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THE MEMBERSHIP PROBLEM IN FINITE FLAT HYPERGRAPH ALGEBRAS

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The membership problem asks whether a finite algebra belongs to the variety generated by another finite algebra. In some sense the β-function is the measure of the complexity of the membership problem. We investigate the β-function for finite flat hypergraph algebras and prove that in general it is not bounded by any polynomial.

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1. Introduction

Computability of algebraic properties has become more popular since the first computer was built. Our major interest in this article is the finite algebra membership problem in varieties: for any variety of algebras $\mathcal{V}$ we want to decide whether a given algebra $\mathbf{B}$ belongs to the variety. In the sequel $\mathcal{V}$ is assumed to be generated by a single finite algebra $\mathbf{A}$. Varieties are equational classes. Thus the membership problem can be decided by equation testing. To get a decision we may test some or all of the equations of the variety in the input algebra. The question arises naturally: what can be the complexity of such an equation testing. A complexity measure can be established for finite algebras via the notion of equational bound, defined by McNulty. The β-function is a map from the positive integers into the natural numbers. The value of $\beta(n)$ is less than or equal to k if for the decision whether an algebra of size less than n belongs to the variety it is enough to check the equations of length less than k. Székely [7] has shown an algebra of at least sublinear β-function. Kozik's dissertation [4] provides a construction of a finite algebra with PSPACE-hard membership problem and at least doubly exponential β-function. In this paper for arbitrary $d \geq 2$ we construct a finite flat hypergraph algebra, $\mathbf{A}$, such that for the corresponding β-function the following holds. There is a polynomial of degree d dominating $\beta_{\mathbf{A}}$. Moreover there exists another polynomial of the same degree dominated by $\beta_{\mathbf{A}}$ for infinitely many values of n.

2. Basic Definitions

An *algebra* $\mathbf{A} = \langle A, F \rangle$ is a nonempty set equipped with a system of finitary *fundamental operations*, $F = \langle f_i : i \in I \rangle$. A system of fundamental operation symbols F such that a nonnegative integer is assigned to each member of F is the *signature* of the algebra. An algebra is *finite*, if the *underlying* set is finite (i.e. $|A| < \infty$). $\mathbf{A}$ is of *finite signature*, if the system of fundamental operations is finite (i.e. $|F| < \infty$).

Let u be a *term* of some signature. The *length* of this term, $l(u)$, can be defined recursively. The length of a variable is 1. Suppose that the lengths of $u_1, u_2, \ldots, u_n$ are defined, and f is an n-ary fundamental operation symbol. Then $l(f(u_1, u_2, \ldots, u_n)) = 1 + \sum_{i=1}^{n} l(u_i)$. The *length* of an equation $l(u \approx v) = l(u) + l(v)$. The *rank* of an equation is the number of different variables occurring on the two sides.

We denote by $\mathbf{A} \models u \approx v$ that the algebra satisfies the equation $u \approx v$. If Σ is a set of equations, then $\mathbf{A} \models \Sigma$ denotes that every equation in Σ holds in $\mathbf{A}$. Let Σ^l denote the equations of length less than l in Σ, and $\Sigma_{\mathbf{A}}$ the set of all equations satisfied in $\mathbf{A}$. A *variety* $\mathcal{V}$ is a class of algebras axiomatized by some set of equations $\Sigma_{\mathcal{V}}$, i.e. $\mathbf{A} \in \mathcal{V} \Leftrightarrow \mathbf{A} \models \Sigma_{\mathcal{V}}$. A variety *generated* by an algebra $\mathbf{A}$ is the variety axiomatized by $\Sigma_{\mathbf{A}}$. By Birkhoff's famous theorem [1] $\mathcal{V} = HSP(\mathcal{V})$, and $\mathcal{V}(\mathbf{A}) = HSP(\mathbf{A})$, where H, S and P denote the operations forming homomorphic images, subalgebras and direct products, respectively. A variety is said to be *locally finite*, if every finitely generated algebra in the variety is finite.

We say, that a nontrivial equation $(u \approx v)$ follows from a set of equations Σ, if all algebras satisfying all equations in Σ satisfy $u \approx v$ as well. An algebra is said to be *finitely based*, if $\Sigma_{\mathbf{A}}$ is a consequence of some finite set of its equations, that is called the *equational basis* for $\mathbf{A}$. A variety is *finitely based*, if it can be axiomatized by a finite set of its equations. Clearly, $\mathbf{A}$ is finitely based, if and only if $\mathcal{V}(\mathbf{A})$ is finitely based. An algebra, or a variety is *nonfinitely based*, if it is not finitely based. A variety is *inherently nonfinitely based*, if it is locally finite, but contained in no locally finite finitely based variety. A *congruence* of an algebra is an equivalence relation which is compatible with the fundamental operations (i.e. if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbf{A}$, $(a_1, b_1) \in \Theta, (a_2, b_2) \in \Theta, \ldots, (a_n, b_n) \in \Theta$, and f

is a fundamental operation then $(f(a_1, a_2, \ldots, a_n), f(b_1, b_2, \ldots, b_n)) \in \Theta$. An algebra is *subdirectly irreducible*, if it has a unique minimal, nontrivial congruence. An algebra is a *subdirect product* of the family $\{\mathbf{A}_i : i \in I\}$, if there is an embedding $\iota : \mathbf{A} \to \prod_{\mathbf{i}\in\mathbf{I}} \mathbf{A_i}$, such that the image of $\iota(\mathbf{A})$ for every projection π_i is $\mathbf{A}_i$ $(i \in I)$. As it is stated in [1] every algebra can be built up from subdirectly irreducible ones, i.e. every algebra is a subdirect product of subdirectly irreducible algebras.

3. The Membership Problem

The *membership problem* for a given variety asks whether a finite algebra $\mathbf{B}$ belongs to the variety. In the sequel we will assume, that the variety is generated by a finite algebra $\mathbf{A}$ of finite signature. By definition $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ if and only if $\mathbf{B} \models \Sigma_{\mathbf{A}}$. So the membership problem can be answered by equation testing. Sometimes to decide whether $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ it is enough to check if a part of $\Sigma_{\mathbf{A}}$ holds in $\mathbf{B}$. For example, if $\mathbf{A}$ is finitely based, then we only have to check the equational basis of $\mathbf{A}$. Or if $|B| = n$, then we only have to check the equations of rank at most n in $\Sigma_{\mathbf{A}}$. So the rank of those equations we must check is bounded by $|B| = n$. Similar questions arise for the maximal length of the necessary equations.

The *β-function* or *equational bound* is a function $\beta : \mathbb{N} \to \mathbb{N}$ such that $\beta_{\mathbf{A}}(n) = \beta(n)$ is the maximal length of those equations that are necessary to decide whether an algebra of size less than n belongs to the variety. Precisely,

$$\beta(n) = \min\left\{l : \forall\, |\mathbf{B}| < n\, , \mathbf{B} \in \mathcal{V}(\mathbf{A}) \Leftrightarrow \mathbf{B} \models \Sigma^l_{\mathbf{A}}\right\}.$$

Or in another way,

$$\beta(n) = \max\left\{l : \exists\, |\mathbf{B}| < n\, , \mathbf{B} \notin \mathcal{V}(\mathbf{A}) \text{ but } \mathbf{B} \models \Sigma^l_{\mathbf{A}}\right\} + 1.$$

Clearly, these definitions give us the same function. By the second formula one can see that the β-function exists and it is uniquely determined for any variety $\mathcal{V} = \mathcal{V}(\mathbf{A})$, where $\mathbf{A}$ is a finite algebra of finite signature and it is recursive (it can be algorithmically computed).

A variety is said to be *constantly bounded*, if the β-function can be bounded by a constant: $\beta(n) \le C$ $(n \in \mathbb{N})$. An algebra is *constantly bounded*, if the corresponding variety is constantly bounded. Clearly, if an algebra is finitely based, then it is constantly bounded as well. As far as the converse statement is concerned only a weaker version is proved by Székely [7]. A similar result was proved by Cacioppo [2] for pseudovarieties of semigroups.

Proposition 1 (Székely). *Let* $\mathbf{A}$ *be a finite*, *constantly bounded algebra of finite signature. Then* $\mathbf{A}$ *is either finitely based or inherently nonfinitely based.*

However, the existence of an inherently nonfinitely based algebra which is constantly bounded is still an open problem. This question was firstly posed by Schützenberger and Eilenberg [5] in the context of pseudovarieties.

In what follows we investigate the β-function for some class of algebras called hypergraph algebras.

4. Hypergraph Algebras and Flat Hypergraph Algebras

Let $\mathbf{R} = \langle R, \alpha \rangle$ be a relational structure with one m-ary symmetric relation: $\alpha \subseteq R^m$ such that $(r_1, r_2, \ldots, r_m) \in \alpha \Leftrightarrow (r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(m)}) \in \alpha$ for every permutation π. These relational structures are some kind of hypergraphs, referred to as *m-uniform hypergraphs*. The elements of R are called *vertices* and the members of α are *edges*. A hypergraph is said to be *connected* if for any two vertices $r, s \in R$ there exists a sequence of edges $E_1, E_2, \ldots, E_l$ such that $r \in E_1$, $E_i \cap E_{i+1} \neq \emptyset$ $(1 \leq i < l)$ and $s \in E_l$. A *connected component* of a hypergraph is a maximal connected sub-hypergraph. The connected components give a partition of the vertex set R. A *path* in a hypergraph is a sequence of edges $E_1, E_2, \ldots, E_l$ such that $E_i \cap E_{i+1} \neq \emptyset$ $(1 \leq i < l)$. A *cycle* or a *closed path* is a path such that $E_1 = E_l$. Note that in a connected hypergraph there always exists a walk of size $\mathcal{O}(|\alpha|)$ containing every edge of the hypergraph:

Remark 2. If $\mathbf{R} = \langle R, \alpha \rangle$ is an m-uniform hypergraph, then there exists a cycle of size at most $2|\alpha|$, containing every edge of $\mathbf{R}$

Proof. We introduce a graph on the edges $\mathbf{G} = \langle \alpha, \epsilon \rangle$. For $E, F \in \alpha$ let $(E, F) \in \epsilon \Leftrightarrow E \cap F \neq \emptyset$. Observe that a cycle in $\mathbf{G}$ containing all of the vertices defines a required path in $\mathbf{R}$. It is well known that such a path in $\mathbf{G}$ of length at most $2|\alpha|$ exists. □

The *m-hypergraph algebra* belonging to $\mathbf{R}$ is $\mathbf{A_R} = \langle A_\mathbf{R}, f, 0 \rangle$ with one m-ary operation f and $A_\mathbf{R} = R \cup \{0\}$, where $0 \notin R$ is an absorbing element, i.e., if $0 \in \{x_1, x_2, \ldots, x_m\}$ then $f(x_1, x_2, \ldots, x_m) = 0$, and for $x_1, x_2, \ldots, x_m \in R$

$$f(x_1, x_2, \ldots, x_m) = \begin{cases} x_1, & \text{if } (x_1, x_2, \ldots, x_m) \in \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $m = 2$ we get a *graph algebra* introduced by Shallon [6]. A *flat m-hypergraph algebra* $\mathbf{F_R} = \langle F_\mathbf{R}, \wedge, f, 0 \rangle$ of $\mathbf{R}$ is defined as follows: $F_\mathbf{R} = R \cup \{0\}$ where $0 \notin R$ is an absorbing element, and for $x_1, x_2, \ldots, x_m \in R$

$$f(x_1, x_2, \ldots, x_m) = \begin{cases} x_1, & \text{if } (x_1, x_2, \ldots, x_m) \in \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

For $x, y \in R$

$$x \wedge y = \begin{cases} x, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\langle F_\mathbf{R}, \wedge, 0 \rangle$ is a semilattice of height 1. These semilattices are called *flat*. Also, $\langle F_\mathbf{R}, f, 0 \rangle$ is the m-hypergraph algebra of $\mathbf{R}$.

The algebras we present are flat hypergraph algebras. Willard in [10] gave a description of subdirectly irreducible algebras of flat algebras with absorbing element in general. We use his results in the context of flat hypergraph algebras to describe the subdirectly irreducible algebras in a variety generated by a single finite flat m-hypergraph algebra.

Theorem 3. *Let $\mathbf{F_R} = \langle F_\mathbf{R}, \wedge, f, 0\rangle$ be a finite flat m-hypergraph algebra and let $\mathbf{D} \in \mathcal{V}(\mathbf{F_R})$ be any finite algebra. Then the following are equivalent*:

(1) $\mathbf{D}$ *is subdirectly irreducible.*
(2) $\mathbf{D}$ *is a finite flat m-hypergraph algebrabelonging to a connected m-uniform hypergraph $\mathbf{S}$ such that $\mathbf{S} \leq \mathbf{R}^t$, an induced sub-hypergraph of the direct power $\mathbf{R}^t$ for some $t > 0$.*
(3) $\mathbf{D}$ *is simple.*

Thanks to this description, we only have to deal with hypergraphs in the sequel.

5. The Hypergraph r-Coloring Problem

A (*proper*) *r-coloring* of a hypergraph $\mathbf{R} = \langle R, \alpha\rangle$ is a mapping $c : R \to \{1, \ldots, r\}$ such that for every edge $(x_1, x_2, \ldots, x_m) \in \alpha$ the size of the set $|\{c(x_1), \ldots, c(x_m)\}| > 1$ (i.e. the edges of $\mathbf{R}$ are not monochromatic). $\mathbf{R}$ is said to be *r-colorable* if there exists such an r-coloring of $\mathbf{R}$.

An m-uniform hypergraph $\mathbf{R} = \langle R, \alpha\rangle$ is called *r-critical* if $\mathbf{R}$ is not r-colorable, but removing any of the edges of $\mathbf{R}$ results in an r-colorable m-uniform hypergraph (i.e. $\mathbf{R}' := \langle R, \alpha \setminus \{(a_1, a_2, \ldots, a_m)\}\rangle$ is r-colorable for any $(a_1, a_2, \ldots, a_m) \in \alpha$).

Let $M_r^m(n)$ denote the maximal number of edges possible in an r-critical m-uniform hypergraph having n vertices. Let $\mathbf{M}_r^m(n) = \langle V_n, \gamma_r^m\rangle$ denote an r-critical m-uniform hypergraph, with maximal number of edges. Toft [9] obtained some bounds on $M_m^r(n)$. Among other things he proved the following.

Theorem 4 (Toft). *For all $r \geq 4$ and all $m \geq 2$ there exists a positive constant c_r^m such that for infinitely many values of n the inequalities $c_r^m n^m \leq M_r^m(n)$ hold.*

In the sequel we will construct an m-uniform hypergraph $\mathbf{R}_m^r = \langle R_m^r, \alpha_m^r\rangle$ such that any finite m-uniform hypergraph $\mathbf{S} = \langle S, \gamma\rangle$ is r-colorable if and only if $\mathbf{S}$ is an induced sub-hypergraph of $(\mathbf{R}_m^r)^t$ for some finite $t > 0$. The vertex set of $\mathbf{R}_m^r$ is $R_m^r = \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r\}$ and the relation is defined as follows. $(x_1, x_2, \ldots, x_m) \in \alpha$ if and only if none of the following relations hold:

(1) $\{x_1, x_2, \ldots, x_m\} \subseteq \{b_1, b_2, \ldots, b_r\}$,
(2) $\{x_1, x_2, \ldots, x_m\} \subseteq \{a_i, b_i\}$ for any $1 \leq i \leq r$.

Note that for $m = 2$ this construction is the same as the one described by Székely in [8]. Clearly $\mathbf{R}_m^r$ is r-colorable, and $\mathbf{R}_m^r$ is a *universally r-colorable* m-hypergraph in the following sense.

Theorem 5. *Let* $\mathbf{S} = \langle S, \gamma \rangle$ *denote a finite* m*-uniform hypergraph. Then the following are equivalent*:

(1) $\mathbf{S}$ *is* r*-colorable.*
(2) $\mathbf{S} \leq (\mathbf{R}_m^r)^t$ *for a finite* $t > 0$.

Proof. A power of an r-colorable hypergraph is r-colorable (a product of hypergraphs is r-colorable if one of the factors is r-colorable). Obviously, an induced sub-hypergraph of any r-colorable hypergraph is r-colorable, as well.

For the converse, let $\mathbf{S} = \langle S, \gamma \rangle$ be an r-colorable m-uniform hypergraph and let $c : S \to \{1, \dots, r\}$ be a proper r-coloring of $\mathbf{S}$. For any (unordered) m-tuple $(x_1, x_2, \dots, x_m) \subseteq S^m$ and any pair in an arbitrary order (x, y) $(x \neq y, x, y \in S)$ we define a coordinate. Thus, we have $t = \binom{|S|}{m} + \binom{|S|}{2}$. We will inject S into $(R_m^r)^t$ by $\iota : S \hookrightarrow (R_m^r)^t$. The coordinates of ι are denoted by $\iota_j : \mathbf{S} \to \mathbf{R}_m^r$ for $1 \leq j \leq \binom{|S|}{m} + \binom{|S|}{2}$. Let $(x_1, x_2, \dots, x_m) \in S^m$ be the jth tuple, then for $s \in S$ we define

$$\iota_j(s) = \begin{cases} a_{c(s)}, & \text{if } (x_1, x_2, \dots, x_m) \in \gamma \quad \text{or } s \notin \{x_1, x_2, \dots, x_n\}; \\ b_{c(s)}, & \text{otherwise.} \end{cases}$$

For the jth pair, (x, y) we define

$$\iota_{\binom{|S|}{m}+j}(s) = \begin{cases} b_{c(s)}, & \text{if } s = x; \\ a_{c(s)}, & \text{otherwise.} \end{cases}$$

We claim that the image of $\iota : s \mapsto (\iota_1(s), \dots, \iota_t(s))$, $\iota(S) \subseteq (R_m^r)^t$ is an induced sub-hypergraph of $(R_m^r)^t$. Indeed, as $\mathbf{S}$ is r-colorable, if $(s_1, s_2, \dots, s_m) \in \gamma$ then for every $1 \leq j \leq t$ we have $(\iota_j(s_1), \dots, \iota_j(s_m)) \in \alpha$, and if $(s_1, s_2, \dots, s_m) \notin \gamma$, then for the coordinate $j = (s_1, s_2, \dots, s_m)$ the tuple $(\iota_j(s_1), \dots, \iota_j(s_m)) \notin \alpha$. The map ι is injective: for every element $x, y \in S, x \neq y$ the images $\iota(x)$ and $\iota(y)$ differ in the coordinate corresponding to the pair (x, y). □

6. Bounds on the β-Function

In general it is enough to check a bound on the β-function for subdirect irreducible algebras. The lower bound obviously follows from the second formula for the equational bound. And for the upper bound, it is because an algebra belongs to the variety if and only if its subdirectly irreducible factors are in $\mathcal{V}$, and the size of the subdirectly irreducible factors do not exceed the size of the original algebra.

From Theorems 3 and 5 we get an exact description of the subdirect irreducibles in the variety generated by $\mathbf{F}_{\mathbf{R}_m^r}$.

Theorem 6. *The finite subdirectly irreducible algebras of* $\mathcal{V}(\mathbf{F}_{\mathbf{R}_m^r})$ *are exactly those flat* m*-hypergraph algebras which belong to some connected* r*-colorable* m*-uniform hypergraph.*

6.1. *Lower bound*

From Theorems 4 and 6 we can give lower bounds on β-function.

Theorem 7. *Let $\beta^r_m(n) = \beta_{\mathbf{F}_{\mathbf{R}^r_m}}(n)$. Then for all $r \geq 4$ and all $m \geq 2$ there exists a positive constant c^m_r such that $c^m_r(n-2)^m < \beta^r_m(n)$ for infinitely many values of n.*

Proof. For the sake of simplicity let $\mathbf{A} = \mathbf{F}_{\mathbf{R}^r_m}$ and $\mathbf{B} = \mathbf{F}_{\mathbf{M}^m_r(n-2)}$. Note that $|B| = n-1$. As $\mathbf{M}^m_r(n-2)$ is not r-colorable $\mathbf{B} \notin \mathcal{V}(\mathbf{A})$. So, there exists an equation $p \approx q$ such that $\mathbf{A} \models p \approx q$, but $\mathbf{B} \not\models p \approx q$. We will prove that $l(p \approx q) \geq M^m_r(n-2)$. Since $\mathbf{B} \not\models p \approx q$, there is an evaluation e from the variable set of $p \approx q$ to $\mathbf{B}$ such that $e(p) \neq e(q)$. Suppose that there is an edge $(a_1, a_2, \ldots, a_m) \in \gamma^m_r$ of $\mathbf{M}^m_r(n-2)$ for which the term $f(a_1, a_2, \ldots, a_m)$ does not occur while evaluating $e(p)$ and $e(q)$. If such a thing happens, then the evaluation would be the same over $\widehat{\mathbf{B}} = \mathbf{F}_{\widehat{\mathbf{M}}^m_r(n-2)}$, where $\widehat{\mathbf{M}}^m_r(n-2) = \langle V_{n-2}, \gamma^m_r \setminus (a_1, a_2, \ldots, a_m)\rangle$. So $\widehat{\mathbf{B}} \not\models p \approx q$. But $\widehat{\mathbf{M}}^m_r(n-2)$ is r-colorable, so $\widehat{\mathbf{B}} \in \mathcal{V}(\mathbf{A})$, thus $\widehat{\mathbf{B}} \models p \approx q$, which is impossible. So in the computation of $f(a_1, a_2, \ldots, a_m)$, for every edge $(a_1, a_2, \ldots, a_m)$ must come up in the evaluation of either p or q. This means that the operation f must occur at least $M^m_r(n-2)$ times in the terms p and q. Therefore $l(p \approx q) \geq M^m_r(n-2)$. And for $M^m_r(n-2)$ we have the desired lower bound for infinitely many values of n. $\square$

As a straightforward consequence of the above theorem we have:

Corollary 8. *There is no polynomial upper bound on the $\beta_{\mathbf{A}}$ for all choices of the finite algebra $\mathbf{A}$.*

6.2. *Upper bound*

A natural way to get an upper bound on the β-function is to bound the length of representation of its terms. What is more, as is proved by Székely [7], this bound need only be valid in the algebra that generates the variety.

Lemma 9. *Let n be an integer and $\mathbf{A} = \langle A, F\rangle$ of maximal arity a. Suppose, that for every polynomial p of rank n there exists an other polynomial $\tilde{p}$ of length at most $b(= b(\mathbf{A}, n))$ such that $\mathbf{A} \models p \approx \tilde{p}$.*

Consider an algebra $\mathbf{B} = \langle B, F\rangle$ of the same signature generated by n elements, where $\mathbf{B} \notin \mathcal{V}(\mathbf{A})$. Then there exists an equation $p \approx q$ of length at most $(a+1)b+1$ such that $\mathbf{A} \models p \approx q$ and $\mathbf{B} \not\models p \approx q$

Corollary 10. $\beta_{\mathbf{A}}(n) \leq (a+1)b+1$

Székely deduced his result from Birkhoff's [1] paper. For this reason Székely refers to such upper bounds on β-function as *Birkhoff's bounds*.

To get upper bounds on the representations of the terms we will give a sufficient condition for the equivalence of two terms. The following lemmas and definitions are very technical. Let $\mathbf{F_S}$ be an arbitrary flat hypergraph algebra belonging to $\mathbf{S} = \langle S, \gamma \rangle$, and let p be a term of this signature. Let X denote the variable set of p. We can get the value of $e(p)$ at $e : X \to \mathbf{F_S}$ by iteration. We would like to describe those subterms which appear while evaluating.

Definition 11. The *reductions* of p are those terms that can be obtained from p by applying the following operations finitely many times.

(1) Writing u_1 instead of $f(u_1, u_2, \ldots, u_m)$, where $f(u_1, u_2, \ldots, u_m)$ is a subterm of some reduction of p.
(2) Writing u or v instead of $u \wedge v$, where $u \wedge v$ is a subterm of some reduction of p.

Let R_p denote the set of reductions of p. A reduction which is only a variable is the *beginning* of p. The set of the beginnings is denoted by X_p.

As the length of a reduction is less then the length of p, there are finitely many reductions of p. A reduction of a reduction of p is a reduction of p as well. Note, that if $e(p) \neq 0$ than $e(x) = e(y)$ for any $x, y \in X_p$, and thus $e(p) = e(x)$ for any $x \in X_p$. Now we define an equivalence relation θ_p on the set of variables. In essence two variables shall be equivalent if they agree at every nonzero evaluation of p.

Definition 12. $x\widetilde{\theta}_p y \Leftrightarrow x \wedge y$ is a subterm of a reduction of p. θ_p is the transitive closure of $\widetilde{\theta}_p$.

The equivalence class of x is denoted by $[x]$

Observe that X_p is contained in a single equivalence class of θ_p. Thus one of the equivalence classes of θ_p contains X_p. Let us denote this equivalence class by C_p. In the sequel we only deal with the equivalence classes: X/θ_p. To determine whether $e(p) \neq 0$ we need the following definition as well.

Definition 13. The m-uniform hypergraph belonging to p is $\mathbf{S_p} = \langle S_p, \gamma_p \rangle$, where $S_p = X/\theta_p$, and for the equivalence classes $(C_1, C_2, \ldots, C_m) \in \gamma_p$ if and only if there exists $x_i \in C_i$ $(1 \leq i \leq m)$ and a permutation such that $f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)})$ is a reduction of p.

Note that $\mathbf{S}_p$ is connected. One can get the above definitions by induction on the formula p. For this we introduce some new notations: Let $A_1, A_2, \ldots, A_l$ be sets of terms, then $(A_1, A_2, \ldots, A_l) = \{(a_1, a_2, \ldots, a_l) : a_i \in A_i \ (1 \leq i \leq l)\}$ and if g is an l-ary operation symbol, then $g(A_1, A_2, \ldots, A_l) = \{g(a_1, a_2, \ldots, a_l) : a_i \in A_i \ (1 \leq i \leq l)\}$.

Definition 14.

(1) If $p = x$ is a variable, then
   (a) $R_p = \{x\}$;
   (b) $X_p = \{x\}$;

(c) $\theta_p = 0$;
(d) $\mathbf{S_p} = \langle S_p, \gamma_p\rangle$ is defined by $S_p = \{[x]\}$ and $\gamma_p = \emptyset \subseteq S_p^m$.

(2) Let $p = f(p_1, p_2, \ldots, p_m)$ and suppose that everything is defined for $p_1, p_2, \ldots, p_m$. Then

(a) $R_p = R_{p_1} \cup f(R_{p_1}, R_{p_2}, \ldots, R_{p_m})$;
(b) $X_p = X_{p_1}$;
(c) $\theta_p = \theta_{p_1} \vee \theta_{p_2} \vee \ldots \vee \theta_{p_m}$;
(d) $\mathbf{S_p} = \langle S_p, \gamma_p\rangle$ is defined by $S_p = S_{p_1} \cup S_{p_2} \cup \ldots \cup S_{p_m}$ and $\gamma_p = \gamma_{p_1} \cup \gamma_{p_2} \cup \ldots \cup \gamma_{p_m} \cup (X_{p_1}, X_{p_2}, \ldots, X_{p_m})$.

(3) Let $p = p_1 \wedge p_2$ and suppose that everything is defined for p_1, p_2. Then

(a) $R_p = R_{p_1} \cup R_{p_2} \cup (R_{p_1} \wedge R_{p_2})$;
(b) $X_p = X_{p_1} \cup X_{p_2}$;
(c) $\theta_p = \theta_{p_1} \vee \theta_{p_2} \vee (X_{p_1}, X_{p_2})$;
(d) $\mathbf{S_p} = \langle S_p, \gamma_p\rangle$ is defined by $S_p = S_{p_1} \cup S_{p_2}$ and $\gamma_p = \gamma_{p_1} \cup \gamma_{p_2}$.

Now, we are able to state:

Lemma 15. *Let $e : X \to \mathbf{F_S}$ be an evaluation of p.*

(1) *Suppose that e is constant on the equivalence classes of θ_p, then it naturally defines a map $\widetilde{e} : \mathbf{S}_p \to \mathbf{S}$.*
(2) *Then $e(p) \neq 0$ if and only if it is constant on the equivalence classes, and $\widetilde{e} : \mathbf{S}_p \to \mathbf{S}$ is a hypergraph homomorphism.*
(3) *If $e(p) \neq 0$ then $e(p) = e(x)$, for any $x \in C_p$, which is well defined by the definition of C_p.*

So θ_p, C_p and $\mathbf{S}_p$ determine the value of p. Thus if $\theta_p = \theta_q$, $C_p = C_q$ and $\mathbf{S}_p = \mathbf{S}_q$ then $p \approx q$ over every flat hypergraph algebra.

Thanks to the above lemma we can define a short representation of a term of rank at most n.

Lemma 16. *For any term p of rank at most n there is another term $\widetilde{p}$, for which $\theta_p = \theta_{\widetilde{p}}$, $C_p = C_{\widetilde{p}}$, $\mathbf{S}_p = \mathbf{S}_{\widetilde{p}}$ and $l(\widetilde{p}) \leq (3m) \cdot n^m$. Thus p has a short representation.*

Proof. First we will construct a term p' such that the variable set of p' is X/θ_p, $\{C_p\} = X_{p'}$, $\theta_{p'} = 0$, and $\mathbf{S}_p = \mathbf{S}_{p'}$. Let $E_1, E_2, \ldots, E_l$ be a path of length at most $2|\gamma|$, containing every edge of S_p. Suppose that $C_p \in E_1$. We define p' by recursion. If $E_1 = (C_1^1, C_2^1, \ldots, C_m^1)$ where $C_p = C_1^1$, then $p_1' = f(C_1^1, C_2^1, \ldots, C_m^1)$. Suppose we have defined p_i'. $E_{i+1} = (C_1^{i+1}, C_2^{i+1}, \ldots, C_m^{i+1})$ and for example $C_j^i = C_1^{i+1} \in E_i \cap E_{i+1}$, then we get p_{i+1}' from p_i', by replacing C_j^i with $f(C_1^{i+1}, C_2^{i+1}, \ldots, C_m^{i+1})$. Then $p' = p_l'$ will be as required. Now we construct $\widetilde{p}$ just by replacing one C with $\wedge C$, and the other C's by any variable $x \in C$ for every equivalence class C of θ_p. Obviously $\widetilde{p}$ satisfies the conditions, and $l(\widetilde{p}) \leq m \cdot 2|\gamma| + n \leq 3m \cdot n^m$. □

From Lemmas 9 and 16 we can state:

Theorem 17. *For any m-hypergraph algebra*

$$\beta_{\mathbf{F_S}}(n) \leq 3m(m+1) \cdot n^m + 1.$$

Finally we have:

Theorem 18. *Let* $\beta_m^r(n) = \beta_{\mathbf{F}_{\mathbf{R}_m^r}}(n)$, *and let* c_r^m *defined as in Theorem* 4. *Then we have*:

(1) *For all* m, n *and* r

$$\beta_m^r(n) \leq 3m(m+1) \cdot (n+1)^m + 1.$$

(2) *For all* $r \geq 4$ *and all* $m \geq 2$ *there exists a positive constant* c_r^m *such that for infinitely many values of* n

$$c_r^m n^m \leq \beta_m^r(n).$$

7. Concluding Remarks

In [3] the first author and Kozik proved the analog of Theorem 6 for (di)graph algebras, without a flat-structure. Also a characterization of subdirectly irreducibles in the variety generated by graph algebras can be found here. Using these techniques similar results can be obtained for hypergraph algebras as well. But we postpone this for another occasion.

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References

[1] G. Birkhoff, On the structure of abstract algebras, *Proc. Cambridge Philos. Soc.* **31** (1935) 435–454.
[2] R. Cacioppo, Non-finitely based pseudovarieties and inherently non-finitely based varieties, *Semigroup Forum* **47**(2) (1993) 223–226.
[3] M. Kozik and G. Kun, The subdirectly irreducible algebras in the variety generated by graph algebras, *Algebra Univ.* (2006), submitted.
[4] M. Kozik, On some complexity problems in finite algebras, Ph.D Thesis, Vanderbilt University (2004).
[5] M. P. Schützenberg and S. Eilenberg, On pseudovarieties, *Adv. Math.* **19**(3) (1976) 413–418.
[6] C. Shallon, *Nonfinitely based finite algebras derived from lattices*, Ph.D. Thesis, University of California at Los Angeles (1979).

[7] Z. Székely, Complexity of the finite algebra membership problem for varieties, Ph.D Thesis, University of South California (1998).

[8] Z. Székely, Computational complexity of the finite algebra membership problem for varieties, *Int. J. Algebra Comput.* **12**(6) (2002) 811–823.

[9] B. Toft, On colour-critical hypergraphs, *Colloq. Math. Soc. János Bolyai* (1973) 1445–1457.

[10] R. Willard, On McKenzie's method, *Period. Math. Hungarica* **32** (1996) 149–165.