The Singularity Theorems in Low Regularity

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Overview

Long-term project on

Lorentzian geometry and general relativity
with metrics of low regularity

jointly with

- ‘theoretical branch’ (Vienna & U.K.):
  Melanie Graf, James Grant, Günther Hörmann, Mike Kunzinger, Clemens Sämann, James Vickers

- ‘exact solutions branch’ (Vienna & Prague):
  Jiří Podolský, Clemens Sämann, Robert Švarc
Contents

1 Remarks on low regularity
2 The $C^{1,1}$-singularity theorems
3 Key issues of the $C^{1,1}$-proofs
4 Outlook
Table of Contents

1. Remarks on low regularity
2. The $C^{1,1}$-singularity theorems
3. Key issues of the $C^{1,1}$-proofs
4. Outlook
Remarks on low Regularity

Why low regularity?

1. Physics: Realistic matter models $g \in C^{1,1}$ (derivs. loc. Lip.)
2. Analysis: ivp $g \in H^{5/2}(M), C^{1}(\Sigma)$, recent big improvements

The challenge

Physics and Analysis vs. Lorentzian geometry

want/need low regularity

needs high regularity

Regularity matters

[Hartman&Wintner, 1951]

$g_{ij}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |x|^\lambda \end{pmatrix}$ $1 < \lambda < 2, \ g \in C^{1,\lambda-1}$

- minimising curves not unique, even locally
- geodesics that are non-minimising between any of its points
The Lorentzian character matters

Riemannian case (added regularity of geodesics)

- $g \in C^0 \implies$ shortest (Lipschitz) curves exist [Hilbert, 1899]
- $g \in C^{0,\alpha} \implies$ all shortest curves are $C^{1,\beta}$ with $\beta = \frac{\alpha}{2-\alpha}$ (optimal) [Calabi, Hartman, 70], [Lytchak, Yaman, 06]
  in particular $\alpha = 1 = \beta$ and $\ddot{\gamma} = 0$ a.e.

Lorentzian case

- no length structure $\sim$ use geodesic equation.
- $g \in C^{0,1} \implies$ geodesics in the sense of Fillipov are $C^{1,1}$ [S., 2014]
- Is there an analogue of the Lytchak-Yaman result?
- causality goes severely wrong,
  light cones “bubble up” below $C^{0,1}$ [CG,12]
Table of Contents

1 Remarks on low regularity

2 The $C^{1,1}$-singularity theorems

3 Key issues of the $C^{1,1}$-proofs

4 Outlook
Regularity for the singularity theorems of GR

Pattern singularity theorem [Senovilla, 98]

In a $C^2$-spacetime the following are incompatible

(i) Energy condition     (iii) Initial or boundary condition
(ii) Causality condition (iv) Causal geodesic completeness

Theorem allows (i)–(iv) and $g \in C^{1,1}$. But $C^{1,1}$-spacetimes

- are physically reasonable models
- are not really singular (curvature bounded)
- still allow unique solutions of geodesic eq. $\leadsto$ formulation sensible

Moreover below $C^{1,1}$ we have

- unbded curv., non-unique geos, no convexity $\leadsto$ ‘really singular’

Hence $C^{1,1}$ is the natural regularity class for singularity theorems!
The classical Theorems

Theorem [Hawking, 1967]

A $C^2$ -spacetime is future timelike geodesically incomplete, if

(i) $\text{Ric} (X, X) \geq 0$ for every timelike vector $X$
(ii) There exists a compact space-like hypersurface $S$ in $M$
(iii) The unit normals to $S$ are everywhere converging

Theorem [Penrose, 1965]

A $C^2$ -spacetime is future null geodesically incomplete, if

(i) $\text{Ric} (X, X) \geq 0$ for every null vector $X$
(ii) There exists a non-compact Cauchy hypersurface $S$ in $M$
(iii) There exists a trapped surface $T$
    (cp. achronal spacelike 2-srf. w. past-pt. timelike mean curvature)
The $C^{1,1}$-Theorems

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<tr>
<th>Theorem</th>
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Obstacles in the $C^{1,1}$-case

- No appropriate version of calculus of variations available (second variation, maximizing curves, focal points, index form, . . . )

- $C^2$-causality theory rests on local equivalence with Minkowski space. This requires good properties of exponential map.

  ➔ big parts of causality theory have to be redone

  - Ricci tensors is only $L^\infty$

  ➔ problems with energy conditions

strategy:

- Proof that the exponential map is a bi-Lipschitz homeo

- Re-build causality theory for $C^{1,1}$-metrics
  regularisation adapted to causal structure replacing calculus of var.

- use surrogate energy condition
Table of Contents

1 Remarks on low regularity

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4 Outlook
The exponential map in low regularity

**Optimal regularity**

- \( g \in \mathcal{C}^{1,1} \Rightarrow \exp_p \text{ local homeo} \) [Whitehead, 1932]
- \( g \in \mathcal{C}^{1,1} \Rightarrow \exp_p \text{ local bi-Lipschitz homeo} \)

- [KSS,14]: regularisation & comparison methods [LeFloch&Chen,08]

**Consequences in \( \mathcal{C}^{1,1} \)**

- Convexity is okay (remember [HW, 51]!)
- Gauss lemma holds

\( \leadsto \) bulk of causality theory remains true [CG,12, KSSV,14, Ming.,15]
Chrusciel-Grant regularization of the metric

Regularisation adapted to the causal structure [CG,12], [KSSV, 14]

Sandwich null cones of $g$ between null cones of two approximating families of smooth metrics so that

$$\tilde{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon.$$ 

- applies to continuous metrics
- local convolution plus small shift

Properties of the approximations for $g \in C^{1,1}$

(i) $\tilde{g}_\varepsilon, \hat{g}_\varepsilon \to g$ locally in $C^1$

(ii) $D^2\tilde{g}_\varepsilon, D^2\hat{g}_\varepsilon$ locally uniformly bded. in $\varepsilon$, but $\text{Ric}[g_\varepsilon] \not\to \text{Ric}[g]$
Surrogate energy condition (Hawking case)

Lemma

Let \((M, g)\) be a \(C^{1,1}\)-spacetime satisfying the energy condition

\[ \text{Ric}[g](X, X) \geq 0 \]

for all timelike local \(C^{\infty}\)-vector fields \(X\).

Then for all \(K \subset \subset M\)

\[ \forall C > 0 \quad \forall \delta > 0 \quad \forall \kappa < 0 \quad \forall \varepsilon \text{ small} \]

\[ \text{Ric}[\tilde{g}_\varepsilon](X, X) > -\delta \quad \forall X \in TM|_K : \tilde{g}_\varepsilon(X, X) \leq \kappa, \|X\|_h \leq C. \]

Proof.

- \(\tilde{g}_\varepsilon - g * \rho_\varepsilon \to 0\) in \(C^2\) \(\sim\) only consider \(g_\varepsilon := g * \rho_\varepsilon\)
- \(R_{jk} = R^i_{jki} = \partial_{x^i} \Gamma^i_{kj} - \partial_{x^k} \Gamma^i_{ij} + \Gamma^i_{im} \Gamma^m_{kj} - \Gamma^i_{km} \Gamma^m_{ij}\)
- Blue terms\(|_\varepsilon\) converge uniformly
- For red terms use variant of Friedrich’s Lemma:
  \[
  \rho_\varepsilon \geq 0 \implies (\text{Ric}[g](X, X)) * \rho_\varepsilon \geq 0
  \]
  \[ (\text{Ric}[g](X, X)) * \rho_\varepsilon - \text{Ric}[g_\varepsilon](X, X) \to 0 \text{ unif.} \]
The $C^{1,1}$-proof (Hawking case)

- $D^+(S) \subseteq D^+_{\check{g}_\varepsilon}(S)$:

- Limiting argument $\Rightarrow \exists$ maximising $g$-geodesic $\gamma$ for all $p \in D^+(S)$ and $\gamma = \lim \gamma_{\check{g}_\varepsilon}$ in $C^1$

- Surrogate energy condition for $\check{g}_\varepsilon$ and Raychaudhury equation
  $\Rightarrow D^+(S)$ relatively compact
  otherwise $\exists$ $\check{g}_\varepsilon$-focal pt. too early
  $\Rightarrow H^+(S) \subseteq \overline{D^+(S)}$ compact

- Derive a contradiction as in the $C^\infty$-case using $C^{1,1}$-causality
Surrogate energy condition (Penrose case)

Lemma [KSV, 15]

Let \((M, g)\) be a \(C^{1,1}\)-spacetime satisfying the energy condition

\[
\text{Ric}[g](X, X) \geq 0 \quad \text{for every local Lip. null vector field } X.
\]

Then for all \(K \subset\subset M\) \(\forall C > 0 \ \forall \delta > 0 \ \exists \eta > 0\) s.t. we have

\[
\text{Ric}[\hat{g}_\varepsilon](X, X) > -\delta
\]

for all \(p \in K\) and all \(X \in T_p M\) with \(\|X\|_h \leq C\) which are close to a \(g\)-null vector in the sense that

\[
\exists Y_0 \in TM|_K \ g\text{-null, } \|Y_0\|_h \leq C, \ d_h(X, Y_0) \leq \eta.
\]
The $C^{1,1}$-proof (Penrose case)

- Choose $\hat{g}_\varepsilon$ globally hyperbolic (stability [NM,11], [S,15])
- Surrogate energy condition is strong enough to guarantee that

$$E^+_\varepsilon(T) = J^+_\varepsilon(T) \setminus I^+_\varepsilon(T)$$

is relatively compact in case of null geodesic completeness

- $\hat{g}_\varepsilon$ globally hyperbolic $\Rightarrow$

$$E^+_\varepsilon(T) = \partial J^+_\varepsilon(T)$$

is a $\hat{g}_\varepsilon$-achronal, compact $C^0$-hypersrf.

- $g < \hat{g}_\varepsilon$ $\Rightarrow$ $E^+_\varepsilon(T)$ is $g$-achronal
- derive usual (topological) contradiction
Table of Contents

1 Remarks on low regularity

2 The $C^{1,1}$-singularity theorems

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4 Outlook
Outlook

Related results, comparison geometry

- volume estimates for nullcones [Grant, 2011]
- volume comparison with (warped product) model spacetimes
  $\sim$ new $C^\infty$-proof of Hawking’s theorem [Grant, Treude, 2013]
- comparison geometry proof of $C^{1,1}$-Hawking theorem [Graf, 2016]
- rigidity results for singularity thms [Graf, 2016]

Current research

- Comparison approach to Penrose singularity theorem
  Evolve trapped surface along null geodesics, quantify area,
  should also give new proof in $C^{1,1}$.

- Mid term goal: [Hawking & Penrose singularity theorem] in $C^{1,1}$:
  Will require completely new methods.
Some related Literature


[S,14] R. Steinbauer, Every Lipschitz metric has $C^1$-geodesics. CQG 31, 057001 (2014)

Thank you for your attention!
Lemma [Hawking and Penrose, 1967]

In a causally complete $C^2$-spacetime, the following cannot all hold:

1. Every inextendible causal geodesic has a pair of conjugate points
2. $M$ contains no closed timelike curves and
3. there is a future or past trapped achronal set $S$

Theorem

A $C^2$-spacetime $M$ is causally incomplete if Einstein’s eqs. hold and

1. $M$ contains no closed timelike curves
2. $M$ satisfies an energy condition
3. Genericity: nontrivial curvature at some pt. of any causal geodesic
4. $M$ contains either
   - a trapped surface
   - some $p$ s.t. convergence of all null geodesics changes sign in the past
   - a compact spacelike hypersurface
Lemma  
[Hawking and Penrose, 1967]

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