ and all $u \in \Gamma(E)$

$$\tilde{\nabla}\nabla(fu) = \tilde{\nabla}(df \otimes u + f\nabla u) = \nabla_g(df) \otimes u + 2df \otimes \nabla u + f\,\tilde{\nabla} \circ \nabla u.$$

Hence we obtain for the connection d'Alembertian

$$\Box^\nabla(fu) = (\Box_g f)u - 2\nabla_{\mathrm{grad} f} u + f\,\Box^\nabla u. \tag{3.2}$$

Now we prove *uniqueness*. Suppose P satisfies (3.1), then $B_p = P - \Box^\nabla \in \Gamma(\mathrm{Hom}(E,E))$ and we find using (3.2)

$$f(Pu - \Box^\nabla u) = (P - \Box^\nabla)(fu) = P(fu) - (\Box_g f)u + 2\nabla_{\mathrm{grad} f} u - f\,\Box^\nabla u.$$

So we obtain

$$\nabla_{\mathrm{grad} f} u = \frac{1}{2}(fPu - P(fu) + (\Box_g f)u). \tag{3.3}$$

Recalling that at any $x \in M$ every $X \in T_x M$ can be written as $\mathrm{grad} f(x)$ for some suitable $f \in \mathcal{C}^\infty(M)$ we see by (3.3) that ∇ is uniquely determined in terms of P and $\Box_g$.
Finally we obtain existence by using (3.3) as the definition of ∇ on E since then our above calculations show that ∇ satisfies (3.1). $\Box$

Remark. Having seen the proof one might wonder where and how the first order terms in $P - \Box^\nabla$ have disappeared. The answer is, that they have been put into the connection ∇. To see this more explicitly denote by ∇' any connection on E and write for the first order operator

$$P - \Box^{\nabla'} = A \circ \nabla' + B$$

with $A \in \mathcal{C}^\infty(M, \mathrm{Hom}(T^*M \otimes E, E)$ and $B \in \mathcal{C}^\infty(M, \mathrm{Hom}(E, E))$. Then setting

$$\nabla_X u = \nabla'_X u - \frac{1}{2} A(X^\flat \otimes u)$$

does the trick as can be shown by an explicit calculation, see [BGP07, Lem. 1.5.5]. Here $X^\flat$ denotes the metric equivalent one-form to X.

3.4. Where to go from here.

The *existence theory* of normally hyperbolic operators is well-understood. As you should expect from the very name, the Cauchy problem is well posed—not only locally but also globally provided the manifold has "good causal structure". Local fundamental solutions can be constructed using the so-called Hadamard parametrix construction (see [Hör94, Sec. 14.7] as well as [BGP07, Ch. 2] and [Fri75, Ch. 4–5]). From there one obtaines a (semi-)global existence theorem ([BGP07, Ch. 3]). (For the notions from Lorentzian geometry appearing below also see Mike Kunzinger's lecture.)

Theorem. Let P be a normally hyperbolic operator on E and let the underlying manifold M be globally hyperbolic with spacelike Cauchy hypersurface S. Denote by ν the future directed timelike unit normal vector field along S.

(i) (Existence and uniqueness) For each u_0, $u_1 \in \Gamma_c(S, E)$ and for each $f \in \Gamma_c(E)$ (i.e., all of them compactly supported) there exists a unique solution $u \in \Gamma(E)$ of the Cauchy problem

$$Pu = f, \qquad u|_S = u_0, \qquad \nabla_\nu u|_S = u_1.$$

(ii) (Finite propagation speed) The solution from (i) satisfies

$$\mathrm{supp}(u) \subseteq J(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)),$$

where $J(A)$ denotes the union of the causal future and causal past of a set A.

(iii) (Continuous dependence on the data) The mapping

$$\Gamma_c(E) \oplus \Gamma_c(S, E) \oplus \Gamma_c(S, E) \ni (f, u_0, u_1) \mapsto u \in \Gamma(E)$$

is linear and continuous.

The local fundamental solutions of normally hyperbolic operators with respect to a point x share the following property with the ordinary wave operator: they are supported in the forward or backward causal cone of x. Recall that in case of the wave operator on $\mathbb{R}^{1+n}$ we know more, depending on whether n is even or odd. Namely, in odd space dimensions we have sharp wave propagation, meaning that the fundamental solution is supported on the light cone only, while in even space dimensions we have residual waves. Now a *Huygens operator* is a normally hyperbolic operator that has sharp wave propagation. In 1923 Hadamard posed the problem of finding all Huygens operators, which remains unsolved to date. From the general theory of fundamental solutions one can infer that on a manifold of dimension n there are no Huygens operators if n is odd or $n = 2$. The characterisation of Huygens operators in even dimensions $n \geqslant 4$ is a very rich theory; for an overview see [BK96].

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