A Regularisation Approach to Causality Theory for Non-smooth Lorentzian metrics

Roland Steinbauer

Faculty of Mathematics, University of Vienna

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General Topic

General Relativity and semi-Riemannian geometry with metrics of low regularity

Prelude: The basic setup of General Relativity

- Albert Einstein’s theory of gravity created exactly 99 years ago
- current description of gravitation in physics
- geometric theory due to Galileo’s principle of equivalence: all bodies fall the same in a gravitational field
  ~ gravitational field as property of the surrounding space
- Gravitational field influences how we measure lengths and angles hence the curvature of space and time
The mathematical setup of GR

**Lorentzian geometry (basic geometric setup)**

- smooth 4-dimensional space-time manifold $M$
- smooth space-time metric $g \in \Gamma^0_2(M)$: at any $T_pM$ symmetric, non-degenerate scalar product with signature $(-, +, +, +)$

**Field equations (basic physical/analytical setup)**

- Einstein Equations
  \[
  G_{ij}[g] := R_{ij}[g] - \frac{1}{2} R[g] g_{ij} = 8\pi T_{ij}
  \]
- Ricci-tensor $R_{ij}$, curvature scalar $R$ built from
  \[
  R^m_{ikp} = \partial_k \Gamma^m_{ip} - \partial_p \Gamma^m_{ik} + \Gamma^a_{ip} \Gamma^m_{ak} - \Gamma^a_{ik} \Gamma^m_{ap}
  \]
  and Christoffel symbols
  \[
  \Gamma^i_{jk} = g^{il} \Gamma_{ljk} = \frac{1}{2} g^{il} (\partial_k g_{lj} + \partial_j g_{kl} - \partial_l g_{jk})
  \]
  \[
  \Rightarrow R_{ij}, R \sim \partial^2 g + (\partial g)^2
  \]
- coupled system of 10 quasi-linear PDEs of 2nd order for $g$
Why Low Regularity?

(1) Physics

- want discontinuous matter configurations \( T \notin C^0 \implies g \notin C^2 \)
- finite jumps in \( T \sim g \in C^{1,1} \)
- standard approach: \( g \) piecewise \( C^3 \), globally only \( C^1 \)
- more extreme situations (impulsive waves): \( g \) piecew. \( C^3 \), globally \( C^0 \)

(2) Initial value problem—Analysis

- 3 + 1-split: \( M = \Sigma \times \{t\} \); C. data \((\Sigma_0, g_0, k)\) with \( \Sigma_0 = \{t = 0\}, \ g(.,0) = g_0, \ \partial_t g(.,0) = k \)
- Local existence and uniqueness Thms. \((g_0, k) \in H^s \times H^{s-1}(\Sigma_0) \implies g \in H^s(\Sigma)\)
  - classical \([CB,HKM]\): \( s > 5/2 \implies g \in C^1(\Sigma) \)
  - recent big improvements \([K,R,M,S]\): \( g \in C^0(\Sigma) \)
### GR and low regularity

#### The big quest

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<th>Physics and Analysis</th>
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<td>want/need low regularity</td>
<td>needs high regularity to maintain standard results</td>
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#### Lorentzian geometry and regularity

- Classically, $g \in C^\infty$, for all practical purposes $g \in C^2$
  - Exponential map works
  - Existence of totally normal nbhds. $\Rightarrow$ geodesically convex
  - Causality theory works [C]
  - Needed for singularity thms. [S]
- Things go wrong below $C^2$
  - Causality goes wrong, light cones “bubble up” for $g \in C^0$ [CG12]
  - Convexity goes wrong for $g \in C^{1,\alpha}$ ($\alpha < 1$) [HW], see M.K.’s talk
- Threshold $g \in C^{1,1}$: Unique solvability of geodesics eq. suffices ???

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A Regularisation Approach to Causality Theory for Non-smooth Lorentzian metrics
Causality theory

What is CT?

- essentially the theory of future & past
- tells how signals propagate, in particular how fields propagate
  \( \leadsto \) PDE, see talks of G.H. and C.S.

Simplest ex: Minkowski space

\( (M, g) = (\mathbb{R}^4, \eta) \),
where \( \eta = \text{diag}(-1, 1, 1, 1) \)

- \( \eta(X, X) < 0 \): timelike
- \( \eta(X, X) = 0 \): null (lightlike)
- \( \eta(X, X) \leq 0 \): causality
- \( \eta(X, X) > 0 \): spacelike
Local causality in a general space-time

Definitions

**Timelike (causal) curve:** $\gamma \in C^{0,1}$

with $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) < 0 \ (\leq 0)$ a.e.

**Timelike/causal future $I^+(p)/J^+(p)$:**

points reachable by future directed timelike (causal) curve

Expectation and classically true:

Locally the causality in any space-time is Minkowskian, in part.

- the local causal structure is given by the image of the lightcone under the exponential map
- the push up principles hold, in particular: any curve from $p$ to $\partial I^+(p)$ is a null geodesic
The exponential map in low regularity

The exponential map

- \( \exp_p : T_p M \ni \nu \mapsto \gamma_{\nu}(1) \in M \)
  - where \( \gamma_{\nu} \) is the (unique) geodesic starting at \( p \) in direction of \( \nu \)
  - maps rays through \( 0 \in T_p M \) to geodesics through \( p \in M \)

Regularity

- \( g \in C^2 \Rightarrow \exp_p \) local diffeo
- \( g \in C^{1,1} \Rightarrow \exp_p \) loc. homeo [W32]
- \( g \in C^{1,1} \Rightarrow \exp_p \) bi-Lipschitz homeo [KSS14],[M13]
Tools for causality theory with $C^{1,1}$-metrics

**Theorem (Maximal regularity of $\exp$ [KSS14])**

If $g \in C^{1,1}$ then $\forall p \in M$ there exist open nbhd. $\tilde{U}$ of $0 \in T_pM$ and $U$ of $p$ in $M$ such that $\exp_p : \tilde{U} \rightarrow U$ is a bi-Lipschitz homeo.

**Theorem (Existence of totally normal nbhd. [KSS14])**

If $g \in C^{1,1}$ then all $p \in M$ possess a basis of totally normal nbhd.

**Theorem (The Gauss Lemma [KSSV14])**

If $g \in C^{1,1}$ then all $p \in M$ possess a basis of normal nbhd. $U$ with $\exp_p : \tilde{U} \rightarrow U$ a bi-Lipschitz homeo. and for almost all $x \in \tilde{U}$, if $v_x, w_x \in T_x(T_pM)$ and $v_x$ is radial, then

$$\langle T_x \exp_p(v_x), T_x \exp_p(w_x) \rangle = \langle v_x, w_x \rangle.$$
Method of proof
(details: M. Stojković’s poster)

[KSS14], [KSSV14] use

- **regularisation technique**
  approximate $g \in C^{1,1}$ by smooth $g_\varepsilon$ gained via convolution
  $\Rightarrow g_\varepsilon \rightarrow g \in C^1$ and $\text{Riem}[g_\varepsilon]$ locally uniformly bounded
  Beware: $\text{Riem}[g_\varepsilon] \nrightarrow \text{Riem}[g]$

- **comparison geometry**
  new methods from Lorentzian comparison geometry [LeFC,08]

Alternative approach by E. Minguzzi [M13] uses

- careful ODE-analysis based on Picard-Lindelöf approximations
- inverse function theorem for Lipschitz maps

Merrits: [M13] gives somewhat stronger results but techniques do not extend below $C^{1,1}$. 
Causality for $C^{1,1}$-metrics

**Theorem (Local causality [KSSV14])**

If $g \in C^{1,1}$ then all $p \in M$ possess a basis of normal nbghds. 

$\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeomorphism and 

\[
I^+(p, U) = \exp_p(I^+(0) \cap \tilde{U}), \quad J^+(p, U) = \exp_p(J^+(0) \cap \tilde{U}) \\
\partial I^+(p, U) = \partial J^+(p, U) = \exp_p(\partial I^+(0) \cap \tilde{U}).
\]

**Theorem (Push up principles [CG12])**

If $g \in C^{0,1}$ then we have

- If there is a timelike curve from $p$ to $q$ and a causal curve from $q$ to $r$ then there is a timelike curve from $p$ to $r$.

- If a causal curve $\alpha$ from $p$ to $q$ has is timelike piece then there exists a timelike curve from $p$ to $q$ arbitrarily close to $\alpha$. 
Main technique

Regularisations of the metric adapted to the causal structure

[CG12],[KSSV14]

If $g \in C^0$ then for any $\varepsilon > 0$ there exist smooth metrics $\mathring{g}_\varepsilon$ and $\hat{g}_\varepsilon$ with

$\mathring{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon,$

$$d_h(\mathring{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$$

where $d_h(g_1, g_2) :=$

$$\sup_{0 \neq X, Y \in TM} \frac{|g_1(X, Y) - g_2(X, Y)|}{\|X\|_h \|Y\|_h}$$

and $h$ is some Riem. backgrd metr.

$g \prec h \iff g(X, X) \leq 0 \Rightarrow h(X, X) < 0$
$C^{1,1}$-causality theory works!

- Fundamental constructions (local causality, push up principles) of causality theory remain valid for $g \in C^{1,1}$.
- Accumulation curves of causal curves are causal. \[CG12\]
- This allows to obtain all of standard causality theory for $g \in C^{1,1}$ following the classical proofs. \[KSSV14\]

Outlook:
This (finally) puts us into a position to try to prove singularity theorems for $g \in C^{1,1}$. 

\[\]
Geodesics in impulsive gravitational waves

Impulsive gravitational waves

- model short but strong pulses of gravitational radiation propagating in constant curvature backgrounds
- frequently described by a $g \in C^{0,1}$

Line element in the non-expanding case (coords ($U, V, Z, \bar{Z}$))

$$ds^2 = \frac{2|dZ + U_+(H,Z\bar{Z}dZ + H,\bar{Z}\bar{Z}d\bar{Z})|^2 - 2dUdV}{[1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - U_+G)]^2}$$

(1)

where $G(Z, \bar{Z}) \equiv H - ZH,Z - \bar{Z}H,\bar{Z}$.

- curvature concentrated on the null hypersurface $\{U = 0\}$
- relevant models of ultrarelativistic particles
Geodesics: regularity, matching, completeness
(details: A. Lecke’s poster)

$C^1$-matching of the geodesics in impulsive grav. waves

- Physicists like to derive the geodesics by matching the geodesics of the background across the wave-surface.
- This is only possible if the geodesics
  — cross the wave-surface at all, and
  — are $C^1$ across the wave-surface

Task: Prove that these space-times are geodesically complete with $C^1$-geodesics.

Problem: Geodesic eqs. are ODEs with discontinuous r.h.s.

\[
\ddot{\gamma}^j(t) + \Gamma^j_{kl}(\gamma(t)) \dot{\gamma}^k(t) \dot{\gamma}^l(t) = 0
\]

$g_{ij} \in C^{0,1} \Rightarrow \Gamma^j_{kl} \in L^\infty_{loc}$
Regularity of geodesics in imp. grav. waves

The case $\Lambda = 0$ \cite{LSŠ14}

- simple structure of the metric $\Rightarrow$ equations can be written as non-autonomous system with $U$ as “time”-parameter
- Geodesic equations possess unique globally defined solutions in the sense of Carathéodory and the solutions are $C^1$-curves.
  $\Rightarrow$ geodesic completeness and $C^1$-matching is ok!

The case $\Lambda \neq 0$

- $U$ is not a parameter $\Rightarrow$ use Filippov’s solution concept.
- Observation \cite{S14}: $g \in C^{1,0} \Rightarrow$ geodesic equations possess solutions in the sense of Filippov which are $C^1$-curves.
- \cite{PSSŠ14}: In case of (1) solutions are unique and globally defined.
  $\Rightarrow$ geodesic completeness and $C^1$-matching is ok!

Work in progress: expanding impulsive waves
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**HVALA!**