Distributional curvature and the focusing of geodesics

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The wider topic

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- physically resonsable assumptions lead to singularities of spacetime
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Here: bring out the analysis underneath the geometry

- classical: analysis of Riccati equation, comparison results
- low regularity: distributional curvature, regularsiation, focusing

The structure of the singularity theorems

Pattern theorem [José Senovilla, 1998]

A spacetime (M, g) is singular if it satisfies:

(I) A suitable initial condition,

(E) an energy or curvature condition,

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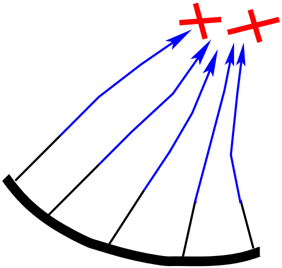
(I) $\rightsquigarrow$ causal geodesics start focusing
(E) $\rightsquigarrow$ focusing goes on (Raychaud. Riccati)
$\rightsquigarrow$ focal pt. $\rightsquigarrow$ geos. stop maximising
(C) $\rightsquigarrow$ there are maximising causal geos
resolution: some causal geodesics stop existing before conj. pt.



Geodesics, maximisers & Jacobi tensors

- Geodesics: $\gamma : I \to M$ with $\nabla_{\dot\gamma}\dot\gamma = 0$

$$\ddot\gamma^i(s) = \Gamma^i_{jk}\big(\gamma(s)\big)\,\dot\gamma^j(s)\,\dot\gamma^k(s) \qquad \text{with } \Gamma \sim g^{-1}\,\partial g$$

- For data $p \in M$, $v \in T_pM$ unique max. extended sol.
- Locally causal geodesics ($g(\dot\gamma,\dot\gamma) \leq 0$) **maximise** Lor. distance

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- Locally causal geodesics ($g(\dot\gamma, \dot\gamma) \leq 0$) **maximise** Lor. distance
- Stop maximising after first **conjuagte point**, i.e. $\gamma(a)$ such that exists Jacobi tensor $A(s) : \dot\gamma^\perp(s) \to \dot\gamma^\perp(s)$, i.e. unique sol. of

$$\ddot A + R\,A = 0, \text{ with } \textbf{tidal force op. } R : v \mapsto \mathrm{Riem}(v, \dot\gamma)\dot\gamma$$

with $A(0) = 0$, $\dot A(0) = \mathrm{id}$ satisfies $\boxed{\ker A(a) \neq \{0\}}$

- **Key idea:** estimates on curvature say when geos stop maximising

Geodesic focusing

- Raychaudhuri eq. for **expansion** $\theta := \mathrm{tr}(\dot{A}A^{-1}) = (\det A)^{-1} (\det A)^{\cdot}$

$$\dot{\theta} = -\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \mathrm{tr}(\sigma^2) - \frac{\theta^2}{d}$$

(SEC) $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \geq 0$ "generates" conjugate points:

$\theta(0) < 0$ then $\theta \to -\infty$ for $t \in [0, -d/\theta(0))$.

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Advanced focusing

All Lagrange[1] Jacobi tensors with $[A(0)] = \mathrm{id}$ and $\theta(0) \leq 0$ become singular for some $t > 0$.

- Needs analysis of full matrix Riccati eq. for $B = \dot{A}A^{-1}$:

$$\dot{B} + B^2 + R = 0$$

- Needs (SEC) and **genericity condition**:

tidal force operator R nontrivial at some $\gamma(t)$ (GC)

[1] $W(A,A) := \dot{A}^{\dagger}A - A^{\dagger}\dot{A} = 0$

The Hawking-Penrose theorem

Theorem [Hawking & Penrose, 1970]

Let (M, g) be a spacetime such that

(E) (SEC) holds, i.e., $\mathrm{Ric}(X, X) \geq 0$ for all X timelike, and (GC) holds along any causal geodesic γ

(C) it is chronological.

Moreover, assume it contains at least one of the following:

(I1) a compact achronal set without edge, (cosmological case)

(I2) a closed future trapped surface P, (grav. collapse)

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Then M is causal geodesically incomplete.

Low regularity: Why & How?

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- Basis: chartwise regularisation by convolution

$$g_\varepsilon(x) := g \star_M \rho_\varepsilon(x) := \sum \chi_i(x)\, \psi_i^*\Big(\big(\psi_{i\,*}(\zeta_i \cdot g)\big) * \rho_\varepsilon\Big)(x).$$

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Lemma 1 (Reg. and conv. for $g \in C^1$) [Graf, 20]

There are smooth $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ with $\check{g}_\varepsilon,\ \hat{g}_\varepsilon \to g$ in C^1, and

$$\|g - g_\varepsilon\|_{\infty,K} \leq c_K \varepsilon \quad \text{and} \quad \|\check{g}_\varepsilon - g_\varepsilon\|_{\infty,K} \leq c_K \varepsilon.$$

Analogously for $\hat{g}_\varepsilon$ and inverses g^{-1}, g_ε^{-1}, $(\check{g}_\varepsilon)^{-1}$, and $(\hat{g}_\varepsilon)^{-1}$.

C^1-focusing: The rough guide (1)

1. Formulate suitable (E) for $g \in C^1$
2. Derive surrogate (E) for $\check{g}_\varepsilon$: $\mathrm{Ric}[\check{g}_\varepsilon](X,X) > -\delta$ (on K cp.)
3. still show advanced focusing for $\check{g}_\varepsilon$
4. show that geodesics of g stop maximising.

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   $\rightsquigarrow$ cannot carry $\mathrm{Ric}[g](X,X) \geq 0$ through construction

   Solution: $\mathrm{Ric}[g] \star_M \rho_\varepsilon - \mathrm{Ric}[g_\varepsilon]$, $\mathrm{Ric}[g_\varepsilon] - \mathrm{Ric}[\check{g}_\varepsilon] \to 0$ loc. unif.

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Friedrichs Lemma for Γ-terms in Ric ($\Gamma \sim g^{-1}\,\partial g$)

$\partial g \in C^0$, $g_\varepsilon^{-1}, g_\varepsilon^{-1} \in C^1$ with $|g_\varepsilon^{-1} - g^{-1}|_{\infty,K} \leq C\varepsilon$ (Lem. 1!)
Then $g_\varepsilon^{-1}(\partial g \star \rho_\varepsilon) - (g^{-1}\partial g) \star \rho_\varepsilon \to 0$ in C^1.

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2. Derive (SEC), $R[\check{g}_\varepsilon](t) > \mathrm{diag}(c, -C, \ldots, -C)$ (SGC). ✓
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3. **Proposition 1** (Advanced smooth focusing under (SEC) & (SGC))
   There is $\delta > 0$, $T > 0$ such that any causal geo. γ with (r suitable)

   (i) $\mathrm{Ric}(\dot\gamma, \dot\gamma) \geq -\delta$ on $[-T, T]$ (ii) (SGC) holds on $[-r, r]$

   possesses a pair of conjugate points on $[-T, T]$.

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$\rightsquigarrow$ initial condition $\tilde{B}(t_1) = \tilde{\beta}(t_1)\,\mathrm{id}$, with $\tilde{\beta}(t_1) \geq$ largest eigenvalue of $B(t_1)$

$\Longrightarrow B \leq \tilde{B}$ on $[t_1, r]$ [Eschenburg & Heintze, 1990]

Equation for $\tilde{B}$ diagonal $\rightsquigarrow$ explicit solution with $B > \tilde{B}$ ↯

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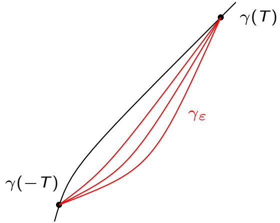
Stragtegy: Ass. γ glob. g-maximiser

- approximate γ by $\check{g}_\varepsilon$-geos. γ_ε
  $\Rightarrow \gamma_\varepsilon$ are $\check{g}_\varepsilon$-maximisers
- on cp. ngbhd. of γ turn
  (DSEC) & (DGC) into (i), (ii)
- Prop. 1 for δ small, T large
  $\Rightarrow \gamma_\varepsilon$ has conj. pts. $\Rightarrow$ ↯



Problem: How to approximate non-unique sols. of g-geo. eq.
by solutions to smooth $\check{g}_\varepsilon$ geo. eq. ???

Solution: Geodesic non-branching assumption

- well motivated from metric geometry
- adds novel aspect to interpretation of HE & GL sing. thms.

The Hawking-Penrose theorem

Theorem [Hawking & Penrose, 1970]

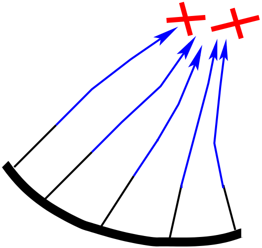
Let (M, g) be a spacetime such that

(E) (SEC) & (GC) hold (C) it is chronological.

Moreover, assume it contains at least one of the following:

(I1) a compact achronal set without edge, (cosmological case)
(I2) a closed future trapped surface P, (grav. collapse)
(I3) a future trapped point

Then M is causal geodesically incomplete.



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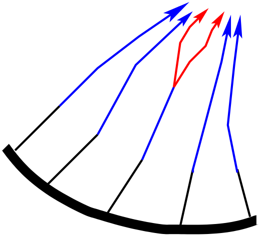

The C^1-Hawking-Penrose theorem

Theorem [KOSS, 22]

Let (M, g) be a C^1-spacetime such that

(E) (DSEC) & (DGC) hold (C) it is causal.

Moreover, assume it contains at least one of the following:
(I1) a compact achronal set without edge, (cosmological case)
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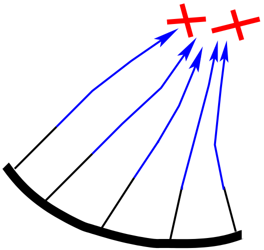
(I3) a future trapped point all in the sense of support manifolds
Then M is causal geodesically incomplete.



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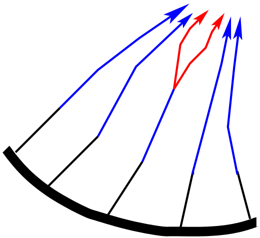

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Happy Birthday Nenad

Puno ti hvala