Non-smooth spacetimes & Lorentzian length spaces

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FWF Österreichischer Wissenschaftsfonds

excellent = austria

Curvature beyond smooth spacetimes

- Curvature is the essential quantity in GR & beyond
- *non-smooth spacetime:* smooth manifold with *metric below $g \in \mathcal{C}^2$*
  e.g. $\mathcal{C}^{1,1}$, Hölder, $\mathcal{C}^1$, Lipschitz, $\mathcal{C}$, Geroch-Traschen: $H^1 \cap L^\infty$
- *Lorentzian length spaces:* akin metric length spaces, *no manifold at all*
  causality axiomatic, curvature(bounds) synthetic

Why should you care?

- physically relevant *models* (matched spacetimes, impulsive wave, etc.)
- *PDE* point-of-view, initial value problem
- *singularities* vs *curvature blow-up* — *CCH* of Penrose
- approaches to *Quantum Gravity* (no metric, discreteness)

What are the main issues?

Basic *geometry* & Lorentzian *causality* change dramatically.

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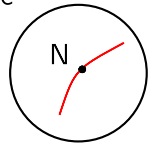
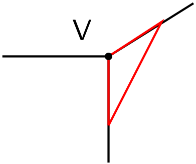
Change of basic geometric features

Example 1: Walking on a sphere vs. walking on a cube

Sphere



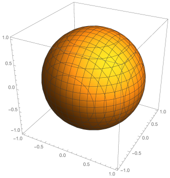
Cube

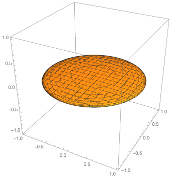


It is always shorter to deviate to the right face than to go along the edges.

Example 2: Squeezing the sphere

Convexity fails for metrics of Hölder regularity $g \in C^{1,\alpha}$ $(\alpha < 1)$.





Equator still geodesic but shorter to deviate into hemispheres. [Hartman-Wintner 52]

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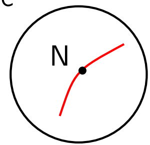

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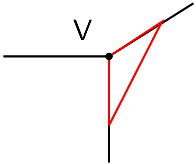

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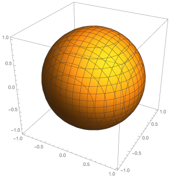

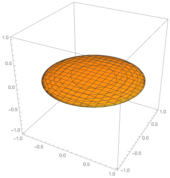

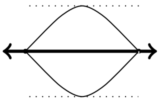

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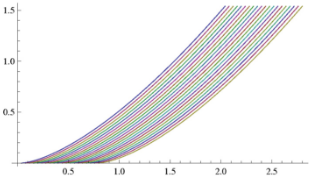
Change of basic Lorentzian causality

Example 3: Lightcones bubble up [Chrusciel-Grant 12]



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Non-uniqueness of null geodesics
$\rightsquigarrow$ null cone has full measure.

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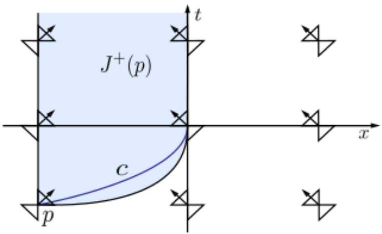

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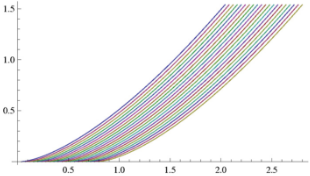

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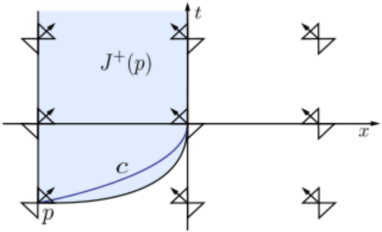

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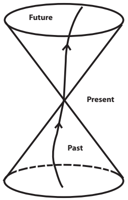
Causality w/o metric: Lorentzian (pre) length spaces



Spacetime: causal relations & time separation

$p \ll q$ ($p \leq q$) $:\Leftrightarrow$ connected by timelike (causal) curve

$\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\}$
lower semi-continuous, and reverse Δ-inequality

$$\tau(p,q) + \tau(q,r) \leq \tau(p,r)$$

Definition: Lorentzian pre-length space [Kunzinger-Sä, 18]

X metrizable space with generalised time function

$$\ell \colon X \times X \to \{-\infty\} \cup [0,\infty] \qquad \text{with } \ell(x,x) \geq 0$$

Define: $\ll := \ell^{-1}((0,\infty))$, $\leq := \ell^{-1}([0,\infty))$, $\tau := \max(\ell, 0)$
Demand

$$\tau \text{ is l.s.c. and } \tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z),$$

Based on [Kronheimer-Penrose, 67]
similar recent approaches by [Braun-McCann], [Minguzzi-Suhr], [Müller]

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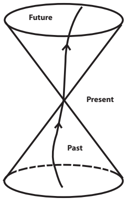

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Examples

- Smooth/Lip. *spacetimes* (M,g) with usual time separation
- *Lorentz-Finsler* spacetimes of *low regularity*
- *directed graphs* (causal sets)

Notions

- *causal* curves and their length
- *geodesics* as locally *maximising* causal curves
- *causality* theory (causal ladder, global hyperbolicity, ...)

Lorentzian length space: If τ is *intrinsic*, that is

$$\tau(p,q) = \sup\{L(\gamma) : \gamma \text{ is a future-directed causal curve from } p \text{ to } q\}$$

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Sectional curvature bounds: Lorentzian, synthetic

Recall: Sectional curvature $\mathrm{Sec}(X,Y) = \dfrac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$

[Kulkarni, 79] If $\mathrm{Sec}(g)$ is bounded below (above), then it is constant.

$\triangle abc$ and their comparison $\triangle\bar a\bar b\bar c$ in 2D space of const. curvature K (Minkowski, (anti-)de Sitter) and all p, q resp. $\bar p$, $\bar q$

$$d_{\mathrm{signed}}(p,q) \geq \bar d_{\mathrm{signed}}(\bar p,\bar q).$$

Definition (Synthetic curvature bounds) [Kunzinger-Sä, 18]

A LLS has *timelike curvature* $\geq K$ if all points have nhd. U such that for all *timelike triangles* Δabc in U and their comparison $\Delta\bar a\bar b\bar c$ in M_K and all p, q resp. $\bar p$, $\bar q$

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Sectional curvature bounds: Lorentzian, synthetic

Definition ("Correct" curvature bounds) [Andersson-Howard, 98]

A smooth Lorentzian manifold has $\mathrm{Sec} \geq K$ if *spacelike* sectional curvatures $\geq K$ and *timelike* sectional curvatures $\leq K$.

Theorem (Lorentzian Toponogov) [Alexander-Bishop, 08]

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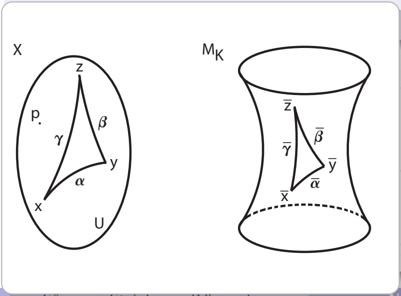

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Selected results (1/2)

Theorem [Kunzinger-Sä 19, Beran-Sä 22]

In a strongly causal Lorentzian pre-length space with *timelike curvature bounded below* timelike geodesics do *not branch*.

Theorem [Grant-Kunzinger-Sä 19]

A timelike geodesically complete spacetime (or LLS) is *inextendible as a regular LLS*, i.e., any LLS-extension necessarily has unbounded curvature.

Extends [Beem-Ehrlich 80s] and C^0-result [Galloway-Ling-Sbierski 18].

Splitting theorem [Beran-Ohanyan-Rott-Solis 23]

Let X be a globally hyperbolic LLS with global timelike $K \geq 0$. If X contains a complete timelike line (+ some technical conditions) then it splits into a product $\mathbb{R} \times S$ with S a metric length space with $K \geq 0$.

Generalises smooth Lorentzian & synthetic Riemannian results.

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Selected results (2/2)

1. *Generalized cones*: Lorentzian warped products of metric spaces with 1-dim base and singularity thms (Alexander-Graf-Kunzinger-Sä. '23)
2. *Ricci curvature bounds* via optimal transport
   timlike case (McCann '18, Mondino-Suhr '23, Cavalletti-Mondino '24)
   null case (McCann '24, Ketterer '24, Cavalletti-Manini-Mondino '24)
3. (vacuum) *Einstein equations* (Mondino-Suhr '23)
4. *Time functions* (Burtscher-García-Heveling '21)
5. *Null distance & LLS* (Kunzinger-St. '22)
6. Lorentzian analog of Hausdorff *dimension*, *measure* (McCann-Sä. '22)
7. *Gluing* of Lorentzian length spaces (Beran-Rott '24, Rott '23)
8. Hyperbolic *angles* (Barrera-Montes de Oca-Solis '22, Beran-Sä. '23)
9. *Causal boundaries* (Ake Hau-Burgos-Solis '23, '25)
10. *sympl./cont. geo.* (Abbondand.-Benedetti-Polterovich '22, Hedicke '24)
11. *Machine learning* in spacetimes (Law-Lucas '23)
12. *Causal differential calculus* and *non-smooth splitting theorem* (Beran-Braun-Calisti-Gigli-McCann-Ohanyan-Rott-Sä. '24), (Braun-Gigli-McCann-Ohanyan-Sä. '24, '25)

Approaches to Lorentzian Gromov–Hausdorff convergence

1. Noldus 2004 for *compact spaces*
2. *(Bounded) Lorentzian metric spaces*: Minguzzi–Suhr 2024, Bykov–Minguzzi–Suhr 2024
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Metric Gromov–Hausdorff convergence

(X, d), $(X_n, d_n)_n$ (compact) metric spaces ($n \in \mathbb{N}$)

Definition (Gromov–Hausdorff convergence)

$\exists$ sequence of correspondences $(R_n)_n$ of X, X_n s.t. $\mathrm{dis}(R_n) \to 0$

$\epsilon > 0$, $S \subseteq X$ is *ϵ-net* if $X = \bigcup_{s \in S} B_\epsilon(s)$

$X_n \to X$ *Gromov–Hausdorff* iff $\forall \epsilon > 0$ $\exists$ finite ϵ-net $S \subseteq X$, ϵ-net $S_n \subseteq X_n$ s.t. $S_n \to S$, i.e., $d_n(s_i^n, s_j^n) \to d(s_i, s_j)$ $\forall s_i^n, s_j^n \in S_n, \forall s_i, s_j \in S$

Gromov's precompactness theorem

$\mathfrak{X}$ class of compact metric spaces s.t. $\exists D > 0$, $N\colon (0, \infty) \to \mathbb{N}$ w/

1. $\mathrm{diam}(X) \leq D < \infty$ $\forall X \in \mathfrak{X}$
2. $\forall X \in \mathfrak{X}$ $\forall \epsilon > 0$ $\exists \epsilon$-net S for X w/ $|S| \leq N(\epsilon)$

then every sequence in $\mathfrak{X}$ has *converging subsequence*

$\rightsquigarrow$ Alexandrov spaces, Ricci limit spaces, (R)CD-spaces...

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Geometric precompactness

(M_n, g_n) sequence of (compact) RMF (of same dim) w/ $\mathrm{Sec}(g_n) \geq K$ or *Ric*$(g_n) \geq K$ for all $n \Rightarrow \exists$ *converging subsequence*

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Correspondences and distortions

Definition (Correspondences, distortions)

X, Y sets, binary relation $R \subseteq X \times Y$ is a *correspondence* if

1. $\forall x \in X \; \exists y \in Y$ w/ $(x, y) \in R$
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distortion of correspondence R between two Lorentzian pre-length spaces (X, ℓ), (Y, ρ) is

$$\operatorname{dis}(R) := \sup_{(x,y),(x',y') \in R} |\ell(x, x') - \rho(y, y')|$$

Definition (Composition of correspondences)

X, Y, Z sets, $R \subseteq X \times Y$ correspondence between X, Y and $Q \subseteq Y \times Z$ correspondence between Y, Z; the *composition* $Q \circ R$ of R and Q is

$$Q \circ R := \{(x, z) \in X \times Z : \exists y \in Y \text{ w/ } (x, y) \in R, (y, z) \in Q\}$$

$\rightsquigarrow \operatorname{dis}(Q \circ R) \leq \operatorname{dis}(Q) + \operatorname{dis}(R)$

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Lorentzian Gromov–Hausdorff conv. & precompactness

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In a Lorentzian pre-length space (X, ℓ): $S = \{J_i := J(p_i, q_i) : i \in \Omega\}$ *family of causal diamonds*
the *set of vertices of S* is

$$V(S) := \{x \in X : x \text{ vertex of a causal diamond of } S, \\ i.e.,\ x = p_i \text{ or } x = q_i\}$$

Definition (ϵ-net)

$\epsilon > 0$, $A \subseteq X$: An *ϵ-net S for A* is collection of causal diamonds $S = (J_i)_{i \in \Omega}$ s.t.:

1. $\tau(J_i) \leq \epsilon \ \forall i \in \Omega$
2. $A \subseteq \bigcup_{i \in \Omega} J_i$

(WLOG $J_i \cap A \neq \emptyset \ \forall i \in \Omega$)

Convergence of subsets

Definition (LGH-convergence of subsets)

(X_n, ℓ_n),(X, ℓ) Lorentzian pre-length spaces, $\forall n \in \mathbb{N}$, $A_n \subseteq X_n$, $A \subseteq X$;
A_n converges to A in LGH-sense ($A_n \xrightarrow{\mathrm{LGH}} A$) if $\forall \epsilon > 0$ $\exists n_0 \in \mathbb{N}$ and finite ϵ-nets S for A in X and S_n for A_n in X_n ($\forall n \geq n_0$) s.t.

1. $|S_n| = |S|$
2. $\forall n \geq n_0$ $\exists$ correspondence R_n of $V(S_n)$ and $V(S)$ w/ $\mathrm{dis}(R_n) \to 0$
3. *extension* of correspondences of $\frac{1}{l}$-nets to $\frac{1}{l+1}$-nets
4. *forward density of vertices*, i.e., $\mathcal{V}$ total set of vertices, then $\forall x \in A \backslash \mathcal{V}$, $\exists (x_k)_k \in \mathcal{V}$ s.t. $x_k \leq x_{k+1} \leq x$ and $x_k \to x$

$A_n \xrightarrow{\mathrm{LGH}} A$ *strongly* if $x_k \ll x_{k+1} \ll x$ above — *timelike forward density*

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Convergence of Lorentzian pre-length spaces

there is no canonical cover of a pointed, unbounded, Lorentzian space!

$\rightsquigarrow$ specify a cover

Definition (pLGH-convergence of covered Lorentzian pre-length spaces)

$(X, \ell, o, \mathcal{U})$, $((X_n, \ell_n, o_n, \mathcal{U}_n))_{n\in\mathbb{N}}$ covered Lorentzian pre-length spaces w/ $\mathcal{U} = (U_{k,\infty})_{k\in\mathbb{N}}$ and $\mathcal{U}_n = (U_{k,n})_{k\in\mathbb{N}}$; $((X_n, \ell_n, o_n, \mathcal{U}_n))_{n\in\mathbb{N}}$ *converges* to $(X, \ell, o, \mathcal{U})$ in the (resp. strong) *pointed Lorentzian Gromov-Hausdorff sense* (pLGH) written

$$(X_n, \ell_n, o_n, \mathcal{U}_n) \xrightarrow{\mathrm{pLGH}} (X, \ell, o, \mathcal{U}) \quad \textit{(resp. strongly)}$$

if $\forall k \in \mathbb{N}$: $U_{k,n} \xrightarrow{\mathrm{LGH}} U_{k,\infty}$ (resp. strongly)

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Geometric precompactness

Theorem (Geometric pre-compactness)

$C\colon (0,\infty) \to (0,\infty)$, $N\colon (0,\infty) \to \mathbb{N}$; family $\mathcal{M}_{C,N}$ of g.h. spacetimes

$$\begin{aligned}\mathcal{M}_{C,N} := \{(\mathbb{R}\times\Sigma, -\beta \mathrm{d}t^2 + h_t) :\ &\Sigma \text{ is a compact smooth manifold,}\\ &\beta\colon \mathbb{R}\times\Sigma \to (0,1] \text{ is a smooth function,}\\ &\forall \epsilon > 0\, \exists \epsilon\text{-net } S \text{ in } \Sigma \text{ w.r.t. } \mathsf{d}^{h_0} \text{ with } |S| \leq N(\epsilon),\\ &\forall T > 0 :\ -C(T)^2 \mathrm{d}t^2 + h_0 \preceq -\beta \mathrm{d}t^2 + h_t \text{ on } [-T,T]\times\Sigma\}\end{aligned}$$

then, $\forall T > 0\ \exists$ *uniform bound* on cardinality of Lorentzian ϵ-nets for $[-T,T]\times\Sigma$ i.e. for $T>0$, $\epsilon>0$, for every $(\mathbb{R}\times\Sigma, -\beta\mathrm{d}t^2 + h_t) \in \mathcal{M}_{C,N}$ $\exists$ Lorentzian ϵ-net for $[-T,T]\times\Sigma$ of cardinality at most

$$\left\lceil \frac{2T}{3\epsilon} \right\rceil \cdot N\left(\frac{C(T)\epsilon}{3}\right)$$

$\mathcal{M}_{C,N}$ is *sequentially precompact*; at most one g.h. spacetime as limit

$g \preceq g'$ if $g(v,v) \leq 0 \Rightarrow g'(v,v) \leq 0\ \forall v \in TM$

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Applications of pLGH-convergence

Theorem (pLGH-convergence for continuous spacetimes)

(M, g) *continuous*, causally plain (or use time separation of [Ling:24]), g.h. spacetime, fix $o \in M$, then $\exists \hat{g}_n \to g$ locally uniformly s.t. $g \preceq \hat{g}_{n+1} \preceq \hat{g}_n$ $\forall n \in \mathbb{N}$ and $\exists$ coverings $\mathcal{U}$, $\mathcal{U}_n$ of M w.r.t $g, \hat{g}_n$ s.t. $(M, \ell_{\hat{g}_n}, o, \mathcal{U}_n) \xrightarrow{\text{pLGH}} (M, \ell_g, o, \mathcal{U})$ strongly

Theorem (Stability of lower timelike sectional curvature bounds)

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Outlook on applications to Quantum Gravity

1. in *positive* signature, sectional curvature bounds for discrete metric spaces and *Ollivier Ricci curvature* (and more)
2. timelike sectional curvature bounds for *discrete spaces* via *four-point conditions* (Beran-Kunzinger-Rott '24, Beran '25+)
3. directly gives *comparison configurations* — no need for curves
4. applies for example for *graphs* or *lattices*, hence e.g. to *discrete Causal Fermion systems* (cf. Sect. 5.2 [Finster-Kindermann-Treude '24]) or *causal sets*
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